Riv. Mat. Univ. Parma (2) 9 (1968), 327-337

S. S. DALAL (*)

On Entire and Meromorphic Functions. (**)

§ 1. - Theorems.

Let $F(z)$ be a meromorphic function of order ϱ $(0 < \varrho < +\infty)$ and let

$$S(t) = \frac{1}{\pi} \int_0^t \int_0^{2\pi} \left\{ \frac{|F'(r e^{i\theta})|}{1 + |F(r e^{i\theta})|^2} \right\}^2 r \, dr \, d\theta .$$

Then it is known that [1] $T(r, F) = T(r)$, the NEVANLINNA characteristic function of $F(z)$, is given by

$$T(r) = \int_0^r \{S(t)/t\} \, dt ,$$

(see L. V. AHLFORS [2]).

It is easy to observe that $T(r)$ is an increasing convex function of $\log r$.

Theorem 1. *Let $f(z)$ be an entire function of order ϱ with genus p. Let*

$$N_p(r) = r^p \int_0^r \{n(x)/x^{p+1}\} \, dx, \qquad Q_p(r) = r^{p+1} \int_r^\infty \{n(x)/x^{p+2}\} \, dx .$$

Then

$$\limsup_{r \to \infty} \{N_p(r)/Q_p(r)\} \geqslant \{(p+1-\varrho)/(\varrho - p)\} .$$

If we take $p = 0$, the result of R. P. BOAS [3] follows as a particular case.

(*) Indirizzo: Department of Mathematics, University of Vienna, Vienna, Austria.
(**) Ricevuto: 5-XII-1967.

Theorem 2. *Let $F(z)$ be a meromorphic function of non integral order ϱ with genus p.*
Then

$$\limsup_{r\to\infty} \frac{N_p(r,\ a) + N_p(r,\ b)}{T(r,\ F)} \geqslant 1 + p - \varrho$$

for all a and b ($a \neq b$; $0 \leqslant |a| \leqslant +\infty$, $0 \leqslant |b| \leqslant +\infty$).

The result of S. K. Singh [4] follows as a particular case if we take $p = 0$.

Theorem 3.

i) $$\lim_{r\to\infty} \frac{\{T(ar)\}^c}{\{S(br)\}^d} = \begin{cases} \infty & \text{if } c\lambda > d\varrho \\ 0 & \text{if } c\varrho < d\lambda, \end{cases}$$

ii) $$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\{T(ar)\}^c}{\{S(br)\}^d} = \begin{cases} \infty & \text{if } c > d \\ 0 & \text{if } c < d, \end{cases}$$

where a, b, c and d are constants.

Above results improve the results of K. Manjanathaih [5].

Theorem 4. *Let*

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{T(r)}{r^\varrho} = \begin{cases} A \\ B, \end{cases} \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{S(r)}{r^\varrho} = \begin{cases} C \\ D. \end{cases}$$

Then

$$(1) \qquad \begin{cases} D \leqslant A\lambda, \qquad C \geqslant B\varrho, \\ \varrho/\lambda \leqslant (A/B)\{1 + \log(C/D)\} \leqslant (AC)/(BD). \end{cases}$$

The result of Hari Shankar [6] follows as a particular case, if we put $A = B$, $C = D$, $\varrho = 1$ in (1).

Theorem 5. *Let $F(z)$ be a meromorphic function of order ϱ and lower order λ. Then*

$$\limsup_{r\to\infty} \sum_{i=1}^{q} \{n(r,\ a_i)/r^{\lambda(r)}\} \geqslant (q-2)\ \lambda$$

and

$$\limsup_{r\to\infty} \sum_{i=1}^{q} \{n(r,\ a_i)/r^{\varrho(r)}\} \geqslant (q-2)\ \varrho\ ,$$

where $\varrho(r)$, $\lambda(r)$ *are proximate order and lower proximate order respectively, with respect to* $T(r, F)$.

Theorem 6. *A meromorphic function is minimal type if and only if* $N_1(r) = o(r^\varrho)$, *provided* $N(r, 1) = o(r^\varrho)$.

And of mean type if and only if $N_1(r) = O(r^\varrho)$, *provided* $N(r, 1) = O(r^\varrho)$, *where* $N_1(r) = N(r, 0) + N(r, \infty)$.

Theorem 7. *If* $F(z)$ *is a meromorphic function of zero order such that*

$$T(r, F) = O(\log r)^p ,$$

then

$$N_1(r) = O(\log r)^p , \qquad Q_1(r) = O(\log r)^{p-1} .$$

And conversly if

$$Q_1(r) = O(\log r)^{p-1} ,$$

then

$$T(r) = O(\log r)^p , \qquad N_1(r) = O(\log r)^p ,$$

where

$$Q_1(r) = r \int_r^\infty \{n_1(t)/t^2\}\, dt , \qquad N_1(r) = \int_0^r \{n_1(t)/t\}\, dt ,$$

and

$$n_1(t) = n(t, 0) + n(t, \infty) .$$

§ 2. - Proof of Theorems.

2.1. - Proof of Theorem 1.

Let

(2) $$\limsup_{r \to \infty} \{N_p(r)/Q_p(r)\} = A .$$

If ϱ is an integer, $A = \infty$, then theorem is trivially true. So we assume that ϱ is not an integer and $A < \infty$.

Then, for $B > A$ we have

$$N_p(r) < Q_p(r)\, B ,$$

i. e.,

$$N_p(r) < B\, r^{p+1} \int_r^\infty \{n(x)/x^{p+2}\}\, \mathrm{d}x$$

$$< B\, r^{p+1} \int_r^\infty \{(N_p(x)/x^p)'/x\}\, \mathrm{d}x$$

$$< B\, r^{p+1}[N_p(x)/x^{p+1}]_r^\infty + B\, r^{p+1} \int_r^\infty \{N_p(x)/x^{p+2}\}\, \mathrm{d}x$$

$$< - B\, N_p(r) + B\, r^{p+1} \int_r^\infty \{N_p(x)/x^{p+2}\}\, \mathrm{d}x\,.$$

[Because $N_p(r)/r^{p+1} \to 0$, as $r \to \infty$.]

Hence

$$(1 + B)\, N_p(r) < B\, r^{p+1} \int_r^\infty \{N_p(x)/x^{p+2}\}\, \mathrm{d}x\,.$$

So:

$$\frac{B}{(1+B)\, r} > \frac{N_p(r)}{r^{p+2} \int_r^\infty \{N_p(x)/x^{p+2}\}\, \mathrm{d}x}$$

and

(3)
$$\frac{B}{(1+B)\, r} + \frac{(\mathrm{d}/\mathrm{d}r) \int_r^\infty \{N_p(t)/t^{p+2}\}\, \mathrm{d}t}{\int_r^\infty \{N_p(x)/x^{p+2}\}\, \mathrm{d}x} > 0\,.$$

Now

$$\int_{r_0}^\infty \{N_p(t)/t^{p+1}\}\, \mathrm{d}t > N_p(r_0) \int_{r_0}^\infty t^{-p-1}\, \mathrm{d}t = \{N_p(r_0)/(p\, r_0^p)\} > K\,.$$

Hence, integrating (3) from r_0 to r, we get

$$\int_r^\infty \{N_p(t)/t^{p+2}\}\, \mathrm{d}t > K_1\, r^{-B/(1+B)}\,.$$

So

$$\limsup_{r\to\infty} \{N_p(r)\, r^{-(p+1+Bp)/(1+B)}\} > 0.$$

Hence

$$N_p(r) > K_2\, r^{(p+1+Bp)/(1+B)}\,,$$

$$\frac{\log N_p(r)}{\log r} > \frac{p+1+Bp}{1+B} \frac{\log r}{\log r} + o(1)\,.$$

So, taking limit,

$$\varrho \geqslant (Bp + p + 1)/(1 + B)\,, \quad \text{i. e.,} \quad \varrho \geqslant (Ap + p + 1)/(1 + A)\,,$$

hence

$$A \geqslant \frac{p + 1 - \varrho}{\varrho - p}\,.$$

2.2. - Proof of Theorem 2.

Without any loss of generality we can take $a = 0$ and $b = \infty$. It is known [1] that

$$F(z) = z^{\varkappa}\, e^{Q(z)}\, \{\pi_1(z)/\pi_2(z)\}\,,$$

where $Q(z)$ is a polynomial of degree $q \leqslant p$ and π_1, π_2 are canonical products of genera p at most.

We known [7] that

$$T(r,\ F) < T(r,\ \pi_1) + T(r,\ \pi_2) + O(\log r + r^q)\,,$$

$$T(r,\ F) < \log M(r,\ \pi_1) + \log M(r,\ \pi_2) + O(\log r + r^q)\,,$$

$$T(r,\ F) < \int\limits_0^\infty \frac{n(x,\ 0)\, r^{p+1}}{(x + r)\, x^{p+1}}\, dx + \int\limits_0^\infty \frac{n(x,\ \infty)\, r^{p+1}}{(x + r)\, x^{p+1}}\, dx + O(r^p) + O(\log r + r^q)\,,$$

$$T(r,\ F) < \int\limits_0^\infty \frac{\{n(x,\ 0) + n(x,\ \infty)\}\, r^{p+1}}{(x + r)\, x^{p+1}}\, dx + O(r^p + \log r)\,.$$

Now let $n(x,\ 0) + n(x,\ \infty) = n(x)$. So:

$$T(r,\ F) < r^p \int\limits_0^\infty \{n(x)/x^{p+1}\}\, dx + r^{p+1} \int\limits_r^\infty \{n(x)/x^{p+2}\}\, dx + O(r^p + \log r)\,,$$

$$T(r,\ F) < N_p(r) + Q_p(r) + O(r^p + \log r)\,,$$

$$T(r,\ F)/N_p(r) \leqslant 1 + \{Q_p(r)/N_p(r)\} + o(1)\,,$$

$$\liminf_{r \to \infty} \frac{T(r,\ F)}{N_p(r)} \leqslant 1 + \frac{1}{\limsup\limits_{r \to \infty} \{N_p(r)/Q_p(r)\}}\,.$$

Now let

$$\limsup_{r\to\infty} \{N_p(r)/Q_p(r)\} = A .$$

Hence

$$\liminf_{r\to\infty} \{T(r, F)/N_p(r)\} \leqslant 1 + 1/A ,$$

but

$$A \geqslant (p + 1 - \varrho)/(\varrho - p) .$$

So

$$\limsup_{r\to\infty} \{N_p(r)/T(r, F)\} \geqslant A/(1 + A) = 1 + p - \varrho ,$$

therefore

$$\limsup_{r\to\infty} \frac{N_p(r, a) + N_p(r, b)}{T(r, F)} \geqslant 1 + p - \varrho .$$

2.3. - Proof of Theorem 3.

i) We know

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log T(r)}{\log r} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log S(r)}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix} .$$

So

$$T(r) > r^{\lambda-\varepsilon} \quad \text{for} \quad r \geqslant r_0 , \qquad T(r) < r^{\varrho+\varepsilon} \quad \text{for} \quad r \geqslant r_0 ,$$

hence

$$\{T(a\, r)\}^c > (a\, r)^{c(\lambda-\varepsilon)} , \qquad \{S(b\, r)\}^d < (b\, r)^{d(\varrho+\varepsilon)} .$$

So

$$\{T(a\, r)\}^c/\{S(b\, r)\}^d > K\, r^{c\lambda - d\varrho - \varepsilon} .$$

Therefore

$$\liminf_{r\to\infty} \{T(a\, r)\}^c/\{S(b\, r)\}^d = \infty \qquad \text{if} \qquad c\lambda > d\varrho .$$

Similarly

$$\limsup_{r\to\infty} \{T(a\, r)\}^c/\{S(b\, r)\}^d = 0 \qquad \text{if} \qquad c\varrho < d\lambda .$$

ii) Now

$$T(r) > r^{\varrho - \varepsilon} \qquad \text{for a sequence of values of } r \to \infty;$$

hence

$$\{T(a\,r)\}^c > (a\,r)^{c(\varrho-\varepsilon)} \text{ for a sequence of values of } r \to \infty,$$

$$\{S(b\,r)\}^d < (b\,r)^{d(\varrho+\varepsilon)} \qquad \text{for } r \geqslant r_0 .$$

So

$$\{T(a\,r)\}^c / \{S(b\,r)\}^d > K\, r^{(c-d)\varrho - \varepsilon},$$

therefore

$$\limsup_{r\to\infty} \{T(a\,r)\}^c / \{S(b\,r)\}^d = \infty \qquad \text{if} \quad c > d .$$

Similarly

$$\liminf_{r\to\infty} \{T(a\,r)\}^c / \{S(b\,r)\}^d = 0 \qquad \text{if} \quad c < d .$$

2.4. - Proof of Theorem 4.

i) It is

$$T(r) < (A + \varepsilon)\, r^{\varrho}, \qquad S(r) > (D - \varepsilon)\, r^{\varrho}, \qquad \text{for } r \geqslant r_0 ,$$

hence

$$T(r)/S(r) < (A + \varepsilon)/(D - \varepsilon) .$$

So

$$\limsup_{r\to\infty} \{T(r)/S(r)\} \leqslant A/D ,$$

but we know

$$\liminf_{r\to\infty} \{T(r)/S(r)\} \leqslant 1/\varrho \leqslant 1/\lambda \leqslant \limsup_{r\to\infty} \{T(r)/S(r)\} .$$

Therefore

$$D \leqslant A\lambda .$$

Similarly

$$C \geqslant B\varrho .$$

So

$$D \leqslant B\varrho \leqslant C\,, \qquad \text{as} \qquad D \leqslant B\varrho \quad \text{by [8]}.$$

ii) We know [8]

$$D \leqslant B\varrho \leqslant \{1 + \log (C/D)\}D\,.$$

So

$$\varrho/\lambda \leqslant (D/B)\{1 + \log (C/D)\}(A/D) \leqslant (AC)/(BD),$$

hence

$$\varrho/\lambda \leqslant (A\,C)/(B\,D)\,.$$

2.5. - Proof of Theorem 5.

Let

$$\limsup_{r\to\infty} \sum_{i=1}^{q} n(r,\ a_i)\, r^{-\lambda(r)} = K\,.$$

So

$$\sum_{i=1}^{q} n(r,\ a_i) < (K + \varepsilon)\, r^{\lambda(r)} \qquad \text{for } r \geqslant r_0\,,$$

and

$$\sum_{i=1}^{q} N(r,\ a_i) < (K + \varepsilon)\, r^{\lambda(r)}/\lambda\,, \qquad \sum_{i=1}^{q} N(r,\ a_i) < (K + \varepsilon)\, T(r,\ F)/\lambda\,.$$

Therefore

$$\sum_{i=1}^{q} \{N(r,\ a_i)/T(r,\ F)\} < (K + \varepsilon)/\lambda\,.$$

But from [9]

$$(q - 2)\, T(r,\ F) \leqslant \sum_{i=1}^{l} N(r,\ a_i) + S(r)\,,$$

where $S(r) = O(\log r - \log T(r,\ F))$, except over a set of values of r of finite measure and $S(r) = O(\log r)$ for all $r \to \infty$, when $F(z)$ is of finite order and $S(r)/T(r) = o(1)$.

So

$$q - 2 \leqslant \sum_{i=1}^{q} \{N(r,\ a_i)/T(r,\ F)\} + o(1)\,,$$

$$q - 2 \leqslant K/\lambda\,, \qquad \text{i. e.} \qquad K \geqslant (q - 2)\lambda\,.$$

Similarly other part follows.

2.6. - Proof of Theorem 6.

We know [10]

$$(q-2)T(r,\ F) \leqslant N(r,\ a) + N(r,\ b) + N(r,\ c) + O(\log r)\,.$$

Put $a = 0,\ b = \infty,\ c = 1,\ q = 3$. So

$$T(r,\ F)/r^{\varrho} \leqslant \{N(r,\ 0) + N(r,\ \infty)\}/r^{\varrho} + \{N(r,\ 1)/r^{\varrho}\} + o(1)\,,$$

$$N_1(r)/r^{\varrho} = o(1) \qquad \Longrightarrow \qquad T(r,\ F)/r^{\varrho} \to 0 \quad \text{as} \quad r \to \infty\,,$$

provided $N(r,\ 1)/r^{\varrho} \to 0$ as $r \to \infty$.

Conversely let $T(r,\ F)/r^{\varrho} \to 0$ as $r \to \infty$, then

$$N_1(r)/r^{\varrho} = \{N(r,\ 0) + N(r,\ \infty)\}/r < A\,\{T(r)/r^{\varrho}\} \to 0 \quad \text{as} \quad r \to \infty\,.$$

Similarly other part follows.

2.7. - Proof of Theorem 7.

It is known [11]

$$N_1(r) < K\,T(r)\,, \qquad\qquad N_1(r) < K_1\,(\log r)^p\,,$$

so

$$N_1(r) = O(\log r)^p\,.$$

Now

$$N_1(r^2) > n_1(r)\log r\,,$$

therefore

$$n_1(r)\log r < N_1(r^2) < A\,(\log r)^p\,,$$

and

$$n_1(r) = O(\log r)^{p-1}\,.$$

But

$$Q_1(r) = r\int_r^{\infty}\{n_1(t)/t^2\}\,dt < r\,A\int_r^{\infty}\{(\log t)^{p-1}/t^2\}\,dt\,.$$

Then

$$Q_1(r) < r\,A \int_r^\infty \{(\log t)/t^l\}^{p-1}\, t^{-2+(p-1)l}\, dt\,.$$

But, $\log (t/t^l)$ is a decreasing function. So

$$Q_1(r) < r\,A \left(\frac{\log r}{r^l}\right)^{p-1} \frac{1}{\{1-(p-1)l\}\, r^{1-(p-1)l}}\,.$$

Hence

$$Q_1(r) = O(\log r)^{p-1}\,,$$

and conversely, if

$$Q_1(r) = O(\log r)^{p-1}\,,$$

we prove

$$T(r) = O(\log r)^p\,.$$

First of all

$$Q_1(r) = r \int_r^\infty \{n_1(t)/t^2\}\, dt \geqslant \{r\, n_1(r)/r\} = n_1(r)\,.$$

So

$$n_1(r) = O(\log r)^{p-1}\,,$$

but

$$N_1(r) \leqslant n_1(r) \log r\,,$$

hence

$$N_1(r) = O(\log r)^p\,.$$

Again

$$T(r) < N_1(r) + Q_1(r)\,,$$

so

$$T(r) = O(\log r)^p\,,$$

or

$$T(r)/N_1(r) < 1 + \{Q_1(r)/N_1(r)\}\,,$$

and as

$$Q_1(r)/N_1(r) \to 0, \qquad T(r) < N_1(r) \qquad \text{for} \qquad r \geqslant r_0\,.$$

Hence

$$T(r) = O(\log r)^p .$$

Finally I wish to thank Prof. Dr. HANS HORNICH for his kind interest and helpful criticism and « Govt of Austria » for awarding me as a scholarship.

References.

[1] R. NEVANLINNA, *Le Théorèm de Picard-Borel et la Théorie des Fonctions Meromorphes*, Gauthier-Villars, Paris 1929.

[2] L. AHLFORS, *Beiträge zur Theorie der meromorphen Funktionen*, VII Congr. Math. Scand. Oslo (1929), 84-88.

[3] R. P. BOAS, *Some elementary theorems on entire functions*, Rend. Circ. Mat. Palermo (2) **1** (1952), 323-331.

[4] S. K. SING, *A note on entire and meromorphic functions*, Proc. Amer. Math. Soc. **9** (1958), 6-10.

[5] K. MANJANATHAIH, *A note on entire and meromorphic functions*, Thesis, 1963 (cf. pp. 99-117).

[6] HARI SHANKAR, *On the characteristic function of a meromorphic function*, Agra Univ. J. **3** (1958), 203-209.

[7] K. MANJANATHAIH, *A note on entire and meromorphic functions*, Thesis, 1963 (cf. pp. 74-86).

[8] S. K. SINGH, *A note on meromorphic functions*, Proc. Nat. Inst. Sci. India (Parte A) **28** (1962).

[9] K. S. SURYANARAYANAN, *Composite meromorphic functions*, Journ. Indian Math. Soc. (2) **1** (1934-35), 241-246.

[10] W. K. HAYMAN, *Meromorphic Functions*, Claredon Press, Oxford 1964.

[11] A. EDREI and W. H. J. FUCHS, *On the growth of meromorphic functions wiht several deficient values*, Trans. Amer. Math. Soc. **93** (1959), 292-328.

Summary.

Here we have proved the results connecting zeros and the maximum function. The result of K. Manjanathaih is generalised. We have also obtained the lower bound for $N_p(r)/Q_p(r)$ *for entire functions of finite order, where*

$$N_p(r) = r^p \int_0^r \{n(x)/x^{p+1}\}\,dx , \qquad Q_p(r) = r^{p+1} \int_r^\infty \{n(x)/x^{p+2}\}\,dx .$$

* * *