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On a Theorem of Louis de Branges. (**)

1. – LOUIS DE BRANGES [1] proved the following theorem for HANKEL transform of order ν.

Theorem. A necessary and sufficient condition for the functions $g(x)$, $f(x) \in L_2$ to be the HANKEL transforms of one another is that the equation

$$(1.1)\qquad \int_0^\infty f(t)\,(xt)^{\nu+(1/2)}\,e^{-x^2 t^2/2}\,dt = \int_0^\infty g(t)\,(x^{-1}t)^{\nu+(1/2)}\,e^{-t^2/(2x^2)}\,x^{-1}\,dt\,,$$

holds for all $x > 0$, $\nu > -1$, where $x^{\nu+(1/2)}\,e^{-x^2/2}$ is defined as a fundamental self-reciprocal function for the HANKEL transformation of order ν.

In this paper we generalise the above theorem for any transformation, the kernel function of which is a symmetrical FOURIER kernel.

2. – Now we give certain results used in the following sections.

Let $s = \sigma + it$ be a complex variable. Following TITCHMARSH ([6], p. 252) the author [4] has established the following result.

A necessary and sufficient condition that a function $f(x), \in A(\alpha,\, a)$ ([6], p. 252) should be its own k-transform, where $k(x)$, the kernel function of the transform is such that its MELLIN transform, $K(s)$ is $O(1)$, $K_1(s)$ is $O(e^{\lambda|t|})$ and $K_1(s)$ satisfies the relation

$$(2.1)\qquad K(s) = K_1(s)/K_1(1-s)$$

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is that $f(x)$ should be of the form

(2.2) $$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_1(s)\, \psi(s)\, x^{-s}\, ds ,$$

with

(2.3) $$\psi(s) = \psi(1-s),$$

where $\psi(s)$ and $K_1(s)$ are regular in the strip

(2.4) $$a < \sigma < 1 - a, \qquad a < 1/2 ,$$

$\psi(s)$ is $O(e^{(-\lambda-\alpha+\eta)|t|})$ for every positive η and uniformly in the strip (2.4) and c in any value of σ in the strip (2.4).

In the equation

(2.5) $$g(x) = \int_0^\infty f(g)\; k(xy)\, dy$$

we can conclude from L_2 theory following TITCHMARSH ([**6**], p. 221) that $g(x)$, $f(x) \in L_2$ if $k(x)$ is a symmetrical FOURIER kernel.

Denoting the MELLIN transform of $f(x)$ by $F(s)$, the PARSEVAL relation ([**3**], p. 391) for MELLIN trasform, with parameter x $(x > 0)$ is given by

(2.6) $$\int_0^\infty f_1(ux)\, f_2(u)\, du = \frac{1}{2\pi i} \int_{(1/2)-i\infty}^{(1/2)+i\infty} F_1(s)\, F_2(1-s)\, x^{-s}\, ds,$$

where $f_1(x)$, $f_2(x) \in L_2$ and $F_1(s)$, $F_2(s) \in L_2(\frac{1}{2} - i\,\infty,\ \frac{1}{2} + i\infty)$.

We now extend the definition of fundamental self-reciprocal function for the HANKEL transform of order ν to k-transform.

If in (2.2) $\psi(s) = 1$, we have

(2.7) $$f(x) = k_f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_1(s)\, x^{-s}\, ds,$$

then $k_f(x)$ is called the fundamental self-reciprocal function for the k-transform.

3. – Theorem. *A necessary and sufficient condition for the functions* $g(x)$, $f(x) \in A(\alpha, a)$ *and* L_2, $K(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, *and* $K_1(s)$ *satisfies* (2.1) *and* $\in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ *to be the k-transform of one another is that the equation*

$$(3.1) \qquad \int_0^\infty f(t)\, k_f(xt)\, dt = \int_0^\infty g(t)\, k_f(t/x)\, \frac{dt}{x}$$

holds good for all $x > 0$, *where* $k_f(x)$ *is the fundamental self-reciprocal function for the k-transform.*

Proof. Condition is necessary. We have

$$(3.2) \qquad g(x) = \int_0^\infty f(y)\, k(xy)\, dy\,.$$

As $f(y) \in L_2$ and $K(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ we can use PARSEVAL relation (2.6) on the R.H.S. of (3.2) thus we have

$$(3.3) \qquad g(x) = \frac{1}{2\pi i} \int_{(1/2)-i\infty}^{(1/2)+i\infty} K(s)\, F(1-s)\, x^{-s}\, ds\,, \qquad x > 0.$$

As $g(x) \in L_2$ the MELLIN transform of $g(x)$ exists and (3.3) reduces to

$$(3.4) \qquad G(s) = F(1-s)\, K(s)\,,$$

using (2.1) in (3.4) we have

$$(3.5) \qquad G(s)\, K_1(1-s) = F(1-s)\, K_1(s)\,.$$

As $K_1(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ so taking the inverse MELLIN transform of both sides of (3.5), we have

$$\frac{1}{2\pi i} \int_{(1/2)-i\infty}^{(1/2)+i\infty} G(s)\, K_1(1-s)\, x^{-s}\, ds = \frac{1}{2\pi i} \int_{(1/2)-i\infty}^{(1/2)+i\infty} K_1(s)\, F(1-s)\, x^{-s}\, ds\,.$$

Using the PARSEVAL relation (2.6) and (2.7), we get

$$(3.6) \qquad \int_0^\infty g(xu)\, k_f(u)\, du = \int_0^\infty k_f(tx)\, f(t)\, dt\,.$$

Replacing xu by t in left hand side of (3.6), we have

$$\int_0^\infty g(t)\, k_f(t/x)\, x^{-1}\, dt = \int_0^\infty f(t)\, k_f(xt)\, dt\,. \tag{3.7}$$

Condition is sufficient. Retracing the steps from (3.7) to (3.2) it can be shown that $g(x)$ is the k-transform of $f(x)$. To show that $f(x)$ is the k-transform of $g(x)$ we retrace steps from (3.7) to (3.4), replace s by $1-s$ to obtain

$$G(1-s)\, K_1(s) = F(s)\, K_1(1-s)\,. \tag{3.8}$$

Moving on similar lines from (3.4) to (3.2), we get

$$f(x) = \int_0^\infty g(y)\, k(xy)\, dy$$

instead of (3.2). This concludes sufficiency.

4. – By selecting different kernel functions we get relations for different transforms. We mention few cases,

Corollary 1. *If*

$$k(x) = x^{1/2} J_\nu(x)\,, \qquad \nu > -1\,,$$

then

$$K(s) = 2^{s-(1/2)}\, \Gamma\left(\frac{1}{4} + \frac{\nu}{2} + \frac{s}{2}\right) \Big/ \Gamma\left(\frac{3}{4} + \frac{\nu}{2} - \frac{s}{2}\right),$$

$$K_1(s) = 2^{s/2}\, \Gamma\left(\frac{1}{4} + \frac{\nu}{2} + \frac{s}{2}\right)$$

giving fundamental self-reciprocal function $x^{\nu+(1/2)}\, e^{-x^2/2}$.

Using the theorem, equation (3.1) reduces to (1.1) which is a known result.

Corollary 2. *If* $k(x) = \omega_{\mu,\nu}(x)$ $(\mu > -1,\ \nu > -1)$ *the kernel introduced by* Watson [7], *then*

$$K_1(s) = 2^s\, \Gamma\left(\frac{1}{4} + \frac{\nu}{2} + \frac{s}{2}\right)\ \Gamma\left(\frac{1}{4} + \frac{\mu}{2} + \frac{s}{2}\right),$$

and the fundamental self-reciprocal function for Watson *transform is*

$$2\, G^{2,0}_{0,2}\left(\frac{x^2}{4}\,\middle|\, \frac{1}{2} + \frac{\mu}{2},\ \frac{1}{4} + \frac{\nu}{2}\right).$$

Using the result ([2], p. 434) this becomes

$$2^{(3-\mu-\nu)/2}\, x^{(\mu+\nu+1)/2}\, K_{(\mu-\nu)/2}(x)\,,$$

where $K_\alpha(x)$ denotes the modified Bessel function of order α. Using the theorem, the equation (3.1) assumes the form

$$(4.1)\qquad \int_0^\infty g(t)\, (tx^{-1})^{(\mu+\nu+1)/2}\, K_{(\mu-\nu)/2}(tx^{-1})\, x^{-1}\, dt = \int_0^\infty f(t)\, (xt)^{(\mu+\nu+1)/2}\, K_{(\mu-\nu)/2}(xt)\, dt\,.$$

Corollary 3. *If* $k(x) = \chi_{\nu,k,m}(x)$ *the kernel function studied by* R. Narain ([5], p. 271), *then*

$$(4.2)\qquad K_1(s) = \frac{2^{s/2}\, \Gamma\left(\frac{1}{4} + \frac{\nu}{2} + \frac{s}{2}\right)\ \Gamma\left(\frac{1}{4} + \frac{\nu}{2} + 2m + \frac{s}{2}\right)}{\Gamma\left(\frac{3}{4} + \frac{\nu}{2} + m - k + \frac{s}{2}\right)}\,,$$

$$\operatorname{Re} s \geqslant s_0 > 0\,, \qquad \operatorname{Re}\,(\nu + 1 + 2m \pm 2m) > 0\,,$$

$2m$ *not an integer and the fundamental self-reciprocal function for* $\chi_{\nu,k,m}$*-transform is given by*

$$G^{2\,0}_{1,2}\left(\frac{x^2}{2}\,\middle|\, \begin{matrix} \frac{3}{4} + \frac{\nu}{2} + m - k \\ \frac{1}{4} + \frac{\nu}{2},\ \frac{1}{4} + \frac{\nu}{2} + 2m \end{matrix}\right), \qquad \operatorname{Re}\,(\nu + 1 + 2m \pm 2m) > 0.$$

Applying the theorem, the equation (3.1) becomes

$$(4.3) \qquad \int_0^\infty f(t)\, G_{1,2}^{2,0}\left(\frac{t^2 x^2}{2} \,\middle|\, \begin{matrix} \frac{3}{4} + \frac{\nu}{2} + m - k \\ \frac{1}{4} + \frac{\nu}{2},\ \frac{1}{4} + \frac{\nu}{2} + 2m \end{matrix} \right) dt =$$

$$= \int_0^\infty g(t)\, G_{1,2}^{2,0}\left(\frac{t^2}{2\,x^2} \,\middle|\, \begin{matrix} \frac{3}{4} + \frac{\nu}{2} + m - k \\ \frac{1}{4} + \frac{\nu}{2},\ \frac{1}{4} + \frac{\nu}{2} + 2m \end{matrix} \right) x^{-1}\, dt .$$

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References.

[1] L. de BRANGES, *Self-reciprocal functions*, J. Math. Anal. Appl. **9** (1964), 433-457.

[2] A. ERDELYI et al., *Tables of Integral Transforms*, Vol. 2, McGraw-Hill, New York 1964.

[3] C. FOX, *A formal solution of certain dual integral equations*, Trans. Amer. Math. Soc. **119** (1965), 389-398.

[4] D. G. JOSHI, *On self-reciprocal functions*, Proc. Nat. Acad. Sci. India **38** A (1968), 193-202.

[5] R. NARAIN, *On a generalization of Hankel transform and self-reciprocal functions*, Univ. e Politec. Torino, Rend. Sem. Mat. **16** (1956-57), 269-300.

[6] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, Claredon Press, Oxford 1937.

[7] G. N. WATSON, *Some self-reciprocal functions*, Quart. J. Math. Oxford **2** (1931), 298-309.

Abstract.

In this paper we have generalised the theorem of Hankel transform given by Louis de Branges.

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