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A Note on k-Transforms. (**)

1. – Generalisations of the classical LAPLACE transform

$$\varphi(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{1.1}$$

have been given from time to time by several Mathematicians. The generalisation due to MEIJER [3] is:

$$\varphi(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{st}\, k_\nu(st)\, f(t)\, dt\,, \tag{1.2}$$

where $k_\nu(t)$ is MACDONALD's function. Generally it is called the k-transform. The inversion formula takes the form

$$f(t) = \frac{1}{i\sqrt{2\pi}} \int_{c-i\infty}^{c+i\infty} \sqrt{st}\, I_\nu(st)\, \varphi(s)\, ds\,. \tag{1.3}$$

In the present Note some properties of k-transform, in the form of theorems, have been given which are mainly useful for evaluating infinite integrals.

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2. – Theorem 1. *If $f(x)$ and $g(x)$ are k-transforms of each other, then*

$$\int_0^\infty \frac{f(x)}{x^p}\,\mathrm{d}x = 2^{-(1/2)-p}\,\Gamma_\times\left(\frac{3}{4}-\frac{p}{2}\pm\frac{\nu}{2}\right)\int_0^\infty \frac{g(x)}{x^{1-p}}\,\mathrm{d}x \quad (^1), \tag{2.1}$$

provided the integrals involved converge absolutely.

Proof. We know that

$$f(s) = \int_0^\infty \sqrt{st}\; k_\nu(st)\, g(t)\,\mathrm{d}t\,.$$

Therefore:

$$\int_0^\infty \frac{f(x)}{x^p}\,\mathrm{d}x = \int_0^\infty \frac{\mathrm{d}x}{x^p}\int_0^\infty \sqrt{xt}\,k_\nu(xt)\,g(t)\,\mathrm{d}t. \tag{2.2}$$

Changing the order of integration, if permissible, we get

$$\int_0^\infty \frac{f(x)}{x^p}\,\mathrm{d}x = \int_0^\infty \sqrt{t}\,g(t)\,\mathrm{d}t\int_0^\infty x^{(1/2)-p}\,k_\nu(xt)\,\mathrm{d}x\,. \tag{2.3}$$

Evaluating the inner integral by a known result, [1], namely

$$\int_0^\infty x^{m-1}\,k_n(ax)\,\mathrm{d}x = \frac{2^{m-2}\,\Gamma\left(\frac{m}{2}+\frac{n}{2}\right)\;\Gamma\left(\frac{m}{2}-\frac{n}{2}\right)}{a^m} \qquad (m\pm n>0)\,,$$

we get

$$\int_0^\infty \frac{f(x)}{x^p}\,\mathrm{d}x = 2^{-(1/2)-p}\,\Gamma\left(\frac{3}{4}-\frac{p}{2}+\frac{\nu}{2}\right)\;\Gamma\left(\frac{3}{4}-\frac{p}{2}-\frac{\nu}{2}\right)\int_0^\infty \frac{g(t)}{t^{1-p}}\,\mathrm{d}t \qquad \left(\frac{3}{4}-\frac{p}{2}\pm\frac{\nu}{2}>0\right)\,.$$

The change in the order of integration is permissible as the integrals involved converge absolutely.

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(1) $\Gamma_\times(a\pm b) = \Gamma(a+b)\,\Gamma(a-b)$.

We take in the following examples to illustrate the application of the theorem in evaluating infinite integrals.

Example 1. Let

$$g(t) = t^{\mu-1} e^{-at} \qquad (|\mu| > |\nu| - \tfrac{1}{2})$$

and

$$f(s) = \frac{\sqrt{\pi}\, 2^{\nu} s^{\nu+\frac{1}{2}}}{(a+s)^{\mu+\nu+\frac{1}{2}}} \frac{\Gamma(\mu+\frac{1}{2}+\nu)\,\Gamma(\mu+\frac{1}{2}-\nu)}{\Gamma(\mu+1)} \,\cdot$$

$$\cdot\, {}_2F_1\left\{\mu+\nu+\tfrac{1}{2},\ \nu+\tfrac{1}{2};\ \mu+1;\ \frac{a-s}{a+s}\right\}, \qquad \operatorname{Re}(a+s) > 0.$$

Then from Theorem 1 we have

$$\frac{\sqrt{\pi}\, 2^{\nu}\, \Gamma(\mu+\frac{1}{2}+\nu)\,\Gamma(\mu+\frac{1}{2}-\nu)}{\Gamma(\mu+1)} \int_0^\infty \frac{s^{\nu+\frac{1}{2}-p}}{(a+s)^{\mu+\nu+\frac{1}{2}}}\, {}_2F_1\left\{\begin{matrix}\mu+\nu+\frac{1}{2},\ \nu+\frac{1}{2};\ \frac{a-s}{a+s}\\ \mu+1\end{matrix}\right\} ds$$

$$= 2^{-\frac{1}{2}-p}\,\Gamma\left(\frac{3}{4}-\frac{p}{2}+\frac{\nu}{2}\right)\,\Gamma\left(\frac{3}{4}-\frac{p}{2}-\frac{\nu}{2}\right)\int_0^\infty t^{\mu-1+p-1} e^{-at}\, dt$$

$$= 2^{-\frac{1}{2}-p}\,\Gamma\left(\frac{3}{4}-\frac{p}{2}+\frac{\nu}{2}\right)\,\Gamma\left(\frac{3}{4}-\frac{p}{2}-\frac{\nu}{2}\right)\frac{\Gamma(\mu+p-1)}{a^{\mu+p-1}} \qquad (\mu+p-1>0)\,.$$

Thus we have

$$\int_0^\infty \frac{s^{\nu+\frac{1}{2}-p}}{(a+s)^{\mu+\nu+\frac{1}{2}}}\, {}_2F_1\left\{\begin{matrix}\mu+\nu+\frac{1}{2},\ \ \nu+\frac{1}{2};\ \frac{a-s}{a+s}\\ \mu+1\end{matrix}\right\} ds =$$

$$= \frac{\Gamma(\mu+1)}{\sqrt{\pi}\, 2^{\nu+p+\frac{1}{2}}}\, \frac{\Gamma_{\times}\left(\frac{3}{4}-\frac{p}{2}\pm\frac{\nu}{2}\right)}{\Gamma_{\times}\left(\mu+\frac{1}{2}\pm\nu\right)}\, \frac{\Gamma(\mu+p-1)}{a^{\mu+p-1}}\,.$$

Example 2. Let

$$g(t) = t^{-\frac{1}{2}-2\mu}\exp(-at^2)\,, \qquad \operatorname{Re} a > 0, \qquad 2\operatorname{Re}\mu < 1 - |\operatorname{Re}\nu|$$

and

$$f(s) = \frac{1}{2}\, a^{\mu}\, s^{-1/2}\, \Gamma_{\times}\left(\frac{1}{2} - \mu \pm \frac{\nu}{2}\right) \exp\frac{s^2}{8a} \cdot W_{\mu,\nu/2}\left(\frac{s^2}{4a}\right).$$

Then Theorem 1 gives that

$$\frac{1}{2}\, a^{\mu}\, \Gamma_{\times}\left(\frac{1}{2} - \mu \pm \frac{\nu}{2}\right) \int_0^{\infty} s^{-(1/2)-p} \exp\frac{s^2}{8a} \cdot W_{\mu,\nu/2}\left(\frac{s^2}{4a}\right) \mathrm{d}s$$

$$= 2^{-(1/2)-p}\, \Gamma_{\times}\left(\frac{3}{4} - \frac{p}{2} \pm \frac{\nu}{2}\right) \int_0^{\infty} t^{p-(3/2)-2\mu} \exp(-\,at^2)\, \mathrm{d}t$$

$$= 2^{-(3/2)-p}\, \Gamma_{\times}\left(\frac{3}{4} - \frac{p}{2} \pm \frac{\nu}{2}\right) \frac{\Gamma\left(\frac{p}{2} - \frac{1}{4} - \mu\right)}{a^{(p/2)-(1/4)-\mu}} \qquad \left(\frac{p}{2} - \frac{1}{4} - \mu > 0\right).$$

Thus we have

$$\int_0^{\infty} s^{-(1/2)-p} \exp\frac{s^2}{8a} \cdot W_{\mu,\nu/2}\left(\frac{s^2}{4a}\right) \mathrm{d}s = 2^{-(1/2)-p}\, \frac{\Gamma_{\times}\left(\frac{3}{4} - \frac{p}{2} \pm \frac{\nu}{2}\right)}{\Gamma_{\times}\left(\frac{1}{2} - \mu \pm \frac{\nu}{2}\right)} \, \frac{\Gamma\left(\frac{p}{2} - \frac{1}{4} - \mu\right)}{a^{(p/2)-(1/4)-2\mu}}.$$

If in the above theorem we write $p = \frac{1}{2}$, we get the following corollary:
If $f(s)$ is the k-transform of $g(t)$, then

$$\int_0^{\infty} \frac{f(s)}{\sqrt{s}}\, \mathrm{d}s = \frac{1}{2}\, \Gamma_{\times}\left(\frac{1}{2} \pm \frac{\nu}{2}\right) \int_0^{\infty} \frac{g(t)}{\sqrt{t}}\, \mathrm{d}t.$$

If $\nu = \pm\frac{1}{2}$, these results reduce to LAPLACE transform (1.1).

3. – Theorem 2. *If $f_1(s)$ and $f_2(s)$ are k-transforms of $g_1(t)$ and $g_2(t)$ respectively, then*

$$\int_0^\infty f_1(t)\, g_2(t)\, \mathrm{d}t = \int_0^\infty f_2(t)\, g_1(t)\, \mathrm{d}t\,, \tag{3.1}$$

provided the integrals exist.

This may be called the Parseval theorem for k-transforms.

Proof. We know that

$$f_1(s) = \int_0^\infty \sqrt{st}\, k_\nu(st)\, g_1(t)\, \mathrm{d}t\,, \tag{3.2}$$

and

$$f_2(s) = \int_0^\infty \sqrt{st}\, k_\nu(st)\, g_2(t)\, \mathrm{d}t\,. \tag{3.3}$$

Therefore

$$\int_0^\infty f_1(s)\, g_2(s)\, \mathrm{d}s = \int_0^\infty g_2(s)\, \mathrm{d}s \int_0^\infty \sqrt{st}\, k_\nu(st)\, g_1(t)\, \mathrm{d}t\,. \tag{3.4}$$

Changing the order of integration, if permissible, we get

$$\int_0^\infty f_1(s)\, g_2(s)\, \mathrm{d}s = \int_0^\infty g_1(t)\, \mathrm{d}t \int_0^\infty \sqrt{st}\, k_\nu(st)\, g_2(s)\, \mathrm{d}s = \int_0^\infty g_1(t)\, f_2(t)\, \mathrm{d}t\,,$$

which proves the theorem.

Assuming that the integrals (3.2) and (3.3) are absolutely convergent, the change in the order of integration will be justifiable.

The generalisation of Theorem 2 may be given in the form:

If $f_1(s)$ and $f_2(s)$ are k-transforms of $g_1(t)$ and $g_2(t)$ respectively, then:

$$\frac{1}{a^p}\int_0^\infty f_1\!\left(\frac{s}{a^p}\right) g_2(s\, b^\varepsilon)\, \mathrm{d}s = \frac{1}{b^\varepsilon}\int_0^\infty g_1(t\, a^p)\, f_2\!\left(\frac{t}{b^\varepsilon}\right) \mathrm{d}t\,,$$

provided the integrals exist.

The following example will illustrate the application of this theorem. Let

$$g_1(t) = t^{\varrho-1}, \qquad \operatorname{Re}\varrho > |\operatorname{Re}\nu| - \tfrac{1}{2},$$

$$f_1(s) = 2^{\varrho-(3/2)} s^{-\varrho} \Gamma_\times\left(\frac{\varrho}{2} + \frac{1}{4} \pm \frac{\nu}{2}\right), \qquad \operatorname{Re} s > 0,$$

and

$$g_2(t) = t^{\sigma+(1/2)} J_\mu(at), \qquad \operatorname{Re}(\mu + \sigma) > |\operatorname{Re}\nu| - 2,$$

$$f_2(s) = \frac{2^\sigma a^\mu}{\Gamma(1+\mu)} s^{-\sigma-\mu-(3/2)} \Gamma_\times\left(\frac{\mu+\sigma}{2} + 1 \pm \frac{\nu}{2}\right) \cdot$$

$$\cdot\, {}_2F_1\left\{\frac{\mu+\sigma}{2} + 1 \pm \frac{\nu}{2};\ \mu+1;\ -\frac{a^2}{s^2}\right\}, \qquad \operatorname{Re} s > |\operatorname{Im} a|.$$

Then Theorem 2 gives:

$$2^{\varrho-(3/2)} \Gamma_\times\left(\frac{\varrho}{2} + \frac{1}{4} \pm \frac{\nu}{2}\right) \int_0^\infty s^{\sigma+(1/2)-\varrho} J_\mu(as) ds =$$

$$= \frac{2^\sigma a^\mu}{\Gamma(1+\mu)} \Gamma_\times\left(\frac{\mu+\sigma}{2} + 1 \pm \frac{\nu}{2}\right) \int_0^\infty s^{\varrho-\sigma-\mu-(5/2)} {}_2F_1\left\{\begin{matrix} \frac{\mu+\sigma}{2} + 1 \pm \frac{\nu}{2};\ \frac{-a^2}{s^2} \\ 1+\mu \end{matrix}\right\} ds.$$

Evaluating the integral on the left hand side by a know formula [1], we get:

$$2^{\varrho-(3/2)} \Gamma_\times\left(\frac{\varrho}{2} + \frac{1}{4} \pm \frac{\nu}{2}\right) \int_0^\infty s^{\sigma+(1/2)-\varrho} J_\mu(a\,s)\, ds =$$

$$= \frac{2^{\sigma-1}\, \Gamma\left(\frac{\sigma}{2} - \frac{\varrho}{2} + \frac{\mu}{2} + \frac{3}{4}\right) \Gamma_\times\left(\frac{\varrho}{2} + \frac{1}{4} \pm \frac{\nu}{2}\right)}{\Gamma\left(\frac{\mu}{2} - \frac{\sigma}{2} + \frac{\varrho}{2} + \frac{1}{4}\right)} a^{\varrho-\sigma-(3/2)}.$$

Therefore:

$$\int_0^\infty s^{\varrho-\sigma-\mu-(5/2)}\, {}_2F_1\left\{\frac{\mu}{2}+\frac{\sigma}{2}+1\pm\frac{\nu}{2};\ 1+\mu;\ \frac{-a^2}{s^2}\right\} ds =$$

$$= \frac{1}{2}\,\frac{a^{\varrho-\sigma-\mu-(3/2)}\,\Gamma(1+\mu)\,\Gamma\left(\frac{\sigma}{2}-\frac{\varrho}{2}+\frac{\mu}{2}+\frac{3}{4}\right)\Gamma_{\times}\left(\frac{\varrho}{2}+\frac{1}{4}\pm\frac{\nu}{2}\right)}{\Gamma_{\times}\left(\frac{\mu}{2}+\frac{\sigma}{2}+1\pm\frac{\nu}{2}\right)\Gamma\left(\frac{\mu}{2}-\frac{\sigma}{2}+\frac{\varrho}{2}+\frac{1}{4}\right)}\,.$$

Similarly other examples, which are useful in evaluating infinite integrals, can be obtained with the help of these theorems.

References.

[1] A. Erdély, *Tables of Integral Transforms*, Vol. 2, McGraw-Hill, New York 1954.

[2] A. Erdély, *Higher Trascendental Functions*, Vol. 1, McGraw-Hill, New York 1953.

[3] C. S. Meijer, *Ueber eine Erweiterung der Laplace-Transformation* (I), Nederl. Akad. Wetensch. Proc. (5) 43 (1940), 599-608.

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