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On Hyponormal Operators. (**)

A bounded operator T defined on a Hilbert space H is said to be a hyponormal if $\| Tx \| \geqslant \| T^* x \|$ for any $x \in H$. This definition has an equivalent form: we say that the operator T is hyponormal if $T^* T \geqslant T T^*$. Here we shall utilise the former definition only.

The spectrum of an operator T, to be denoted by $\sigma(T)$, is the set of all complex numbers for which $(T - \lambda I)^{-1}$ does not exist. The approximate point spectrum of an operator T denoted by $\alpha(T)$, is the set of complex numbers λ such that $\| (T - \lambda I) x_n \| \to 0$, for $\| x_n \| = 1$. The numerical range of an operator T denoted by $W(t)$ is the set defined by the relation $W(T) = \{(Tx, x): x \in H, \|x\| = 1\}$. The closure of the set $W(T)$ will be denoted by $\overline{W(T)}$. We defined $\gamma_\sigma(T) = \sup\{| \lambda |: \lambda \in \sigma(T)\}$ and call $\gamma_\sigma(T)$ the spectral radius of T.

Lemma 1. *If* T *is hyponormal, then* $\|T\| = \sup \{ | (Tx, x) |: \| x \| = 1\}$. This Lemma is contained in [2].

Theorem 1. *Let* T *be a hyponormal operator, then*

$$\| T \| = \sup\{ | \lambda |: \quad \lambda \in \sigma(T)\}.$$

Proof. It is well known [3] that, for any operator T, $\lambda \in' \sigma(T)$ if $| \lambda | > \| T \|$. On the other hand if T is hyponormal, then by Lemma 1 there exist a sequence of elements $\{x_n\}$ such that $\| x_n \| = 1$ for all n and $\lim_{n \to \infty} | (Tx_n, x_n) | = \| T \|$.

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If necessary by choosing a subsequence, we may suppose that $\lim_{n\to\infty} (Tx_n, x_n) = \lambda$, where $|\lambda| = \|T\|$. Then

$$\|Tx_n - \lambda x_n\|^2 = (Tx_n, Tx_n) - \lambda(x_n, Tx_n) - \bar{\lambda}(Tx_n, x_n) + \lambda\bar{\lambda}(x_n, x_n)$$

$$\leqslant \|T\|^2 - \lambda(x_n, Tx_n) - \bar{\lambda}(Tx_n, x_n) + |\lambda|^2 .$$

Thus

$$\limsup_{n\to\infty} \|Tx_n - \lambda x_n\|^2 \leqslant \|T\|^2 - \lambda\bar{\lambda} - \bar{\lambda}\lambda + |\lambda|^2 = 0 .$$

Hence $T - \lambda I$ cannot have any bounded inverse, so $\lambda \in \sigma(T)$ and

$$\|T\| \leqslant \sup\{|\lambda| : \lambda \in \sigma(T)\} \leqslant \|T\| .$$

Thus the Theorem is established.

It may be mentioned that Theorem 1 is an indirect proof of a result due to STAMPFLI [4].

Lemma 2. *Let* T *be a hyponormal operator such that* $Tx = \mu x$, $x \neq \varnothing$, *then* $T^* x = \bar{\mu} x$.

This occurs as an exercise in [1].

Lemma 3. *For a hyponormal operator* T, *eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. This also occurs as an exercise in [1]. The proof is given here for the sake of easiness.

Let λ and μ be the eigenvalues of the hyponormal operator T corresponding to the eigen vectors x and y respectively. Then by

$$|(\lambda - \mu)(x, y)| = |(\lambda x - Tx, y) + (x, T^* y - \bar{\mu} y)$$

$$\leqslant \|\lambda x - Tx\| \|y\| + \|x\| \|T^* y - \bar{\mu} y\| = 0.$$

Since by Lemma 2 $Ty = \mu y$ implies $T^* y = \bar{\mu} y$. Hence the proof is complete.

Theorem 2. *Let* T *be a hyponormal operator on the Hilbert space H which possesses a set of eigenvectors fundamental in H and C a closed subspace of H*

which has at least one non-null eigenvector and invariant under T, *then* C *is generated by eigenvectors of* T.

Proof. We have seen in Lemma 2 that the eigenvectors corresponding to different eigenvalues are orthogonal to each other. By our assumption the set of eigenvalues is fundamental in H. So if $N_T(\lambda_i)$ denote the λ_i-th proper subspace of the operator T, that is, $N_T(\lambda_i) = \{x_i: \ Tx_i = \lambda_i x_i, \ x_i \in H\}$, then

$$H = \sum_{i=1}^{\infty} N_T(\lambda_i), \qquad \text{where} \qquad N_T(\lambda_i) \perp N_T(\lambda_j) \qquad \text{for } i \neq j.$$

Let us suppose that M be a closed subspace of C generated by eigenvectors of C. Then our theorem would be established if we show that $M \equiv C$. Let us suppose the contrary, that is $M \not\equiv C$. Then $C' = C \cap M^{\perp}$ is non-vacous. Now if $x \in C'$, then $x \in C$ and also $Tx \in C$, because C is invariant under T. Let $y \in M$, then $Ty = \mu y$ implies $T^* y = \bar{\mu} y$, where μ is a certain scalar. Therefore $(Tx, y) = (x, T^* y) = (x, \bar{\mu} y) = \mu (x, y) = 0$, because $x \in C' \Longrightarrow x \in M^{\perp}$. Hence $Tx \in C'$ and C' is invariant under T. By our assumption C' contains an eigenvector $\Phi \neq \varnothing$ and so $\Phi \perp M$. Now since M is a closed subspace of C generated by eigenvectors of C, so Φ also belongs to M. But this is a contradictory to our supposition. Hence the Theorem is established.

Lemma 4. *For a hyponormal operator* T, $\gamma_\sigma(T) = \| T \|$.
It is contained in [4].

Theorem 3. *Let* T *be a hyponormal operator and* λ *a complex number such that* $\gamma_\sigma(T) = |\lambda|$ *and* $\lambda \in \overline{W(T)}$, *then* $\{p(\lambda, \bar{\lambda}): \lambda \in \overline{W(T)}\} \subset \sigma[p(T, T^*)]$ *where* $p(\lambda, \bar{\lambda})$ *is any polynomial in* $\lambda, \bar{\lambda}$ *and the operator corresponding to it is* $p(T, T^*)$.

Proof. Let $\lambda \in \overline{W(T)}$. Then there exists a sequence of elements $x_n \in H$ such that $(Tx_n, x_n) \to \lambda$. Now, for $\| x_n \| = 1$,

$$\begin{aligned} \| Tx_n - \lambda x_n \|^2 &= (Tx_n, Tx_n) - \bar{\lambda} (Tx_n, x_n) - \lambda (x_n, Tx_n) + \lambda\bar{\lambda} (x_n, x_n) \\ &\leqslant \| T \|^2 - \bar{\lambda} (Tx_n, x_n) - \lambda (x_n, Tx_n) + \lambda\bar{\lambda} \\ &\to \| T \|^2 - \bar{\lambda}\lambda - \lambda\bar{\lambda} + \lambda\bar{\lambda} = 0, \end{aligned}$$

by Lemma 4.

Consider the following identity:

$$T^r T^{*s} - \lambda^r \bar{\lambda}^s = T^r (T^{*s} - \bar{\lambda}^s I) + \bar{\lambda}^s (T^r - \lambda^r I)$$

$$= T^r (T^{*s-1} + \ldots + \bar{\lambda}^{s-1} I)(T^* - \bar{\lambda} I)$$

$$+ \bar{\lambda}^s (T^{r-1} + \ldots + \lambda^{r-1} I)(T - \lambda I).$$

Now as shown above, the hypotheses $\gamma_\sigma(T) = |\lambda|$ and $\lambda \in \overline{W(T)}$ ensure that $\lambda \in a(T)$ and so $\| Tx_n - \lambda x_n \| \to 0$, as shown above. Since T is hyponormal, so is $T - \lambda I$ for any complex number λ, therefore $\|(T^* - \bar{\lambda} I)x_n\| \leqslant \|(T - \lambda I)x_n\| \to 0$. It clearly follows from the above identity that

$$\| (T^r T^{*s} - \lambda^r \bar{\lambda}^s) x_n \| \to 0, \qquad \text{as } n \to \infty.$$

Adding the relations of the above kind, we at once deduce that $\| p(T, T^*)x_n - p(\lambda, \bar{\lambda})x_n \| \to 0$, as $n \to \infty$. From this we conclude that $\{p(\lambda, \bar{\lambda}) : \lambda \in \overline{W(T)}\} \subset \sigma[p(T, T^*)]$.

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