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On the Geometric Means of Entire Functions of Several Complex Variables. (**)

1. – Let

$$f(z_1, z_2) = \sum_{m_1, m_2 \geq 0} a_{m_1 m_2} z_1^{m_1} z_2^{m_2},$$

be an entire function of two complex variables z_1 and z_2, holomorphic in the closed polydisc $\mathscr{P} \equiv \{ |z_j| \leqslant r_j \ (j = 1, 2)\}$. The maximum modulus of $f(z_1, z_2)$ is denoted as

$$M(r_1, r_2) = M(r_1, r_2; f) = \max_{|z_t| \leq r_t} | f(z_1, z_2) | \qquad (t = 1, 2).$$

The finite order ϱ of an entire function $f(z_1, z_2)$ is denoted as ([1], p. 219)

$$\limsup_{r_1, r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} = \varrho. \tag{1.1}$$

Similarly, we can define the lower order λ of $f(z_1, z_2)$ as

$$\liminf_{r_1, r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} = \lambda. \tag{1.2}$$

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The geometric mean $G(r_1, r_2; f)$ of the function $|f(z_1, z_2)|$ for $|z_t| \leq r_t$ $(t = 1, 2)$ has been defined as [2]:

$$(1.3) \qquad G(r_1, r_2; f) = \exp\left\{\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\log|f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|\, d\theta_1\, d\theta_2\right\}.$$

The function $G(r_1, r_2; f)$ is an increasing function of: (i) r_1 for a given r_2, (ii) r_2 for a given r_1, (iii) r_1 and r_2 (both increasing).

Further, I have defined the geometric mean $g_k(r_1, r_2; f)$ as [3]:

$$(1.4) \qquad g_k(r_1, r_2; f) = \exp\left\{\frac{(k+1)^2}{(r_1 r_2)^{k+1}}\int_0^{r_1}\int_0^{r_2}(x_1 x_2)^k \cdot \log G(x_1, x_2; f)\, dx_1\, dx_2\right\},$$

where $0 < k < \infty$.

Similarly, we denote the geometric means of the product of two or more entire functions as

$$(1.5) \qquad G(r_1, r_2; f_1 f_2) = \exp\left\{\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\log|f_1(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})\, f_2(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|\, d\theta_1\, d\theta_2\right\}$$

and

$$(1.6) \qquad g_k(r_1, r_2; f_1 f_2) = \exp\left\{\frac{(k+1)^2}{(r_1 r_2)^{k+1}}\int_0^{r_1}\int_0^{r_2}(x_1 x_2)^k \cdot \log G(x_1, x_2; f_1 f_2)\, dx_1\, dx_2\right\}.$$

In this paper we have investigated properties of the above defined geometric means (1.3), (1.4), (1.5) and (1.6).

2. – Theorem 1. *For a class of entire function $f(z_1, z_2)$ for which*

$$\lim_{r_1, r_2 \to \infty} \frac{\log\log g_k(r_1, r_2; f)}{\log(r_1 r_2)} = \infty,$$

we have

$$(2.1) \qquad \lim_{r_1, r_2 \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log\log g_k(r_1, r_2; f)}{\log(r_1 r_2)} = \frac{\log H_k}{\log h_k},$$

where

$$\lim_{r_1, r_2 \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \left\{ \frac{\log G(r_1, r_2; f)}{\log g_k(r_1, r_2; f)} \right\}^{1/\log(r_1 r_2)} = \frac{H_k}{h_k}, \tag{2.2}$$

provided that $h_k^2 < H_k$.

Proof. Since

$$\frac{\partial^2}{\partial r_1 \partial r_2} [\log \{(r_1 r_2)^{k+1} \cdot \log g_k(r_1, r_2; f)\}] = \frac{\partial}{\partial r_1} \frac{r_2^k \int_0^{r_1} x_1^k \cdot \log G(x_1, r_2; f)\, dx_1}{(r_1 r_2)^{k+1} \cdot \log g_k(r_1, r_2; f)} \tag{2.3}$$

$$= \{(r_1 r_2)^{k+1} \cdot \log g_k(r_1, r_2; f)\}^{-2} [(r_1 r_2)^k \cdot \log G(r_1, r_2; f) \cdot \{(r_1 r_2)^{k+1} \cdot \log g_k(r_1, r_2; f)\}$$

$$- \{r_2^k \int_0^{r_1} x_1^k \cdot \log G(x_1, r_2; f)\, dx_1\} \{r_1^k \int_0^{r_2} x_2^k \cdot \log G(r_1, x_2; f)\, dx_2\}] \leqslant$$

$$\leqslant \frac{1}{r_1 r_2} \frac{\log G(r_1, r_2; f)}{\log g_k(r_1, r_2; f)}.$$

Now integrating both the sides of the above inequality, we get

$$\log \{(r_1 r_2)^{k+1} \cdot \log g_k(r_1, r_2; f)\} \leqslant \int_0^{r_1} \int_0^{r_2} \frac{\log G(x_1, x_2; f)}{\log g_k(x_1, x_2; f)} \frac{dx_1}{x_1} \frac{dx_2}{x_2}.$$

If H_k is finite, then for a positive ε and constants a_1 and a_2, we have

$$\log \{(r_1 r_2)^{k+1} \cdot \log g_k(r_1, r_2; f)\} \leqslant \int_{a_1}^{r_1} \int_{a_2}^{r_2} \frac{\log G(x_1, x_2; f)}{\log g_k(x_1, x_2; f)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} + O(1) \tag{2.4}$$

$$< \int_{a_1}^{r_1} \int_{a_2}^{r_2} (H_k + \varepsilon)^{\log(x_1 x_2)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} + O(1)$$

$$< \frac{(H_k + \varepsilon)^{\log r_1 r_2)}}{\log (H_k + \varepsilon)} + O(1).$$

Since

$$\lim_{r_1, r_2 \to \infty} \frac{\log \log g_k(r_1, r_2; f)}{\log (r_1 r_2)} = \infty.$$

Hence the inequality (2.4) leads to

$$\limsup_{r_1, r_2 \to \infty} \frac{\log\log\log g_k(r_1, r_2; f)}{\log(r_1 r_2)} \leqslant \log H_k . \tag{2.5}$$

Next, integrating (2.3), we have

$$\log\{(4r_1r_2)^{k+1} \cdot \log g_k(2r_1, 2r_2; f)\} >$$

$$> \int_{r_1}^{2r_1}\int_{r_2}^{2r_2} \frac{\log^2 G(x_1, x_2; f)}{\log g_k(x_1, x_2; f)} \frac{1}{\log G(x_1, x_2; f)} \frac{dx_1}{x_1}\frac{dx_2}{x_2} - \int_{r_1}^{2r_1}\int_{r_2}^{2r_2} \frac{\log^2 G(x_1, x_2; f)}{\log^2 g_k(x_1, x_2; f)} \frac{dx_1}{x_1}\frac{dx_2}{x_2} .$$

Now, we note from ([**4**], p. 44) that $\{(r_1 r_2)^{k+1} \cdot \log^2 G(r_1, r_2; f)\}$ is a convex function of $\{(r_1 r_2)^{k+1} \cdot \log g_k(r_1, r_2; f)\}$, when one of the variables r_1 fixed and the other variable r_2 increases, vice-versa or both increase. Therefore, for $0 < H_k < \infty$,

$$\log\{(4r_1r_2)^{k+1} \cdot \log G(r_1, r_2; f)\} >$$

$$> \left[\frac{\log^2 G(r_1, r_2; f)}{\log g_k(r_1, r_2; f)} \frac{1}{\log G(2r_1, 2r_2; f)} - \frac{\log^2 G(2r_1, 2r_2; f)}{\log^2 g_k(2r_1, 2r_2; f)}\right] (\log 2)^2$$

$$> \left[(H_k - \varepsilon)^{\log(r_1r_2)} \frac{\log G(r_1, r_2; f)}{\log G(2r_1, 2r_2; f)} - (h_k + \varepsilon)^{2\log(r_1r_2)}\right] (\log 2)^2$$

$$> (H_k - \varepsilon)^{\log(r_1r_2)} (\log 2)^2 ,$$

since $h_k^2 < H_k$ and for a sequence of values of r_1 and r_2 tending to infinity. This leads to

$$\limsup_{r_1, r_2 \to \infty} \frac{\log\log\log g_k(r_1, r_2; f)}{\log(r_1 r_2)} \geqslant \log H_k . \tag{2.6}$$

Combining the two inequalities (2.5) and (2.6), it gives

$$\limsup_{r_1, r_2 \to \infty} \frac{\log\log\log g_k(r_1, r_2; f)}{\log(r_1 r_2)} = \log H_k .$$

Similarly, it can be proved that

$$\liminf_{r_1, r_2 \to \infty} \frac{\log\log\log g_k(r_1, r_2; f)}{\log(r_1 r_2)} = \log h_k .$$

This completes the proof of the theorem.

3. – Theorem 2. *Let $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ be two entire functions of orders ϱ' and ϱ'' respectively, then*

$$\alpha \equiv \limsup_{r_1, r_2 \to \infty} \frac{\log\log G(r_1, r_2; f_1 f_2)}{\log(r_1 r_2)} = \tag{3.1}$$

$$= \limsup_{r_1, r_2 \to \infty} \frac{\log\log g_k(r_1, r_2; f_1 f_2)}{\log(r_1 r_2)} \equiv \beta = \max(\varrho', \varrho'').$$

In order to prove this theorem, let us first prove the following:

Lemma 1. *Let $f(z_1, z_2)$ be holomorphic in the closed polydisc $\mathscr{P} \equiv \equiv \{|z_j| \leqslant R_j \quad (j = 1, 2)\}$ and if $z_j = r_j e^{i\theta_j}$, $0 \leqslant r_j < R_j$ $(j = 1, 2)$, then*

$$\log|f(z_1, z_2)| \leqslant \tag{3.2}$$

$$\leqslant \frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} \frac{(R_1^2 - r_1^2)(R_2^2 - r_2^2)\cdot\log|f(R_1 e^{i\varphi_1}, R_2 e^{i\varphi_2})|}{(R_1^2 - 2r_1 R_1\cdot\cos(\theta_1 - \varphi_1) + r_1^2)(R_2^2 - 2r_2 R_2\cdot\cos(\theta_2 - \varphi_2) + r_2^2)}\, d\varphi_1\, d\varphi_2 .$$

Proof. For a fixed z_2, let us apply Poisson-Jensen formula to the function $\log f(r_1 e^{i\theta_1}, z_2)$, then we have

$$\log|f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{(R_1^2 - r_1^2)\cdot\log|f(R_1 e^{i\varphi_1}, r_2 e^{i\theta_2})|}{R_1^2 - 2r_1 R_1\cdot\cos(\theta_1 - \varphi_1) + r_1^2}\, d\varphi_1 .$$

Again, we apply this formula to the function $\log f(R_1 e^{i\varphi_1}, r_2 e^{i\theta_2})$, for a given z_1, we get

$$\log|f(R_1 e^{i\varphi_1}, r_2 e^{i\theta_2})| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{(R_2^2 - r_2^2)\cdot\log|f(R_1 e^{i\varphi_1}, R_2 e^{i\varphi_2})|}{R_2^2 - 2r_2 R_2\cdot\cos(\theta_2 - \varphi_2) + r_2^2}\, d\varphi_2 .$$

Since the integrands and the range of integrations on the right hand side of above inequalities are positive, therefore we have

$$\log|f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| \leqslant$$

$$\leqslant \frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} \frac{(R_1^2 - r_1^2)(R_2^2 - r_2^2)\cdot\log|f(R_1 e^{i\varphi_1}, R_2 e^{i\varphi_2})|}{(R_1^2 - 2r_1 R_1\cdot\cos(\theta_1 - \varphi_1) + r_1^2)(R_2^2 - 2r_2 R_2\cdot\cos(\theta_2 - \varphi_2) + r_2^2)}\, d\varphi_1\, d\varphi_2 .$$

Proof of the Theorem. If $M(r_1, r_2; f_1)$ and $M(r_1, r_2; f_2)$ denote the maximum moduli of $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ respectively for $|z_j| \leqslant r_j$ $(j=1, 2)$, then

$$\text{(3.3)}\quad \log G(r_1, r_2; f_1 f_2) = \frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} \log |f_1(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \cdot f_2(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| \, d\theta_1 \, d\theta_2$$

$$\leqslant \log \{M(r_1, r_2; f_1) \cdot M(r_1, r_2; f_2)\}.$$

Let $f(z_1, z_2) = f_1(z_1, z_2) \cdot f_2(z_1, z_2)$. Then, from Lemma 1, we get

$$\log |f_1(z_1, z_2) \cdot f_2(z_1, z_2)| \leqslant$$

$$\leqslant \frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} \frac{(R_1^2 - r_1^2)(R_2^2 - r_2^2) \cdot \log |f_1(R_1 e^{i\varphi_1}, R_2 e^{i\varphi_2}) \cdot f_2(R_1 e^{i\varphi_1}, R_2 e^{i\varphi_2})|}{(R_1^2 - 2r_1 R_1 \cdot \cos(\theta_1 - \varphi_1) + r_1^2)(R_2^2 - 2r_2 R_2 \cdot \cos(\theta_2 - \varphi_2) + r_2^2)} \, d\varphi_1 \, d\varphi_2 .$$

Let us first choose r_1 and r_2, then θ_1 and θ_2, such that

$$\begin{matrix} \log \{M(r_1, r_2; f_1) \cdot |f_2(z_1, z_2)|\} \\ \log \{|f_1(z_1, z_2)| \cdot M(r_1, r_2; f_2)\} \end{matrix} \leqslant \frac{R_1 + r_1}{R_1 - r_1} \, \frac{R_2 + r_2}{R_2 - r_2} \log G(R_1, R_2; f_1 f_2),$$

according as $\varrho' \geqslant \varrho''$ or $\varrho' \leqslant \varrho''$. Taking $R_j = 2r_j$ $(j = 1, 2)$, it leads to

$$\text{(3.4)}\qquad \log G(2r_1, 2r_2; f_1 f_2) \geqslant \begin{matrix} (1/3) \log\{M(r_1, r_2; f_1) \cdot |f_2(z_1, z_2)|\} \\ (1/3) \log\{|f_1(z_1, z_2)| \cdot M(r_1, r_2; f_2)\} \end{matrix},$$

according as $\varrho' \geqslant \varrho''$ or $\varrho' \leqslant \varrho''$.

From (3.3) and (3.4), it follows that $\alpha = \max(\varrho', \varrho'')$. Next, since $G(x_1, x_2; f_1 f_2)$ is an increasing function of: (i) x_1 for a given x_2, (ii) x_2 for a given x_1, (iii) x_1 and x_2 (both increasing),

$$\log g_k(r_1, r_2; f_1 f_2) = \frac{(k+1)^2}{(r_1 r_2)^{k+1}} \int_0^{r_1}\int_0^{r_2} (x_1 x_2)^k \cdot \log G(x_1, x_2; f_1 f_2) \, dx_1 \, dx_2$$

$$\leqslant \log G(r_1, r_2; f_1 f_2),$$

and from this it follows that $\beta \leqslant \alpha$.

Further,

$$\log g_k(2r_1\,,\ 2r_2\ ;\ f_1 f_2) = \frac{(k+1)^2}{(4r_1 r_2)^{k+1}} \int_0^{2r_1}\int_0^{2r_2} (x_1 x_2)^k \cdot \log G(x_1\,,\ x_2;\ f_1 f_2)\, \mathrm{d}x_1\, \mathrm{d}x_2$$

$$\geqslant \frac{(k+1)^2}{(4r_1 r_2)^{k+1}} \int_{r_1}^{2r_1}\int_{r_2}^{2r_2} (x_1 x_2)^k \cdot \log G(x_1\,,\ x_2;\ f_1 f_2)\, \mathrm{d}x_1\, \mathrm{d}x_2$$

$$\geqslant \{1 - 1/2^{k+1}\}^2 \cdot \log G(r_1\,,\ r_2;\ f_1 f_2),$$

which leads to $\beta \geqslant \alpha$. Hence

$$\alpha = \beta = \max(\varrho'\,,\ \varrho'').$$

Theorem 3. *For the class of entire functions for which*

$$\lim_{r_1,\, r_2 \to \infty} \frac{\log\log g_k(r_1\,,\ r_2;\ f_1 f_2)}{\log (r_1 r_2)} = \infty\,,$$

we have

$$\lim_{r_1,\, r_2 \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log\log g_k(r_1,\ r_2;\ f_1 f_2)}{\log (r_1 r_2)} = \frac{\log L_k}{\log l_k}, \tag{3.5}$$

where

$$\lim_{r_1,\, r_2 \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\frac{\log G(r_1\,,\ r_2;\ f_1 f_2)}{\log g_k(r_1\,,\ r_2;\ f_1 f_2)}\right\}^{1/\log(r_1 r_2)} = \frac{L_k}{l_k},$$

provided that $l_k^2 < L_k$.

Proof is similar to that of Theorem 1, and so is omitted.

References.

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