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On the Absolute Cesàro Summability Factors of a Fourier Series. (**)

1. - Definitions and notations.

Let $\sum a_n$ be a given infinite series. Let $S_n = S_n^0$ denote its n-th partial sum, let S_n^k and t_n^k denote respectively the n-th CESÀRO means of order k $(k > -1)$ of the sequences $\{S_n\}$ and $\{n\, a_n\}$. The series $\sum a_n$ is said to be absolutely summable (C, k), or summable $|\,C,\ k\,|$, if the sequence $\{S_n^k\}$ is of bounded variation, that is to say, the infinite series

$$\sum |\, S_n^k - S_{n-1}^k \,| ,$$

is convergent [1].

We use the following well known identities for $k > -1$ [2]:

$$t_n^k = n\,(S_n^k - S_{n-1}^k)\,, \tag{1}$$

$$t_n^k = \frac{1}{A_n^k} \sum_{\nu=1}^{n} A_{n-\nu}^{k-1}\, \nu\, a_\nu\,, \tag{2}$$

where

$$\sum_{n=0}^{\infty} A_n^k x^n = (1-x)^{-k-1} \qquad (|x|<1),$$

$$A_n^k = \binom{n+k}{n} \sim \frac{n^k}{\Gamma(k+1)} \qquad (k \neq -1, -2, \ldots). \tag{3}$$

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(1) FEKETE [5], KOGBETLIANTZ [7].

(2) KOGBETLIANTZ [7], [8].

We write

$$\Delta^0 u_n = u_n , \qquad \Delta u_n = \Delta^1 u_n = u_n - u_{n+1} ,$$

and, for positive integers σ and ϱ,

(4) $$\Delta^\sigma(\Delta^\varrho u_n) = \Delta^{\sigma+\varrho} u_n .$$

We also have, for positive integral σ,

(5) $$\Delta^\sigma u_n = \sum_{\nu=n}^{\infty} A_{\nu-n}^{-\sigma-1} u_\nu ,$$

and we write this in general for all $\sigma > 0$, provided the series on the right converges.

We have, for positive integral σ,

(6) $$\Delta^\sigma (\delta_n u_n) = \sum_{\varrho=0}^{\sigma} \binom{\sigma}{\varrho} \Delta^\varrho \delta_n \; \Delta^{\sigma-\varrho} u_{\varrho+n} .$$

Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Without any loss of generality, we may assume that the constant term in the FOURIER series of $f(t)$ is zero, that is

(7) $$\int_{-\pi}^{\pi} f(t)\, dt = 0 ,$$

and that the FOURIER series of $f(t)$ is given by

(8) $$\sum (a_n \cos nt + b_n \sin nt) = \sum c_n(t) .$$

We use the following further notations:

$$\varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2 f(x)\},$$

(9) $$\begin{cases} \Phi_\alpha(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-u)^{\alpha-1} \varphi(u)\, du & (\alpha > 0), \\ \Phi_0(t) = \varphi(t) & \\ \varphi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\, \Phi_\alpha(t) & (\alpha \geqslant 0), \end{cases}$$

(10) $$K_n^\alpha(t) = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}\, \varepsilon_\nu\, \nu \cos\nu t\,,$$

(11) $$\{F(t)\}_\varrho = \left(\frac{\partial}{\partial t}\right)^\varrho F(t)\,.$$

2. – Generalising the previous works of PRASAD [3], IZUMI and KAWATA [4], CHENG [5], and PATI [6], in 1959, PATI and SINHA [7] proved the following theorem concerning absolute CESÀRO summability factors of FOURIER series:

Theorem A [8]. Let h be a non-negative integer and let sequence $\{\varepsilon_n\}$ be a monotonic non-increasing sequence when $h = 0$, and a hyper-convex sequence of order $h-1$ when $h \geqslant 1$, such that

(i) $$\sum n^{-1}\, \varepsilon_n < \infty\,,$$

(ii) $$\sum n^h\, \Delta^{h+1} \varepsilon_n < \infty\,.$$

Then, if

$$\int_0^t |\, \varphi_h(u)\,| = O(t)\,,$$

as $t \to 0$, $\sum \varepsilon_n\, c_n(x)$ is summable $|\,\mathrm{C},\, h+1+\delta\,|$, $\delta > 0$.

The object of the present paper is to obtain an extension of Theorem A by repalcing h in the order of absolute CESÀRO summability of $\sum \varepsilon_n\, c_n(x)$ by k, which now need not be the same as h.

3. – We establish the following theorem.

Theorem. *Let h and k be non-negative integers and let*

(12) $$\int_0^t |\, \varphi_h(u)\,|\, \mathrm{d}u = O(t)\,,$$

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(3) PRASAD [12].
(4) IZUMI and KAWATA [6].
(5) CHENG [4].
(6) PATI [9].
(7) PATI and SINHA [10].
(8) A more compact proof of this is recently given by PATI and AHMAD [11].

as $t \to 0$. *Then* $\sum \varepsilon_n c_n(x)$ *is summable* $|C, k+1+\delta|$, *for every* $\delta > 0$, *where the sequence* $\{\varepsilon_n\}$ *satisfies the following conditions:*

Case (a), $k > h$:

(i) $$\sum n^k \, | \Delta^{k+1} \varepsilon_n | < \infty ,$$

(ii) $$\sum n^{-1} \, | \varepsilon_n | < \infty .$$

Case (b), $k = h$:

(i) $$\sum n^h \log n \, | \Delta^{h+1} \varepsilon_n | < \infty ,$$

(ii) $$\varepsilon_n \log n = O(1) .$$

Case (c), $k < h$:

(i) $$\sum n^{h-k-1} \, | \varepsilon_n | < \infty ,$$

(ii) $$\sum n^h \, | \Delta^{k+1} \varepsilon_n | < \infty .$$

4. – We need the following lemmas.

Lemma 1 [9]. Let C_n^k denote the n-th CESÀRO sum of order k $(k \geqslant 0)$ corresponding to the infinite series $(\sum_1^\infty \sin nt)_{h+1}$, $h \geqslant 0$.

Then

(i) $$C_n^k = O(n^{k+h+2}) \qquad \text{for} \quad 0 < t \leqslant n^{-1},$$

(ii) $$C_n^k = O(n^{h+1} t^{-k-1}) + O(n^k t^{-h-2}) \qquad \text{for} \quad n^{-1} < t \leqslant \pi.$$

Lemma 2 [10]. Let $\{\lambda_n\}$ be a positive monotonic non-decreasing sequence. If $p \geqslant 0$, $\varepsilon_n \lambda_n = O(1)$, and $\sum n^p \lambda_n \, | \Delta^{p+1} \varepsilon_n | < \infty$, then $\sum n^q \lambda_n \, | \Delta^{q+1} \varepsilon_n | < \infty$, for every q such that $0 \leqslant q \leqslant p$.

Lemma 3 [11]. Let (12) hold. Then

$$\int_{n^{-1}}^{\pi} t^{h-\alpha} \, | \varphi_h(t) | \, dt = O(n^{\alpha-h-1}) \qquad \text{for} \quad \alpha > h+1,$$

$$= O(\log n) \qquad \text{for} \quad \alpha = h+1,$$

$$= O(1) \qquad \text{for} \quad \alpha < h+1.$$

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(9) PATI and SINHA [10].
(10) PATI and AHAMAD [11], cf. BOSANQUET [1].
(11) PATI and SINHA [10].

Lemma 4 (12). If $S^k_{n,\varrho}$ denotes the n-th CESÀRO sum of order k corresponding to the series $\sum (-1)^n n^\varrho$, then

$$S^k_{n,\varrho} = O(n^k) \qquad \text{for} \quad k \geqslant \varrho.$$

Lemma 5 (13). If $k > -1$, $\varrho \geqslant 0$, $p \geqslant 0$, necessary and sufficient conditions for $\sum \varepsilon_n a_n$ to be summable $| C, \varrho |$, whenever $S^k_n = O(n^p)$, are:

$$\text{(i)} \qquad \sum n^{k-\varrho+p} \, | \varepsilon_n | < \infty ,$$

$$\text{(ii)} \qquad \sum n^{-1+p} \, | \varepsilon_n | < \infty ,$$

$$\text{(iii)} \qquad \sum n^{k+p} \, | \Delta^{k+1} \varepsilon_n | < \infty .$$

Lemma 6 (14). If $\sigma > -1$ and $\sigma - \delta > 0$, then

$$\sum_{n=\mu}^{\infty} \frac{A^\delta_{n-\mu}}{n \, A^\sigma_n} = \frac{1}{\mu \, A^{\sigma-\delta-1}_\mu} .$$

Lemma 7 (15). If $0 < k \leqslant 1$ and $1 \leqslant \nu \leqslant n$, then

$$\Big| \sum_{\mu=1}^{\nu} A^{k-1}_{n-\mu} \, C_\mu \Big| \leqslant \max_{1 \leq m \leq \nu} \Big| \sum_{\mu=1}^{m} A^{k-1}_{m-\mu} \, C_\mu \Big| .$$

5. - Proof of the Theorem.

By virtue of the identity (1) and the consistency theorem for absolute CESÀRO summability, we need only prove that, for $k + 1 < \alpha < k + 2$,

$$\sum n^{-1} \, | \zeta^\alpha_n | < \infty ,$$

where

$$\zeta^\alpha_n = \frac{2}{\pi} \int_0^\pi \varphi(t) \, K^\alpha_n(t) \, \mathrm{d}t .$$

(12) PATI and SINHA [10].
(13) BOSANQUET and CHOW [2].
(14) CHOW [3], Lemma 1.
(15) BOSANQUET [1].

Integrating by parts h times, we have

$$\zeta_n^\alpha = \frac{2}{\pi}\left[\sum_{\varrho=1}^{h}(-1)^{\varrho-1}\Phi_\varrho(t)\left(K_n^\alpha(t)\right)_{\varrho-1}\right]_0^\pi + (-1)^h\frac{2}{\pi}\int_0^\pi t^h\varphi_h(t)\left(K_n^\alpha(t)\right)_h \mathrm{d}t$$

$$= \frac{2}{\pi} I + \frac{2}{\pi}\frac{(-1)^h}{h+1}\int_0^\pi \varphi_h(t)\,\overline{K}_n^\alpha(t)\,\mathrm{d}t, \quad \text{say.}$$

Hence it is sufficient for our purpose to show that

$$\sum n^{-1}\,|\,I\,| < \infty, \tag{13}$$

and

$$\sum n^{-1}\left|\int_0^\pi \varphi_h(t)\,\overline{K}_n^\alpha(t)\,\mathrm{d}t\right| < \infty. \tag{14}$$

Proof of (13). We have

$$I = \frac{1}{A_n^\alpha}\left[\sum_{\varrho=1}^{h}(-1)^{3(\varrho-1)/2}\,\Phi_\varrho(t)\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-1}\,\varepsilon_\nu\,\nu^\varrho\cos\nu t\right]_0^\pi =$$

$$= \frac{\Phi_\varrho(\pi)}{A_n^\alpha}\sum_{\varrho=1}^{h}(-1)^{3(\varrho-1)/2}\sum_{\nu=1}^{n}(-1)^\nu A_{n-\nu}^{\alpha-1}\,\varepsilon_\nu\,\nu^\varrho,$$

where ϱ is an odd positive integer.

Hence proving (13) is the same as proving the summability $|\,C,\alpha\,|$, $\alpha > k+1$ of $\sum(-1)^n\varepsilon_n n^{\varrho-1}$, where ϱ is an odd integer such that $1 \leqslant \varrho \leqslant h$. This is verified by virtue of Lemma 4 and Lemma 5, and hypotheses on $\{\varepsilon_n\}$.

Proof of (14). We have

$$\overline{K}_n^\alpha(t) = \frac{t^h}{A_n^\alpha}\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-1}\,\varepsilon_\nu\,(\sin\nu t)_{h+1}$$

$$= \frac{t^h}{A_n^\alpha}\sum_{\nu=1}^{n}\Delta^{k+1}(A_{n-\nu}^{\alpha-1}\,\varepsilon_\nu)\,C_\nu^k$$

$$= \frac{t^h}{A_n^\alpha}\sum_{r=0}^{k}\binom{k+1}{r}\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-r-1}\,\Delta^{k+1-r}\,\varepsilon_{\nu+r}\,C_\nu^k + \frac{t^h}{A_n^\alpha}\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-k-2}\,\varepsilon_{\nu+k+1}\,C_\nu^k$$

$$= \frac{t^h}{A_n^\alpha}\,(M_1+M_2), \quad \text{say.}$$

Thus, it is sufficient to show that [16]

(15) $$J_1 = \sum (n A_n^\alpha)^{-1} \int_0^{\nu^{-1}} t^h |\varphi_h(t)| \, |M_1| \, dt \leqslant K,$$

(16) $$J_2 = \sum (n A_n^\alpha)^{-1} \int_{\nu^{-1}}^{\pi} t^h |\varphi_h(t)| \, |M_1| \, dt \leqslant K,$$

(17) $$J_3 = \sum (n A_n^\alpha)^{-1} \int_0^{\pi} t^h |\varphi_h(t)| \, |M_2| \, dt \leqslant K.$$

Proof of (15). For $r = 0, 1, \ldots, k$, we have

$$J_1 \leqslant K \sum (n A_n^\alpha)^{-1} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1} |\Delta^{k+1-r} \varepsilon_{\nu+r}| \int_0^{\nu^{-1}} t^h |\varphi_h(t)| \, |C_\nu^k| \, dt$$

$$\leqslant K \sum (n A_n^\alpha)^{-1} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1} |\Delta^{k+1-r} \varepsilon_{\nu+r}| \; \nu^{h+k+2} \int_0^{\nu^{-1}} t^h |\varphi_h(t)| \, dt$$

$$\leqslant K \sum (n A_n^\alpha)^{-1} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1} |\Delta^{k+1-r} \varepsilon_{\nu+r}| \, \nu^{k+1} \leqslant K \sum \nu^{k+1} |\Delta^{k+1-r} \varepsilon_{\nu+r}| \sum_{n=\nu}^{\infty} \frac{A_{n-\nu}^{\alpha-r-1}}{n A_n^\alpha}$$

$$\leqslant K \sum \nu^{k-r} |\Delta^{k+1-r} \varepsilon_{\nu+r}| \leqslant K,$$

by virtue of Lemma 2, Lemma 6, and hypotheses.

Proof of (16). For $r = 0, 1, \ldots, k$, we have

$$J_2 \leqslant K \sum (n A_n^\alpha)^{-1} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1-r} |\Delta^{k+1-r} \varepsilon_{\nu+r}| \int_{\nu^{-1}}^{\pi} t^h |\varphi_h(t)| \, |C_\nu^k| \, dt$$

$$\leqslant K \sum (n A_n^\alpha)^{-1} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1-r} |\Delta^{k+1-r} \varepsilon_{\nu+r}| \int_{\nu^{-1}}^{\pi} t^h |\varphi_h(t)| \, (\nu^k t^{-h-2} + \nu^{h+1} t^{-k-1}) \, dt$$

$$= J_{21} + J_{22}.$$

For $r = 0, 1, \ldots, k$, we have

$$J_{21} = K \sum (n A_n^\alpha)^{-1} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1} |\Delta^{k+1-r} \varepsilon_{\nu+r}| \, \nu^k \int_{\nu^{-1}}^{\pi} t^{-2} |\varphi_h(t)| \, dt$$

$\leqslant K$ (by Lemma 3), as in J_1.

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[16] K is a positive constant, not necessarily the same at each occurrence.

Again, for $r=0, 1, \ldots, k$ and $k>h$, we have

$$J_{22}=K\sum (n\,A_n^{\alpha})^{-1}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1-r}\,|\,\Delta^{k+1-r}\,\varepsilon_{\nu+r}\,|\,\nu^{h+1}\int_{\nu^{-1}}^{\pi} t^{h-k-1}\,|\,\varphi_h(t)\,|\,dt$$

$$\leqslant K\sum (n\,A_n^{\alpha})^{-1}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1-r}\,|\,\Delta^{k+1-r}\,\varepsilon_{\nu+r}\,|\,\nu^{k+1}\quad \text{(by Lemma 3)}$$

$$\leqslant K,\quad \text{as in } J_1\,.$$

For $r=0, 1, \ldots, k$ and $k=h$, we have

$$J_{22}\leqslant K\sum (n\,A_n^{\alpha})^{-1}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1}\,|\,\Delta^{h+1-r}\,\varepsilon_{\nu+r}\,|\,\nu^{h+1}\log(\nu+1)$$

$$\leqslant K\sum \nu^{h-r}\log(\nu+1)\,|\,\Delta^{h+1-r}\,\varepsilon_{\nu+r}\,|\leqslant K\,,$$

by Lemma 2, Lemma 6 and hypothesis.

And for $r=0, 1, \ldots, k$ and $k<h$, we have

$$J_{22}\leqslant K\sum (n\,A_n^{\alpha})^{-1}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1}\,|\,\Delta^{k+1-r}\,\varepsilon_{\nu+r}\,|\,\nu^{h+1}\leqslant K\sum \nu^{h-r}\,|\,\Delta^{k+1-r}\,\varepsilon_{\nu+r}\,|\leqslant K,$$

by Lemma 2, Lemma 6 and hypotheses.

Proof of (17). We have

$$M_2=\sum_{\nu=1}^{n}\Delta\varepsilon_{\nu+k+1}\sum_{\mu=1}^{\nu}A_{n-\mu}^{\alpha-k-2}\,C_{\mu}^{k}+\varepsilon_{n+k+2}\sum_{\mu=1}^{n}A_{n-\mu}^{\alpha-k-2}\,C_{\mu}^{k}$$

$$=O\Big(\sum_{\nu=1}^{n}|\,\Delta\varepsilon_{\nu+k+1}\,|\max_{1\leq m\leq\nu}|\,C_m^{\alpha-1}\,|\Big)+O(\varepsilon_{n+k+2}\,|\,C_n^{\alpha-1}\,|)\,.$$

Hence, it is sufficient to show that

$$J_{31}=\sum n^{-1-\alpha}\sum_{\nu=1}^{n}|\,\Delta\varepsilon_{\nu+k+1}\,|\int_{0}^{\nu^{-1}} t^{h}\,|\,\varphi_h(t)\,|\,\max|\,C_m^{\alpha-1}\,|\,dt\leqslant K\,,$$

$$J_{32}=\sum n^{-1-\alpha}\sum_{\nu=1}^{n}|\,\Delta\varepsilon_{\nu+k+1}\,|\int_{\nu^{-1}}^{\pi} t^{h}\,|\,\varphi_h(t)\,|\,\max|\,C_m^{\alpha-1}\,|\,dt\leqslant K\,,$$

$$J_{33}=\sum n^{-1-\alpha}\,|\,\varepsilon_{n+k+2}\,|\int_{0}^{n^{-1}} t^{h}\,|\,\varphi_h(t)\,|\,|\,C_n^{\alpha-1}\,|\,dt\leqslant K\,,$$

$$J_{34} = \sum n^{-1-\alpha} \, | \, \varepsilon_{n+k+2} \, | \int_{n^{-1}}^{\pi} t^h \, | \, \varphi_h(t) \, | \, | \, C_n^{\alpha-1} \, | \, dt \leqslant K \, .$$

$$J_{31} \leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} | \, \Delta \varepsilon_{\nu+k+1} \, | \, \nu^{\alpha+h+1} \int_{0}^{\nu^{-1}} t^h \, | \, \varphi_h(t) \, | \, dt$$

$$\leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} | \, \Delta \varepsilon_{\nu+k+1} \, | \, \nu^{\alpha} \leqslant K \sum | \, \Delta \varepsilon_{\nu+k+1} \, | \leqslant K \, ,$$

by Lemma 2 and hypotheses.

Since, by Lemma 3, we have

$$\int_{\nu^{-1}}^{\pi} t^h \, | \, \varphi_h(t) \, | \, \nu^{h+1} \, t^{-\alpha} \, dt = O(\nu^{\alpha}) \qquad (\alpha > h+1),$$

$$= O(\nu^{h+1}) \qquad (\alpha < h+1),$$

$$J_{32} \leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} | \, \Delta \varepsilon_{\nu+k+1} \, | \int_{\nu^{-1}}^{\pi} t^h \, | \, \varphi_h(t) \, | \, [\nu^{h+1} \, t^{-\alpha} + \nu^{\alpha-1} \, t^{-h-2}] \, dt$$

$$\leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} | \, \Delta \varepsilon_{\nu+k+1} \, | \left\{ \nu^{\alpha} + \nu^{h+1} \right\} \leqslant K \sum_{\nu=1}^{\infty} | \, \Delta \varepsilon_{\nu+k+1} \, | \left\{ 1 + \nu^{h-\alpha+1} \right\}$$

$$\leqslant K \sum_{\nu=1}^{\infty} | \, \Delta \varepsilon_{\nu+k+1} \, | + K \sum_{\nu=1}^{\infty} \nu^{h-k} \, | \, \Delta \varepsilon_{\nu+k+1} \, | \leqslant K \, ,$$

by Lemma 2 and hypotheses.

$$J_{33} \leqslant K \sum n^{-1-\alpha} \, | \, \varepsilon_{n+k+2} \, | \, n^{\alpha+h+1} \int_{0}^{n^{-1}} t^h \, | \, \varphi_h(t) \, | \, dt \leqslant K \sum n^{-1} \, | \, \varepsilon_{n+k+2} \, | \leqslant K \, ,$$

by hypotheses.

$$J_{34} \leqslant K \sum n^{-1-\alpha} \, | \, \varepsilon_{n+k+2} \, | \int_{n^{-1}}^{\pi} t^h \, | \, \varphi_h(t) \, | \, (n^{h+1} \, t^{-\alpha} + n^{\alpha-1} \, t^{-h-2}) \, dt$$

$$\leqslant K \sum n^{-1-\alpha} \, | \, \varepsilon_{n+k+2} \, | \, (n^{\alpha} + n^{h+1}) \leqslant K \sum n^{-1} \, | \, \varepsilon_{n+k+2} \, | + \sum n^{h-k-1} \, | \, \varepsilon_{n+k+2} \, | \leqslant K,$$

by hypotheses.

This completes the proof of the theorem.

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