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On the Absolute Nörlund Summability Factors of Infinite Series. (**)

1.1. – Definitions. Let $\sum a_n$ (1) be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, and let us write

$$P_n = p_0 + p_1 + \dots + p_n, \quad P_{-1} = p_{-1} = 0 .$$

The sequence-to-sequence transformation:

$$t_n = \sum_{\nu=0}^{n} \frac{p_{n-\nu} s_\nu}{P_n} \qquad (P_n \neq 0) \tag{1.1.1}$$

defines the sequence $\{t_n\}$ of NÖRLUND means (2) of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation, that is, $\sum |t_n - t_{n-1}| \leqslant K$ (3). *Absolute harmonic summability* is the particular case of the summability $|N, p_n|$ for $p_n = \dfrac{1}{n+1}$.

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(1) Throughout the paper $\sum$ denotes $\sum\limits_{1}^{\infty}$.

(2) NÖRLUND [6]; see also WORONOI [9], where substantially the same definition was given. This is the English translation of the paper of WORONOI (in Russian) due to TAMARKIN.

(3) MEARS [5]; K denotes here, as elsewhere, a positive constant, not necessarily the same at each occurrence.

For any sequence $\{f_{n,\nu}\}$ depending on n and ν, we write

$$\Delta_\nu f_{n,\nu} = f_{n,\nu} - f_{n,\nu+1};$$

and for any sequence $\{f_n\}$,

$$\Delta f_n = f_n - f_{n+1}.$$

A sequence $\{\lambda_n\}$ is said to be convex (4) if

$$\Delta^2\lambda_n \geqslant 0, \qquad n = 1, 2, \ldots,$$

where

$$\Delta^2\lambda_n = \Delta(\Delta\lambda_n) = \Delta\lambda_n - \Delta\lambda_{n+1}$$

and

$$\Delta\lambda_n = \lambda_n - \lambda_{n+1}.$$

If

$$\sum_{\nu=1}^{n} \frac{|s_\nu|}{\nu} = O(\log n),$$

as $n \to \infty$, then $\sum a_n$ is said to be strongly bounded by logarithmic means with index 1, or bounded $[R, \log n, 1]$ (5).

We generalise the above definition in a natural way as follows. We say that $\sum a_n$ is bounded $[R, P_n, 1]$, if

$$\text{(1.1.2)} \qquad \sum_{\nu=1}^{n} p_\nu |s_\nu| = O(P_n),$$

as $n \to \infty$.

1.2. - Introduction. In 1962, PATI (6) proved the following as a factor theorem connecting boundedness $[R, \log n, 1]$ and summability $|C, 1|$.

Theorem A. *Let λ_n be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent. If $\sum a_n$ is bounded $[R, \log n, 1]$, then $\sum \lambda_n a_n$ is summable $|C, 1|$.*

Now absolute harmonic summability implies summability $|C, \delta|$, for every $\delta > 0$ (7). Naturally therefore, we would expect the factors to be heavier in case we were interested in getting the result that $\sum \lambda_n a_n$ is summable $\left| N, \frac{1}{n+1} \right|$.

This idea led LAL (8) to the following theorem.

(4) For properties of *convex* sequences, see, for example, ZYGMUND [10], vol. I, p. 93.
(5) See PATI [8], p. 293.
(6) PATI [8], Theorem 1.
(7) MC FADDEN [4], p. 175.
(8) LAL [3], Theorem 2.

Theorem B. *If* $\sum a_n$ *is bounded* [R, log n, 1], *then the series* $\sum \log (n+1)\lambda_n a_n/n$, *where* $\{\lambda_n\}$ *is a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent, is summable* $\left| N, \dfrac{1}{n+1} \right|$.

2.1. – The object of this paper is to establish the following general theorem, which contains as a particular case the result of Theorem B, when one takes $p_n = \dfrac{1}{n+1}$.

Theorem. *If* $\sum a_n$ *is bounded* $[R, P_n, 1]$ *then the series* $\sum P_n\lambda_n p_n a_n$ *is summable* $|N, p_n|$ *under the following conditions:*

(a) $\{\lambda_n\}$ *is a positive, convex and bounded sequence such that the series* $\sum p_n\lambda_n$ *is convergent and the sequence* $\left\{\dfrac{1}{p_n} P_n \Delta\lambda_n\right\}$ *is bounded,*

(b) $\{p_n\}$ *is positive monotonic non-increasing,*

(c) *there exists a monotonic increasing function of* n, *say* μ_n $(< n-1$, *for sufficiently large* $n)$ *such that, as* $n \to \infty$,

(i)$_a$ $\quad P_n - P_{n-\theta_n} = O(1)$ (⁹), $\qquad$ (i)$_b$ $\quad p_{n-\theta_n} = O(p_n)$;

(ii) $$p_{\theta_n} = O(p_n),$$

(d) $$\frac{p_n}{p_{n+1}} = O(1),$$

(e) $$\sum p_n^2 < \infty,$$

(f) $$\frac{\Delta p_n}{p_n^2} = O(1),$$

(g) $$\left(\frac{1}{p_n} - \frac{1}{p_{n-\nu}}\right) p_\nu = O(1) \quad \textit{for } 1 \leqslant \nu \leqslant \theta_n - 1 .$$

2.2. – We shall require the following lemmas for the proof of the theorem.

Lemma 1. *If the sequence* $\{p_n\}$ *satisfies the conditions*

(i) $\dfrac{\Delta p_n}{p_n^2} = O(1)$, *and* (ii) $\dfrac{p_n}{p_{n+1}} = O(1)$,

(⁹) Throughout the paper θ_n denotes the greatest integer not greater than μ_n.

hen

(a) $\Delta \dfrac{1}{p_n} = O(1)$ *and* (b) $\dfrac{\Delta_\nu\, p_{n-\nu}}{p^2_{n-\nu}} = O(1),\ (1 \leqslant \nu \leqslant n-1)\,.$

Proof. We have

$$\Delta \frac{1}{p_n} = \frac{1}{p_n} - \frac{1}{p_{n+1}} = -\frac{\Delta\, p_n}{p_n\, p_{n+1}}$$

$$= \frac{1}{p_n\, p_{n+1}}\, O(p_n^2) = O\left(\frac{p_n}{p_{n+1}}\right) = O(1),$$

by hypotheses (i) & (ii).

We also have $(1 \leqslant \nu \leqslant n-1)$

$$\frac{\Delta_\nu\, p_{n-\nu}}{p^2_{n-\nu}} = \frac{p_{n-\nu} - p_{n-\nu-1}}{p^2_{n-\nu}} = -\frac{p_{n-\nu-1} - p_{n-\nu}}{p^2_{n-\nu-1}} \cdot \frac{p^2_{n-\nu-1}}{p^2_{n-\nu}}$$

$= O(1)\cdot O(1)$, by hypotheses (i) & (ii),

$= O(1)\,.$

Lemma 2. *If* $p_n > 0$, *then*

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n\, P_{n-1}} \leqslant K\,.$$

Proof.

$$\sum_1^m \frac{p_n}{P_n\, P_{n-1}} = \sum_1^m \Delta \frac{1}{P_{n-1}} = \frac{1}{P_0} - \frac{1}{P_m}\,.$$

Since $\{P_n\}$ is monotonic increasing, $\left\{\dfrac{1}{P_n}\right\}$ is monotonic decreasing. Also $\dfrac{1}{P_m} \geqslant 0$. Hence $\left\{\dfrac{1}{P_n}\right\}$ is convergent to a limit, l, say, such that $0 \leqslant l \leqslant \dfrac{1}{P_0}$. Hence

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n\, P_{n-1}} = \frac{1}{P_0} - l \leqslant \frac{1}{P_0}\,.$$

Lemma 3. *If the sequence* $\{\lambda_n\}$ *is non-increasing such that the series* $\sum p_n \lambda_n$ *is convergent, then*

(a) $$P_m \lambda_m = O(1);$$

(b) $$\sum_{n=1}^{m} P_n \Delta\lambda_n = O(1),$$

as $m \to \infty$.

Proof. We have

(2.2.1) $$\lambda_m P_m = \lambda_m \sum_{n=0}^{m} p_n = O(\sum_{0}^{m} \lambda_n p_n) = O(1),$$

as $m \to \infty$, by hypothesis.

Also, by (2.2.1),

$$\sum_{n=1}^{m} P_n \Delta\lambda_n = \sum_{n=1}^{m-1} \Delta P_n \sum_{\nu=1}^{n} \Delta\lambda_\nu + P_m \sum_{\nu=1}^{m} \Delta\lambda_\nu$$

$$= \sum_{n=1}^{m-1} (-p_{n+1})(\lambda_1 - \lambda_{n+1}) + P_m(\lambda_1 - \lambda_{m+1}) = \sum_{n=1}^{m-1} p_{n+1} \lambda_{n+1} + \lambda_1(p_0 + p_1) - P_m \lambda_{m+1}$$

$$= O(1),$$

as $m \to \infty$, by hypothesis.

This completes the proof of Lemma 3.

Since, if $\{\lambda_n\}$ is convex and bounded, then it is non-increasing, therefore the following result follows from Lemma 3.

Corollary. *If* λ_n *is a convex and bounded sequence such that the series* $\sum p_n \lambda_n$ *is convergent, then*

(a) $$P_m \lambda_m = O(1):$$

(b) $$\sum_{n=1}^{m} P_n \Delta\lambda_n = O(1),$$

as $m \to \infty$ (10).

Lemma 4. *If* $\sum a_n$ *is bounded* $[\mathrm{R}, P_n, 1]$, *and* $\{\lambda_n\}$ *is non-increasing such that the series* $\sum p_n \lambda_n$ *is convergent, then*

$$\sum_{n=1}^{m} \lambda_n p_n | s_n | = O(1),$$

as $m \to \infty$.

Proof. By ABEL's transformation, and definition (1.1.2),

$$\sum_{n=1}^{m} \lambda_n p_n | s_n | = O[\sum_{n=1}^{m-1} P_n \Delta \lambda_n + \lambda_m P_m] = O(1),$$

as $m \to \infty$, by Lemma 3.

Corollary 1. *If* $\sum a_n$ *is bounded* $[\mathrm{R}, P_n, 1]$, *and* $\{\lambda_n\}$ *is convex and bounded such that the series* $\sum p_n \lambda_n$ *is convergent, then*

$$\sum_{n=1}^{m} \lambda_n p_n | s_n | = O(1),$$

as $m \to \infty$.

Corollary 2. *If* $\sum a_n$ *is bounded* $[\mathrm{R}, P_n, 1]$, $\{p_n\}$ *is positive monotonic non-increasing such that the series* $\sum p_n^2/P_n$ *is convergent and* χ_n $(< n + 2$, *for sufficiently large* $n)$ *is a monotonic increasing function of* n, *then*

$$\sum_{n=1}^{m} \frac{p_n^2}{P_{n-1}} \, | s_{[\chi_n]-1} | = O(1),$$

as $m \to \infty$.

Proof. Under the hypotheses of Lemma 4, we have,

$$\sum_{n=1}^{m} \lambda_{[\chi_n]-1} \, p_{[\chi_n]-1} \, | s_{[\chi_n]-1} | = O(1) \, ,$$

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(10) In the particular case: $p_n = (n + 1)^{-1}$, compare (a) with CHOW [1], Lemma 4, and (b) with IZUMI e KAWATA [2], § 1 and PATI [7], Lemma 3.

as $m\to\infty$. We observe that $\frac{p_n}{P_n}$ satisfies the hypotheses about λ_n. Hence

$$\sum_{n=1}^{m} \frac{p^2_{[\chi_n]-1}}{P_{[\chi_n]-1}} \,|\, s_{[\chi_n]-1} \,| = O(1)\,,$$

as $m \to \infty$. Hence

$$\sum_{n=1}^{m} \frac{p_n^2}{P_{n-1}} \,|\, s_{[\chi_n]-1} \,| = O(1)\,,$$

as $m \to \infty$.

Lemma 5. *Under the hypotheses* (a), (b), (d) *and* (f) *of our theorem*

$$\sum_{n=1}^{m} \frac{1}{p_n} P_n \, \Delta^2 \lambda_n = O(1)\,,$$

as $m \to \infty$ (11).

Proof. We have

$$\sum_{n=1}^{m} \frac{1}{p_n} P_n \, \Delta^2 \lambda_n = \sum_{n=1}^{m-1} \Delta \, \frac{P_n}{p_n} \sum_{\nu=1}^{n} \Delta^2 \lambda_\nu + \frac{P_m}{p_m} \sum_{n=1}^{m} \Delta^2 \lambda_n$$

$$= -\sum_{n=1}^{m-1} \Delta \, \frac{P_n}{p_n} \, \Delta\lambda_{n+1} - \frac{P_m}{p_m} \, \Delta\lambda_{m+1} + \Delta\lambda_1 \left\{ \sum_{n=1}^{m-1} \Delta \, \frac{P_n}{p_n} + \frac{P_m}{p_m} \right\}$$

$$= -\sum_{n=1}^{m-1} \left\{ P_n \, \Delta \, \frac{1}{p_n} - 1 \right\} \Delta\lambda_{n+1} + O(1) + O(1)$$

$$= O \left[\sum_{n=1}^{m-1} P_n \, \Delta\lambda_{n+1} \right] + O(\lambda_m) + O(1)$$

$$= O(1),$$

as $m \to \infty$, by hypotheses, Lemma 1(a) and Corollary to Lemma 3.

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(11) In the particular case: $p_n = (n+1)^{-1}$, compare with Pati [8], Lemma 2.

Lemma 6. *Under the hypotheses of our theorem* [12]

$$\sum_{\nu=1}^{\theta_n} |\Delta_\nu \{(P_n - P_{n-\nu}) p_{n-\nu} P_\nu p_\nu \lambda_\nu\} s_\nu| = O(p_n).$$

Proof. By hypothesis and Lemma 1 (b),

$$\sum_{\nu=1}^{\theta_n} |\Delta_\nu \{(P_n - P_{n-\nu}) p_{n-\nu} P_\nu p_\nu \lambda_\nu\} s_\nu|$$

$$\leqslant K \sum_{\nu=1}^{\theta_n} p_{n-\nu}^2 P_\nu p_\nu \lambda_\nu |s_\nu| + K \sum_{\nu=1}^{\theta_n} (P_n - P_{n-\nu}) p_{n-\nu}^2 P_\nu p_\nu \lambda_\nu |s_\nu|$$

$$+ K \sum_{\nu=1}^{\theta_n} (P_n - P_{n-\nu}) p_{n-\nu} P_\nu p_\nu^2 \lambda_\nu |s_\nu| + K \sum_{\nu=1}^{\theta_n} (P_n - P_{n-\nu}) p_{n-\nu} P_\nu p_\nu \Delta\lambda_\nu |s_\nu|$$

$$= K \sum_{r=1}^{4} \mathfrak{J}_r \text{, say.}$$

Now

$$\mathfrak{J}_1 \leqslant K p_{n-\theta_n}^2 P_{\theta_n} \sum_{\nu=1}^{\theta_n} p_\nu \lambda_\nu |s_\nu|$$

$$\leqslant K p_n^2 P_n,$$

by hypothesis and Corollary 1 to Lemma 4.

$$\mathfrak{J}_2 \leqslant (P_n - P_{n-\theta_n}) \sum_{\nu=1}^{\theta_n} p_{n-\nu}^2 P_\nu p_\nu \lambda_\nu |s_\nu| \leqslant K \sum_{\nu=1}^{\theta_n} p_{n-\nu}^2 P_\nu p_\nu \lambda_\nu |s_\nu|$$

$$\leqslant K p_n^2 P_n,$$

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(12) Hypotheses (c) (ii), and (g) are not used here. Also it would be sufficient to take $\{\lambda_n\}$ as non-increasing instead of convex and bounded.

as in $\mathfrak{J}_1$. Now, since $\{p_n\}$ satisfies the hypotheses on $\{\lambda_n\}$ in Lemma 3, $P_n p_n = O(1)$, as $n \to \infty$. Thus $\mathfrak{J}_r \leqslant K p_n$ $(r = 1, 2)$.

By Corollary to Lemma 3,

$$\mathfrak{J}_3 \leqslant K(P_n - P_{n-\theta_n}) p_{n-\theta_n} \sum_{\nu=1}^{\theta_n} p_\nu^2 |s_\nu| \leqslant K p_n,$$

by Lemma 4 (on putting $\lambda_n = p_n$), and hypothesis.

By hypothesis,

$$\mathfrak{J}_4 \leqslant K(P_n - P_{n-\theta_n})\, p_{n-\theta_n} \sum_{\nu=1}^{\theta_n} P_\nu\, p_\nu\, \Delta\lambda_\nu |s_\nu| \leqslant K(P_n - P_{n-\theta_n}) p_{n-\theta_n} \sum_{\nu=1}^{\theta_n} p_\nu^2 |s_\nu|$$
$$\leqslant K p_n,$$

as in $\mathfrak{J}_3$.

This completes the proof of the lemma.

Lemma 7. *Under the hypotheses of our theorem* ([13])

$$\sum_{\nu=1}^{\theta_n} |\Delta_\nu (P_{n-\nu} P_\nu\, p_{n-\nu}\, \lambda_\nu) s_\nu| = O(P_n^2 p_n).$$

Proof.

$$\sum_{\nu=1}^{\theta_n} |\Delta_\nu (P_{n-\nu}\, P_\nu\, p_{n-\nu}\, \lambda_\nu) s_\nu|$$
$$\leqslant K \sum_{\nu=1}^{\theta_n} p_{n-\nu}^2\, P_\nu\, \lambda_\nu |s_\nu| + K \sum_{\nu=1}^{\theta_n} P_{n-\nu}\, p_\nu\, p_{n-\nu}\, \lambda_\nu |s_\nu| +$$
$$+ K \sum_{\nu=1}^{\theta_n} P_{n-\nu}\, P_\nu\, p_{n-\nu}^2\, \lambda_\nu |s_\nu| + K \sum_{\nu=1}^{\theta_n} P_{n-\nu}\, P_\nu\, p_{n-\nu}\, \Delta\lambda_\nu |s_\nu|, \quad \text{by Lemma 1 (b)},$$
$$= K \sum_{r=1}^{4} \mathfrak{J}_r', \text{ say.}$$

Now, by hypothesis and Corollary 1 to Lemma 4:

$$\mathfrak{J}_1' \leqslant K\, p_{n-\theta_n}^2\, P_{\theta_n} \sum_{\nu=1}^{\theta_n} \frac{1}{p_\nu}\, p_\nu\, \lambda_\nu |s_\nu| \leqslant K\, p_{n-\theta_n}^2 P_{\theta_n} \frac{1}{p_{\theta_n}} \sum_{\nu=1}^{\theta_n} p_\nu\, \lambda_\nu |s_\nu|$$
$$\leqslant K\, p_n\, P_n;$$

$$\mathfrak{J}_2' \leqslant K\, P_{n-1}\, p_{n-\theta_n} \sum_{\nu=1}^{\theta_n} p_\nu\, \lambda_\nu |s_\nu| \leqslant K\, P_n\, p_n\ ;$$

$$\mathfrak{J}_3' \leqslant K\, P_{n-1}\, P_{\theta_n}\, p_{n-\theta_n}^2 \sum_{\nu=1}^{\theta_n} \frac{1}{p_\nu}\, p_\nu\, \lambda_\nu |s_\nu| \leqslant K\, P_{n-1}\, P_{\theta_n}\, p_{n-\theta_n}^2 \frac{1}{p_{\theta_n}} \sum_{\nu=1}^{\theta_n} p_\nu\, \lambda_\nu |s_\nu|$$
$$\leqslant K\, P_\nu^2\, p_n.$$

([13]) Hypotheses (c) (i), (e) and (g) are not used here. Also it would be sufficient to take $\{\lambda_n\}$ as non-increasing instead of convex and bounded.

Also, by hypotheses,

$$\mathfrak{I}_4' \leqslant K\, P_{n-1}\, p_{n-\theta_n} \sum_{\nu=1}^{\theta_n} P_\nu\, \Delta\lambda_\nu\, |\, s_\nu\, | \leqslant K\, P_{n-1}\, p_{n-\theta_n} \sum_{\nu=1}^{\theta_n} p_\nu\, |\, s_\nu\, |$$

$$\leqslant K\, P_{n-1}\, p_{n-\theta_n} P_{\theta_n} \leqslant K\, P_n^2\, p_n\, .$$

This completes the proof of the lemma.

3.1. – Proof of the Theorem. Since

$$t_n = \frac{P_n\, u_0 + P_{n-1}\, u_1 + \dots + P_0\, u_n}{P_n} \qquad (u_n = P_n\, p_n\, \lambda_n\, a_n),$$

we have

$$t_n - t_{n-1} = \sum_{\nu=0}^{n-1} \left(\frac{P_\nu}{P_n} - \frac{P_{\nu-1}}{P_{n-1}}\right) u_{n-\nu} = \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} (p_\nu\, P_n - p_n\, P_\nu)\, u_{n-\nu}$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} (p_{n-\nu}\, P_n - p_n P_{n-\nu})\, u_\nu = \frac{p_n}{P_n\, P_{n-1}} \sum_{\nu=1}^{n} \left(\frac{P_n}{p_n} - \frac{P_{n-\nu}}{p_{n-\nu}}\right) p_{n-\nu}\, P_\nu\, p_\nu\, \lambda_\nu\, a_\nu\, .$$

Thus, in order to prove the theorem, we have to establish that

$$\sum |\, t_n - t_{n-1}\, | = \sum_{n=1}^{\infty} \frac{p_n}{P_n\, P_{n-1}}\, |\, T\, | \leqslant K, \tag{3.1.1}$$

where

$$T = \sum_{\nu=1}^{n} \left(\frac{P_n}{p_n} - \frac{P_{n-\nu}}{p_{n-\nu}}\right) p_{n-\nu}\, P_\nu\, p_\nu\, \lambda_\nu\, a_\nu\, .$$

Now

$$T = \Bigl(\sum_{\nu=1}^{\theta_n-1} + \sum_{\nu=\theta_n}^{n}\Bigr) \left(\frac{P_n}{p_n} - \frac{P_{n-\nu}}{p_{n-\nu}}\right) p_{n-\nu}\, P_\nu\, p_\nu\, \lambda_\nu\, a_\nu \tag{3.1.2}$$

$$= \frac{1}{p_n} \sum_{\nu=1}^{\theta_n-1} (P_n - P_{n-\nu})\, p_{n-\nu}\, P_\nu\, p_\nu\, \lambda_\nu\, a_\nu \;\; + \sum_{\nu=1}^{\theta_n-1} \left(\frac{1}{p_n} - \frac{1}{p_{n-\nu}}\right) p_\nu\, p_{n-\nu}\, P_{n-\nu}\, P_\nu\, \lambda_\nu\, a_\nu$$

$$+ \frac{1}{p_n} \sum_{\nu=\theta_n}^{n} (p_{n-\nu}\, P_n - p_n\, P_{n-\nu})\, P_\nu\, p_\nu\, \lambda_\nu\, a_\nu = T_1 + T_2 + T_3, \quad \text{say.}$$

Now, by ABEL's transformation,

$$T_1 = \frac{1}{p_n}\Bigg[\sum_{\nu=1}^{\theta_n-2} \Delta_\nu \{ (P_n - P_{n-\nu})\, p_{n-\nu}\, P_\nu\, p_\nu\, \lambda_\nu \} s_\nu$$

$$+ (P_n - P_{n-\theta_n+1})\, p_{n-\theta_n+1}\, P_{\theta_n-1}\, p_{\theta_n-1}\, \lambda_{\theta_n-1}\, s_{\theta_n-1}\Bigg].$$

Hence by hypotheses and Corollary to Lemma 3,

$$(3.1.3) \qquad |T_1| \leqslant K\frac{1}{p_n}\sum_{\nu=1}^{\theta_n-2} |\Delta_\nu \{ (P_n - P_{n-\nu})\, p_{n-\nu}\, P_\nu\, p_\nu\, \lambda_\nu \} s_\nu| + Kp_n |s_{\theta_n-1}|$$

$$\leqslant K + Kp_n |s_{\theta_n-1}|,$$

by Lemma 6.

By ABEL's transformation, since $\Delta_\nu \left\{\left(\frac{1}{p_n} - \frac{1}{p_{n-\nu}}\right) p_\nu\right\} = O(p_\nu)$ by hypotheses,

$$(3.1.4) \qquad |T_2| \leqslant K\sum_{\nu=1}^{\theta_n-2} p_\nu\, p_{n-\nu-1}\, P_{n-\nu-1}\, P_{\nu+1}\, \lambda_{\nu+1} |s_\nu| +$$

$$+ K\sum_{\nu=1}^{\theta_n-2} |\Delta_\nu (p_{n-\nu}\, P_{n-\nu}\, P_\nu\, \lambda_\nu)\, s_\nu| + K\, p_{n-\theta_n+1}\, P_{n-\theta_n+1}\, P_{\theta_n-1}\, \lambda_{\theta_n-1} |s_{\theta_n-1}|$$

$$\leqslant K\, p_n\, P_n^2 + K\, p_n\, P_n |s_{\theta_n-1}|,$$

by hypotheses, Corollary 1 to Lemma 4, Lemma 7 and Corollary to Lemma 3. By ABEL's transformation,

$$T_3 = \frac{1}{p_n}\sum_{\nu=\theta_n}^{n-1} \Delta_\nu \{ (p_{n-\nu}\, P_n - p_n\, P_{n-\nu})\, P_\nu\, p_\nu\, \lambda_\nu \} s_\nu$$

$$- \frac{1}{p_n}(p_{n-\theta_n}\, P_n - p_n\, P_{n-\theta_n})\; P_{\theta_n}\, p_{\theta_n}\, \lambda_{\theta_n}\, s_{\theta_n-1} + (p_0\, P_n - p_n\, P_0)\, P_n\, \lambda_n s_n$$

$$= T_{31} - T_{32} + T_{33}, \qquad \text{say.}$$

Now

$$(3.1.5) \qquad |T_{31}| \leqslant \frac{1}{p_n}\sum_{\nu=\theta_n}^{n-1} (\,|\Delta_\nu\, p_{n-\nu}|\, P_n + p_n\, p_{n-\nu})\, P_\nu\, p_\nu\, \lambda_\nu |s_\nu|$$

$$+ \frac{1}{p_n}\sum_{\nu=\theta_n}^{n-1} (p_{n-\nu-1}\, P_n - p_n\, P_{n-\nu-1})\, p_\nu^2\, \lambda_\nu |s_\nu|$$

$$+ K\frac{1}{p_n}\sum_{\nu=\theta_n}^{n-1} (p_{n-\nu-1}\, P_n - p_n\, P_{n-\nu-1}) P_\nu\, p_\nu^2\, \lambda_\nu \; |s_\nu|$$

$$+ K\frac{1}{p_n}\sum_{\nu=\theta_n}^{n-1} (p_{n-\nu-1}\, P_n - p_n\, P_{n-\nu-1})\, P_\nu\, p_\nu\, \Delta\lambda_\nu \; |s_\nu| = K\sum_{r=1}^{4} T_{31r}, \qquad \text{say.}$$

Now by hypothesis we have:

$$T_{311} \leqslant K \frac{1}{p_n} \sum_{\nu=\theta_n}^{n-1} p_{n-\nu}^2 P_n P_\nu p_\nu \lambda_\nu |s_\nu| \leqslant K \frac{1}{p_n} P_n p_{\theta_n} \sum_{\nu=\theta_n}^{n-1} p_{n-\nu}^2 P_\nu \lambda_\nu |s_\nu|$$

$$\leqslant K P_n \sum_{\nu=1}^{n-1} p_{n-\nu}^2 P_\nu \lambda_\nu |s_\nu|;$$

$$T_{312} \leqslant K \frac{1}{p_n} \sum_{\nu=\theta_n}^{n-1} p_{n-\nu} P_n p_\nu^2 \lambda_\nu |s_\nu| \leqslant K \frac{1}{p_n} P_n p_{\theta_n}^2 \sum_{\nu=\theta_n}^{n-1} p_{n-\nu} \lambda_\nu |s_\nu|$$

$$\leqslant K P_n p_n \sum_{\nu=1}^{n-1} p_{n-\nu} \lambda_\nu |s_\nu|;$$

$$T_{313} \leqslant K \frac{1}{p_n} \sum_{\nu=\theta_n}^{n-1} p_{n-\nu} P_n P_\nu p_\nu^2 \lambda_\nu |s_\nu| \leqslant K \frac{1}{p_n} P_n p_{\theta_n}^2 \sum_{\nu=\theta_n}^{n-1} p_{n-\nu} P_\nu \lambda_\nu |s_\nu|$$

$$\leqslant K P_n p_n \sum_{\nu=1}^{n-1} p_{n-\nu} P_\nu \lambda_\nu |s_\nu|;$$

$$T_{314} \leqslant K \frac{1}{p_n} \sum_{\nu=\theta_n}^{n-1} p_{n-\nu} P_n P_\nu p_\nu \Delta\lambda_\nu |s_\nu| \leqslant K \frac{1}{p_n} P_n p_{\theta_n} \sum_{\nu=\theta_n}^{n-1} p_{n-\nu} P_\nu \Delta\lambda_\nu |s_\nu|$$

$$\leqslant K P_n \sum_{\nu=1}^{n-1} p_{n-\nu} P_\nu \Delta\lambda_\nu |s_\nu| .$$

By Corollary to Lemma 3, we now have,

$$(3.1.6) \qquad |T_{32}| \leqslant K \frac{1}{p_n} p_{n-\theta_n} P_n P_{\theta_n} \lambda_{\theta_n} p_{\theta_n} |s_{\theta_n-1}| \leqslant K P_n p_n |s_{\theta_n-1}| ,$$

by hypothesis.

Also,

$$(3.1.7) \qquad |T_{33}| \leqslant K P_n^2 \lambda_n |s_n| .$$

Hence, collecting the results (3.1.3)-(3.1.7), we have from (3.1.2),

$$(3.1.8) \qquad |T| \leqslant K + Kp_n |s_{\theta_n-1}|$$

$$+ K p_n P_n^2 + K p_n P_n |s_{\theta_n-1}| + K P_n \sum_{\nu=1}^{n-1} p_{n-\nu}^2 P_\nu \lambda_\nu |s_\nu| + K p_n P_n \sum_{\nu=1}^{n-1} p_{n-\nu} \lambda_\nu |s_\nu|$$

$$+ K p_n P_n \sum_{\nu=1}^{n-1} p_{n-\nu} P_\nu \lambda_\nu |s_\nu| + K P_n \sum_{\nu=1}^{n-1} p_{n-\nu} P_\nu \Delta\lambda_\nu |s_\nu| + K P_n^2 \lambda_n |s_n| .$$

Now, from (3.1.1) and (3.1.8), we observe that for the proof of our theorem it suffices to show that

(3.1.9) $$I_r \leqslant K \quad \text{for} \quad r = 1, \dots, 6,$$

where

$$I_1 = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}}, \qquad I_2 = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} p_n P_n^2,$$

$$I_3 = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} p_n P_n \, | s_{\theta_n - 1} |, \qquad I_4 = \sum_{n=2}^{\infty} \frac{p_n}{P_n P_{n-1}} P_n \sum_{\nu=1}^{n-1} p_{n-\nu}^2 P_\nu \lambda_\nu \, | s_\nu | ,$$

$$I_5 = \sum_{n=2}^{\infty} \frac{p_n}{P_n P_{n-1}} P_n \sum_{\nu=1}^{n-1} p_{n-\nu} P_\nu \Delta\lambda_\nu \, | s_\nu |, \qquad I_6 = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} P_n^2 \lambda_n \, | s_n | .$$

We proceed to prove the inequalities (3.1.9).

$I_1 \leqslant K$, by Lemma 2.

$$I_2 = \sum_{n=1}^{\infty} p_n^2 \frac{P_n}{P_{n-1}} = \sum_{n=1}^{\infty} p_n^2 \left(1 + \frac{p_n}{P_{n-1}}\right) \leqslant K \sum_{n=1}^{\infty} p_n^2 \leqslant K,$$

by hypothesis.

$$I_3 = \sum_{n=1}^{\infty} \frac{p_n^2}{P_{n-1}} \, | s_{\theta_n - 1} | \leqslant K,$$

by Corollary 2 to Lemma 4.

$$I_4 = \sum_{n=2}^{\infty} \frac{p_n}{P_{n-1}} \sum_{\nu=1}^{n-1} p_{n-\nu}^2 P_\nu \lambda_\nu \, | s_\nu | = \sum_{\nu=1}^{\infty} P_\nu \lambda_\nu \, | s_\nu | \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_{n-1}} p_{n-\nu}^2$$

$$\leqslant K \sum_{\nu=1}^{\infty} P_\nu \lambda_\nu \, | s_\nu | \frac{p_{\nu+1}}{P_\nu} \sum_{n=\nu+1}^{\infty} p_{n-\nu}^2 \leqslant K \sum_{\nu=1}^{\infty} p_\nu \lambda_\nu | s_\nu | \leqslant K,$$

by Corollary 1 to Lemma 4.

$$I_5 = \sum_{n=2}^{m} \frac{p_n}{P_{n-1}} \sum_{\nu=1}^{n-1} p_{n-\nu} P_\nu \Delta\lambda_\nu \, | s_\nu | \qquad (m \to \infty)$$

$$= \sum_{\nu=1}^{m-1} P_\nu \Delta\lambda_\nu \, | s_\nu | \sum_{n=\nu+1}^{m} \frac{p_n}{P_{n-1}} p_{n-\nu} \leqslant \sum_{\nu=1}^{m} \Delta\lambda_\nu \, | s_\nu | \sum_{n=\nu+1}^{m} p_{n-\nu}^2$$

$$\leqslant K \sum_{\nu=1}^{m} \left(\frac{1}{p_\nu} \Delta\lambda_\nu\right) p_\nu \, |s_\nu| \leqslant K \left\{ \sum_{\nu=1}^{m-1} P_\nu \Delta \left(\frac{1}{p_\nu} \Delta\lambda_\nu\right) + \frac{1}{p_m} \Delta\lambda_m P_m \right\}$$

$$\leqslant K \sum_{\nu=1}^{m-1} P_\nu \left\{ \frac{1}{p_\nu} \Delta^2\lambda_\nu + \Delta\lambda_{\nu+1} \Delta \left(\frac{1}{p_\nu}\right) \right\} + K$$

$$\leqslant K \sum_{\nu=1}^{m-1} P_\nu \frac{1}{p_\nu} \Delta^2 \lambda_\nu + K \sum_{\nu=1}^{m-1} P_\nu \Delta\lambda_{\nu+1} + K \leqslant K ,$$

by hypotheses, lemmas 1(a) and 5, and Corollary to Lemma 3.

$$J_6 \leqslant K \sum_{n=1}^{\infty} p_n \lambda_n | s_n | \leqslant K ,$$

by Corollary 1 to Lemma 4.

This completes the proof of the theorem.

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