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Kummer's Transform of Two Variables. (**)

1. - Introduction.

The object of this paper is to define a new transform which we call KUMMER's transform of two variables. An inversion formula and the uniqueness theorem, for this transform will be given in n. 2 and n. 3 respectively. We have also obtained some theorems for this transform. Particular cases of these theorems give rise to known results given earlier by BOSE [1].

A generalisation of the classical LAPLACE transform

$$\varphi(p) = p \int_0^\infty e^{-pt}\, h(t)\, \mathrm{d}t\,, \tag{1}$$

has been introduced by ERDELYI [3] in the form

$$\varphi(p) = \frac{\Gamma(\alpha)}{\Gamma(\beta)}\, p \int_0^\infty {}_1F_1(\alpha;\ \beta;\ -px)\,(px)^\gamma\, f(x)\, \mathrm{d}x\,, \tag{2}$$

where ${}_1F_1$ denotes the KUMMER's hypergeometric function and $\operatorname{Re} p > 0$; when $\alpha = \beta$ and $\gamma = 0$, (2) reduces to (1).

We define the KUMMER's trasform in two variables by the integral relation

$$\varphi_{\gamma,\delta}^{\alpha,\beta}(p,\ q) = \frac{\Gamma(\alpha)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\delta)}\, p\, q \int_0^\infty\!\!\int_0^\infty {}_1F_1(\alpha;\ \beta; -px)\, {}_1F_1(\gamma;\ \delta;\ -qy)\, f(x,\ y)\, \mathrm{d}x\, \mathrm{d}y, \tag{3}$$

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which will be denoted by

$$\varphi_{\gamma,\delta}^{\alpha,\beta}(p,\ q) \xrightarrow[\alpha,\ \beta;\ \gamma,\ \delta]{F} f(x,\ y)\,;$$

when $\alpha = \beta$, $\gamma = \delta$, then (3) reduces to the LAPLACE transform in two variables

$$\varphi(p,\ q) = pq \int_0^\infty \int_0^\infty e^{-px-qy}\, f(x,\ y)\, \mathrm{d}x\, \mathrm{d}y\,, \tag{4}$$

which will be denoted by

$$\varphi(p,\ q) \doteqdot f(x,\ y)\,.$$

2. - An Inversion Formula.

The following is the theorem which gives the solution of the integral equation (3).

Theorem. *If $g(x, y)$ be a function defined by*

$$g(x,\ y) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} \frac{x^{-r}\, y^{-s}\, \mathrm{d}r\, \mathrm{d}s}{H_1(1-r)\, H_2(1-s)}\,,$$

where

$$H_1(r) = \frac{\Gamma(r)\ \Gamma(\alpha - r)}{\Gamma(\beta - r)}\,, \qquad H_2(s) = \frac{\Gamma(s)\, \Gamma(\gamma - s)}{\Gamma(\delta - s)}\,,$$

then

$$f(x,\ y) = \int_0^\infty \int_0^\infty g(px,\ qy)\ \varphi[f:\ p,\ q]\ \frac{\mathrm{d}p}{p}\,\frac{\mathrm{d}q}{q}\,, \tag{5}$$

provided that $|\,g(x,\ y)\,|$ *exists and* $p^{-c}\, q^{-c_1}\, \varphi[f:\ p,\ q] \in L(0,\ \infty)$,

$$x^{c-1}\, y^{c_1-1}\, f(x,\ y) \in L(0,\ \infty)\,, \qquad c < \mathrm{Re}\,\beta + 1\,, \qquad c_1 < \mathrm{Re}\,\gamma + 1\,.$$

Proof. Substituting the value of $\varphi[f:\ p,\ q]$ from (3), we have

$$\int_0^\infty\int_0^\infty p^{-r-1}\, q^{-s-1}\, \varphi[f:\ p,\ q]\, \mathrm{d}p\, \mathrm{d}q =$$

$$= \int_0^\infty\int_0^\infty p^{-r}\, q^{-s} \left\{\frac{\Gamma(\alpha)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\delta)} \int_0^\infty\int_0^\infty {}_1F_1(\alpha;\beta;-px)\, {}_1F_1(\gamma;\,\delta;\,-qy) f(x,\,y)\, \mathrm{d}x\, \mathrm{d}y\right\} \mathrm{d}p\, \mathrm{d}q\,,$$

which, on changing the order of integration and simplification, yields

$$\frac{\Gamma(\alpha)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\delta)} \int_0^\infty\int_0^\infty x^{r-1}\, y^{s-1}\, f(x,\,y) \left\{\int_0^\infty\int_0^\infty u^{-r} v^{-s}{}_1F_1(\alpha;\,\beta;\,-u){}_1F_1(\gamma;\,\delta;\,-v)\, \mathrm{d}u\, \mathrm{d}v\right\} \mathrm{d}x\, \mathrm{d}y.$$

Since the double integral in x and y and the double integral in u and v are independent of each other, on evaluating u and v integrals, we find that

$$\int_0^\infty\int_0^\infty x^{r-1} y^{s-1}\, f(x,\ y)\, \mathrm{d}x\, \mathrm{d}y = \int_0^\infty\int_0^\infty p^{-r-1} q^{-s-1} \frac{\varphi[f:\ p,\ q]}{H_1(1-r)\ H_2(1-s)}\, \mathrm{d}p\, \mathrm{d}q\,.$$

Applying MELLIN's inversion formula and changing the order of integration again, we obtain

$$f(x,\ y) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} x^{-r} y^{-s}\,\cdot$$

$$\cdot \left\{\int_0^\infty\int_0^\infty p^{-r-1}\, q^{-s-1}\, \frac{\varphi[f:\ p,\ q]}{H_1(1-r)\ H_2(1-s)}\, \mathrm{d}p\, \mathrm{d}q\right\} \mathrm{d}r\, \mathrm{d}s =$$

$$= \int_0^\infty\int_0^\infty \frac{\varphi[f:\ p,\ q]}{pq} \left[\frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} \frac{(px)^{-r}\, (qy)^{-s}\, \mathrm{d}r\, \mathrm{d}s}{H_1(1-r)\ H_2(1-s)}\right] \mathrm{d}p\, \mathrm{d}q\,,$$

or

$$f(x,\ y) = \int_0^\infty\int_0^\infty \frac{\varphi[f:\ p,\ q]}{pq}\, g(px\,,\ qy)\, \mathrm{d}p\, \mathrm{d}q\,.$$

3. – We now prove the following

Lemma. *If*

$$\int_0^\infty\int_0^\infty x^\lambda\, y^\mu\, {}_1F_1(\alpha;\ \beta;\ -px)\, {}_1F_1(\gamma;\ \delta;\ -qy)\, f(x,\ y)\, dx\, dy = 0\ , \tag{6}$$

where $\operatorname{Re}(p,\ q) > 0$, *then* $f(x,\ y) \equiv 0$, *provided that*

(i) $f(x,\ y)$ *be a continuous function of* x *and* y *in* $x,\ y \geqslant 0$,

(ii) $F(x,\ y) = O(x^{a_1} y^{a_2})$ *for small* x *and* y *and* $\operatorname{Re}(a_1+1) > 0$, $\operatorname{Re}(a_2+1) > 0$,

(iii) $\operatorname{Re}\beta > 0$, $\operatorname{Re}\delta > 0$.

Proof. Multiplying (6) by

$$p^{\varrho-1}\, q^{\sigma-1}\, G^{\lambda_1,\mu_1}_{\lambda_1+\mu_1,2\lambda_1}\left[\left(\frac{p}{\lambda_1}\right)^{\lambda_1}\left(\frac{1}{a\mu_1}\right)^{\mu_1}\ \middle|\ \begin{matrix}\Delta(\lambda_1\,,\ 1), & \Delta(\lambda_1\,,\ \alpha-\varrho)\\ \Delta(\lambda_1\,,\ \beta-\varrho), & \Delta(\lambda_1\,,\ 1-\varrho)\end{matrix}\right]\cdot$$

$$\cdot\, G^{\lambda_2,\mu_2}_{\lambda_2+\mu_2,2\lambda_2}\left[\left(\frac{q}{\lambda_2}\right)^{\lambda_2}\left(\frac{1}{b\mu_2}\right)^{\mu_2}\ \middle|\ \begin{matrix}\Delta(\lambda_2\,,\ 1), & \Delta(\lambda_2\,,\ \gamma-\sigma)\\ \Delta(\lambda_2\,,\ \delta-\sigma), & \Delta(\lambda_2\,,\ 1-\sigma)\end{matrix}\right],$$

and integrating with respect to p and q between the limit 0 to ∞, we have

$$\int_0^\infty\int_0^\infty x^\lambda\, y^\mu\, f(x,\ y)\, dx\, dy \int_0^\infty\int_0^\infty p^{\varrho-1}\, {}_1F_1(\alpha;\ \beta;\ -px)\, q^{\sigma-1}\, {}_1F_1(\gamma;\ \delta;\ -qy)\,\cdot$$

$$\cdot\, G^{\lambda_1,\mu_1}_{\lambda_1+\mu_1,2\lambda_1}\left[\left(\frac{p}{\lambda_1}\right)^{\lambda_1}\left(\frac{1}{a\mu_1}\right)^{\mu_1}\ \middle|\ \begin{matrix}\Delta(\lambda_1\,,\ 1), & \Delta(\lambda_1\,,\ \alpha-\varrho)\\ \Delta(\lambda_1\,,\ \beta-\varrho), & \Delta(\lambda_1\,,\ 1-\varrho)\end{matrix}\right]\cdot$$

$$\cdot\, G^{\lambda_2,\mu_2}_{\lambda_2+\mu_2,2\lambda_2}\left[\left(\frac{q}{\lambda_2}\right)^{\lambda_2}\left(\frac{1}{b\mu_2}\right)^{\mu_2}\ \middle|\ \begin{matrix}\Delta(\lambda_2\,,\ 1), & \Delta(\lambda_2\,,\ \gamma-\sigma)\\ \Delta(\lambda_2\,,\ \delta-\sigma), & \Delta(\lambda_2\,,\ 1-\sigma)\end{matrix}\right] dp\, dq = 0\ ,$$

on changing the order of integration which is justified because of the absolute convergence of the integrals under the conditions given above.

On evaluating the integrals with the help of SAXENA's formula ([5], p. 40).

$$(7)\begin{cases} \dfrac{\Gamma(\alpha)}{\Gamma(\beta)}\, ps \displaystyle\int_0^\infty t^{\varrho-1}\, {}_1F_1(\alpha;\ \beta;\ -pt) G^{r,s}_{r+s,2s}\left[\left(\frac{t}{r}\right)^r \left(\frac{1}{as}\right)^s \middle| \begin{matrix} \Delta(r,1), & \Delta(r,\alpha-\varrho) \\ \Delta(r,\beta-\varrho), & \Delta(r,1-\varrho)\end{matrix}\right] \mathrm{d}t = \\ = p^{1-\varrho}\, e^{-ap^{r/s}} (2\pi)^{(s-r)/2}, \end{cases}$$

where $r < s$, $\operatorname{Re}\beta > 0$, $\operatorname{Re} p > 0$, $|\arg a| < ((s-r)/2)\pi$.

On evaluating the q-integral, the last line reduces to

$$(8)\begin{cases} \displaystyle\int_0^\infty\int_0^\infty x^\lambda y^\mu\, \mathrm{d}x\, \mathrm{d}y \int_0^\infty p^{\varrho-1}\, {}_1F_1(\alpha;\ \beta; -px)\cdot \\ \cdot G^{\lambda_1,\mu_1}_{\lambda_1+\mu_1,2\lambda_1}\left[\left(\frac{p}{\lambda_1}\right)^{\lambda_1}\left(\frac{1}{a\mu_1}\right)^{\mu_1} \middle| \begin{matrix} \Delta(\lambda_1,1), & \Delta(\lambda_1,\alpha-\varrho) \\ \Delta(\lambda_1,\beta-\varrho), & \Delta(\lambda_1,1-\varrho)\end{matrix}\right] f_2(x,\ y)\, \mathrm{d}x\, \mathrm{d}y = 0\,. \end{cases}$$

Again on evaluating the p-integral, (8) becomes

$$(9)\qquad \int_0^\infty\int_0^\infty x^\lambda y^\mu x^{-\varrho} y^{-\sigma} e^{-ax^{\lambda_1/\mu_1}} e^{-by^{\lambda_2/\mu_2}} f_1(x,\ y)\, \mathrm{d}x\, \mathrm{d}y = 0\,,$$

where

$$f_1(x,\ y) = \left\{1 + O\left(\frac{1}{ax^{\lambda_1/\mu_1}}\right)\right\} f_2(x,\ y)$$
$$= \left\{1 + O\left(\frac{1}{ax^{\lambda_1/\mu_1}}\right)\right\}\left\{1 + O\left(\frac{1}{by^{\lambda_2/\mu_2}}\right)\right\} f(x,\ y)\,.$$

Now on setting

$$x^{\lambda_1/\mu_1} = \xi\,, \qquad y^{\lambda_2/\mu_2} = \eta\,,$$

(9) transforms into

$$\frac{\mu_1\,\mu_2}{\lambda_1\,\lambda_2}\int_0^\infty\int_0^\infty e^{-a\xi-b\eta}\, \xi^{(\mu_1/\lambda_1)(1+\lambda)-1}\, \eta^{(\mu_2/\lambda_2)(1+\mu)-1}\, f_1(\xi^{\mu_1/\lambda_1},\ \eta^{\mu_2/\lambda_2})\, \mathrm{d}\xi\, \mathrm{d}\eta = 0\,.$$

Now let

$$\int_0^\infty e^{-a\xi}\, \xi^{(\mu_1/\lambda_1)(1+\lambda)-1} f_1(\xi^{\mu_1/\lambda_1},\ \eta^{\mu_2/\lambda_2})\, \mathrm{d}\xi = \psi(\eta).$$

Since the integral converges uniformly in $y \geqslant 0$, $\psi(\eta)$ is a continuous function of η, we then have

$$\int_0^\infty e^{-b\eta}\, \eta^{(\mu_2/\lambda_2)(1+\mu)-1}\, \psi(\eta)\, \mathrm{d}\eta = 0\,.$$

An application of LERCH's theorem gives $\psi(\eta) = 0$, i. e.

$$\int_0^\infty e^{-a\xi}\, \xi^{(\mu_1/\lambda_1)(1+\lambda)-1}\, f_1(\xi^{\mu_1/\lambda_1},\ \eta^{\mu_2/\lambda_2})\, \mathrm{d}\xi \equiv 0\,,$$

since $f_1(\xi^{\mu_1/\lambda_1},\ \eta^{\mu_2/\lambda_2})$ is a continuous function of ξ, η therefore LERCH's theorem again gives $f_1(\xi^{\mu_1/\lambda_1},\ \eta^{\mu_2/\lambda_2}) \equiv 0$ or

$$\left\{1 + O\left(\frac{1}{a\xi}\right)\right\}\left\{1 + O\left(\frac{1}{b\eta}\right)\right\} f(\xi^{\mu_1/\lambda_1},\ \eta^{\mu_2/\lambda_2}) \equiv 0\,.$$

Hence $f(\xi^{\mu_1/\lambda_1},\ \eta^{\mu_2/\lambda_2}) \equiv 0$, i. e. $f(x,\ y) \equiv 0$.

As a direct consequence of the above lemma, we obtain the following

Uniqueness Theorem. *Let $f_1(x,\ y)$ and $f_2(x,\ y)$ be continuous functions of both the variables in the range $x,\ y \geqslant 0$ and*

$$\varphi(p,\ q) \,\frac{F}{\overline{\overline{\alpha,\ \beta;\ \gamma,\ \delta}}}\, f_1(x,\ y)$$

and also

$$\varphi(p,\ q) \,\frac{F}{\overline{\overline{\alpha,\ \beta;\ \gamma,\ \delta}}}\, f_2(x,\ y)\,,$$

then

$$f_1(x,\ y) \equiv f_2(x,\ y)\,. \tag{10}$$

4. – In this section we establish some theorems for the transform defined by (3).

Theorem 1. *If*

$$\varphi_1(p,\ q) \,\frac{F}{\overline{\overline{\alpha,\ \beta;\ \gamma,\ \delta}}}\, f_1(x,\ y)$$

and

$$\varphi_2(p,\ q) \xlongequal[\alpha,\ \beta;\ \gamma,\ \delta]{F} f_2(x,\ y)\,,$$

then

$$\int_0^\infty\!\!\int_0^\infty \varphi_1(u,\ v)\ f_2(u,\ v)\,\frac{\mathrm{d}u}{u}\,\frac{\mathrm{d}v}{v} = \int_0^\infty\!\!\int_0^\infty \varphi_2(s,\ t)\ f_1(s,\ t)\,\frac{\mathrm{d}s}{s}\,\frac{\mathrm{d}t}{t}\,, \tag{11}$$

provided the order of integration may be inverted.

Proof. We have

$$\int_0^\infty\!\!\int_0^\infty \varphi_1(u,\ v)\ f_2(u,\ v)\,\frac{\mathrm{d}u}{u}\,\frac{\mathrm{d}v}{v} =$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\delta)} \int_0^\infty\!\!\int_0^\infty f_2(u,\ v)\left\{\int_0^\infty\!\!\int_0^\infty f_1(s,\ t)\,{}_1F_1(\alpha;\ \beta; -us)\ {}_1F_1(\gamma;\ \delta; -vt)\ \mathrm{d}s\,\mathrm{d}t\right\}\mathrm{d}u\,\mathrm{d}v =$$

$$= \int_0^\infty\!\!\int_0^\infty f_1(s,t)\ st\left\{\frac{\Gamma(\alpha)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\delta)}\int_0^\infty\!\!\int_0^\infty {}_1F_1(\alpha;\ \beta; -\ us)\ {}_1F_1(\gamma;\ \delta; -\ vt) f_2(u,\ v)\ \mathrm{d}u\,\mathrm{d}v\right\}\frac{\mathrm{d}s}{s}\,\frac{\mathrm{d}t}{t} =$$

$$= \int_0^\infty\!\!\int_0^\infty \varphi_2(s,\ t)\ f_1(s,\ t)\,\frac{\mathrm{d}s}{s}\,\frac{\mathrm{d}t}{t}\,.$$

The change of the order of integration is justified provided that the double integrals within brackets above are absolutely and uniformly convergent. This means that

$$\mathrm{Re}\,(\xi_1 + 1) > 0,\ \ \mathrm{Re}\,(\xi_2 + 1) > 0,\ \ \text{where } f_1(s,\ t) = \begin{cases} O(s^{\xi_1}) & \text{for small } s \\ O(t^{\xi_2}) & \text{for small } t \end{cases}$$

and

$$\mathrm{Re}\,(\zeta_1 + 1) > 0,\ \ \mathrm{Re}\,(\zeta_2 + 1) > 0,\ \ \text{where } f_2(u,\ v) = \begin{cases} O(u^{\zeta_1}) & \text{for small } u \\ O(v^{\zeta_1}) & \text{for small } v. \end{cases}$$

When $\alpha = \beta$, $\gamma = \delta$, (11) reduces to BOSE's result [**1**].

Theorem 2. *If*

$$f(x,\ y) = f_1(x)\, f_2(y)\,,$$

and

$$\varphi_1(p) \underset{\alpha,\ \beta}{\overset{F}{=\!=}} f_1(x)\,, \qquad \varphi_2(q) \underset{\gamma,\ \delta}{\overset{F}{=\!=}} f_2(y)\,,$$

then

$$f(x,\ y) \underset{\alpha,\ \beta;\ \gamma,\ \delta}{\overset{F}{=\!=}} \varphi(p,\ q) \underset{\alpha,\ \beta;\ \gamma,\ \delta}{\overset{F}{=\!=}} \varphi_1(p)\, \varphi_2(q)\ , \tag{12}$$

provided that the integrals are absolutely convergent.

The proof is trivial.

Example 1. If we take $f(x,\ y) = x^\lambda y^\mu$, then by virtue of the integral

$$\int_0^\infty x^\varrho\, {}_1F_1(\alpha;\ \beta;\ -px)\, dx = p^{-\varrho} \frac{\Gamma(\alpha - \varrho - 1)\ \ \Gamma(\beta)\ \ \Gamma(\varrho + 1)}{\Gamma(\alpha)\, \Gamma(\beta - \varrho - 1)} \tag{13}$$

we find that

$$x^\lambda y^\mu \underset{\alpha,\ \beta;\ \gamma,\ \delta}{\overset{F}{=\!=}} p^{-\lambda} q^{-\mu} \frac{\Gamma(\alpha - \lambda - 1)\, \Gamma(\gamma - \mu - 1)\, \Gamma(\lambda + 1)\, \Gamma(\mu + 1)}{\Gamma(\beta - \lambda + 1)\, \Gamma(\delta - \mu - 1)}\ ,$$

where $\mathrm{Re}(\lambda + 1) > 0$, $\mathrm{Re}(\mu + 1) > 0$, $\mathrm{Re}(p, q) > 0$.

Example 2. Taking $f(x,\ y) = x^\lambda y^\mu\ e^{-ax-by}$, and using the relation $\Gamma(\alpha)/\Gamma(\beta) \int_0^\infty e^{-ax} x^\varrho\, {}_1F_1(\alpha;\ \beta;\ -px)\, dx = \Gamma(1 + \varrho)\ {}_2F_1(1 + \varrho,\ \alpha;\ \beta; -(p/a))$, we see that

$$\begin{aligned} x^\lambda y^\mu\ e^{-ax-by} \underset{\alpha,\ \beta;\ \gamma,\ \delta}{\overset{F}{=\!=}} pq\, [\Gamma(1 + \lambda)\, \Gamma(1 + \mu)] \cdot \\ \cdot\ {}_2F_1\left(1 + \lambda\,,\ \alpha\ ;\ \beta; -\frac{p}{a}\right) {}_2F_1\left(1 + \mu,\ \gamma;\ \delta;\ -\frac{q}{b}\right), \end{aligned} \tag{14}$$

where $\mathrm{Re}(\lambda + 1) > 0$, $\mathrm{Re}(\mu + 1) > 0$, $\mathrm{Re}(p + a) > 0$, $\mathrm{Re}(q + b) > 0$. When

$\alpha = \beta$, $\gamma = \delta$, results (5) and (6) give the wellknown results in the theory of LAPLACE transform of two variables.

Theorem 3. *If*

$$p(\varphi\ q) \xlongequal[\alpha,\ \beta;\ \gamma,\ \delta]{F} f(x,\ y)\ ,$$

then

$$\varphi\left(\frac{p}{a},\ \frac{q}{b}\right) \xlongequal[\alpha,\ \beta;\ \gamma,\ \delta]{F} f(ax,\ by)\ , \tag{15}$$

provided that the integrals converge absolutely.

The proof is trivial.

Theorem 4. *If*

$$\varphi(p,\ q) \doteqdot f(x,\ y)\ ,$$

then

$$\frac{\varphi(\log p,\ \log q)}{\log p\ \log q} \xlongequal[\alpha,\ \beta;\ \gamma,\ \delta]{F} \frac{\Gamma(\alpha)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\delta)} \int_0^\infty\!\!\int_0^\infty \frac{\Gamma(\beta-s-1)\,\Gamma(\delta-t-1)\,f(s,\ t)\,\mathrm{d}s\,\mathrm{d}t}{\Gamma(\alpha-s-1)\,\Gamma(\gamma-t-1)\,\Gamma(s+1)\,\Gamma(t+1)}\ . \tag{16}$$

Proof. We have

$$\frac{\varphi\,(\log p,\ \log q)}{\log p\ \log q} = \int_0^\infty\!\!\int_0^\infty p^{-s}\ q^{-t}\ f(s,\ t)\ \mathrm{d}s\ \mathrm{d}t \xlongequal[\alpha,\ \beta;\ \gamma,\ \delta]{F}$$

$$\xlongequal[\alpha,\ \beta;\ \gamma,\ \delta]{F} \frac{\Gamma(\alpha)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\delta)} \int_0^\infty\!\!\int_0^\infty \frac{\Gamma(\beta-s-1)\,\Gamma(\delta-t-1)\,f(s,\ t)\,\mathrm{d}s\,\mathrm{d}t}{\Gamma(\gamma-t-1)\ \Gamma(\alpha-s-1)\ \Gamma(s+1)\ \Gamma(t+1)}\ ,$$

on using (13).

When $\alpha = \beta$, $\gamma = \delta$, we get a result given by BOSE [1].

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