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Parallelism and Covariant Differentiation in a Generalized Finsler Space of n Dimensions.

Introduction.

The present paper deals with the problem of parallelism and covariant differentiation in a generalised FINSLER space. A generalised FINSLER space differs from a FINSLER space essentially in the non-symmetric character of the metric tensor. Since FINSLER spaces have been studied mainly in two ways [1], [2], this paper has been divided into two sections. The first Section deals with generalised FINSLER spaces of first kind viz. the spaces obtained as a generalisation of the FINSLER spaces of RUND [1]; the second Section deals with generalised FINSLER spaces of second kind viz. the ones obtained as a generalisation of the FINSLER spaces of CARTAN [2].

Section I.

Let F_n be an n-dimensional manifold in which a point is represented by means of an ordered n-tuple of real numbers.

A transformation of coordinates will be represented by

$$x^{i'} = x^{i'}(x^j) . \tag{1.1}$$

In analogy with FINSLER spaces we postulate that the distance between two neighbouring points $P(x^i)$ and $Q(x^i + dx^i)$ is given by

$$ds = F(x^i, dx^i),$$

where the function F satisfies the same four properties [1].

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Let $\dot{x}^i$ be an arbitrary contravariant vector. With each point x^i of F_n we associate a second order tensor $g_{ij}(x, \dot{x})$ which is, in general, non-symmetric in its indices. $g_{(ij)}$, $g_{[ij]}$ will denote the symmetric and skew-symmetric parts, respectively, of the metric tensor. The symmetric part is supposed to be the same as the metric tensor of a FINSLER space with the same distance function:

$$g_{(ij)}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \, \partial \dot{x}^j} . \tag{1.2}$$

We also assume that

$$\frac{\partial g_{[ij]}}{\partial \dot{x}^k} = 0 . \tag{1.3}$$

Therefore g_{ij} will represent the metric tensor of a FINSLER space if $g_{[ij]} = 0$; and that of a generalised Riemannian space if

$$C_{ijk} = \frac{1}{F} A_{ijk} \equiv \frac{1}{2} \frac{\partial g_{ij}}{\partial \dot{x}^k} = 0 . \tag{1.4}$$

We have

$$C_{ijk} \dot{x}^i = C_{ijk} \dot{x}^j = C_{ijk} \dot{x}^k = 0 . \tag{1.5}$$

The totality of all the contravariant vectors attached to the point P is defined to be the *tangent space* $T_n(P)$ at P. Thus we associate a tangent space with each point P of F_n.

Let $\dot{x}^i$ be any vector of this $T_n(P)$. Then we define the associated covariant vector y_i by

$$y_i = g_{(ij)}(x, \dot{x}) \dot{x}^j. \tag{1.6}$$

The totality of all such covariant vectors y_i attached to the point P is defined to be the *dual tangent space* $T_n'(P)$ at P.

Proceeding in the same way as RUND [1], we can establish one to one correspondence between the elements of $T_n(P)$ and $T_n'(P)$. The conjugate tensor of the tensor $g_{(ij)}$, is given by

$$g_{(ij)} G^{ik} = \delta_j^k . \tag{1.7}$$

Magnitude of a vector: if X^i be the contravariant components of a vector, then

$$g_{ij}(x, X) X^i X^j = F^2(x, X)$$

is defined to be the square of its magnitude.

Angle between two vectors: if X^i and Y^i be the contravariant components of two vectors, then the angle between X^i and Y^i is defined by

$$\cos(X, Y) = \frac{g_{(ij)}(x, X) X^i Y^j}{F(x, X) F(x, Y)}.$$

Evidently this concept of angle is, in geneial, not a symmetrical one.

Parallelism: Let X^i be a vector field defined along a curve $C: x^i = x^i(t)$. Then under the coordinate transformation (1.1), we have

$$X^i = A^i_{i'} X^{i'}, \tag{1.8}$$

where

$$A^i_{i'} = \frac{\partial x^i}{\partial x^{i'}}. \tag{1.9}$$

Differentiating (1.8) with respect to t, we obtain

$$\frac{dX^i}{dt} = A^i_{i'} \frac{dX^{i'}}{dt} + (\partial_{j'} A^i_{i'}) \dot{x}^{j'} X^{i'}, \tag{1.10}$$

where

$$\partial_{j'} A^i_{i'} = \frac{\partial A^i_{i'}}{\partial x^{j'}} \quad \dot{x}^{i'} = \frac{dx^{i'}}{dt}. \tag{1.11}$$

In analogy with other spaces [1], [3] we assume that the vector X^i undergoes parallel displacement along the curve C, if

$$\frac{dX^i}{dt} = - P^i_{hk}(x, \dot{x}) X^h \dot{x}^k, \tag{1.12}$$

where P^i_{hk} are to be determined as functions of x and $\dot{x}$.

Since the concept of parallelism has to be invariant through a transformation of coordinates, we have

$$\frac{\mathrm{d}X^{i'}}{\mathrm{d}t} = - P^{i'}_{h'k'}\, X^{h'}\, \dot{x}^{k'}\,, \tag{1.13}$$

where $P^{i'}_{h'k'}$ are the components of P^{i}_{hk} in the new coordinate system.

Substituting the values of $\frac{\mathrm{d}X^{i}}{\mathrm{d}t}$ and $\frac{\mathrm{d}X^{i'}}{\mathrm{d}t}$ from (1.12) and (1.13) in (1.10), we obtain

$$\left[\frac{\partial^2 x^i}{\partial x^{h'} \partial x^{k'}} + P^{i}_{hk}\, A^{h}_{h'}\, A^{k}_{k'} - P^{i'}_{h'k'}\, A^{i}_{i'}\right] X^{h'}\, \dot{x}^{k'} = 0.$$

In view of the arbitrary character of X^i and hence of $X^{i'}$, we obtain

$$\frac{\partial^2 x^i}{\partial x^{h'} \partial x^{k'}}\, \dot{x}^{k'} + P^{i}_{hk}\, A^{h}_{h'}\, \dot{x}^{k} = P^{i'}_{h'k'}\, A^{i}_{i'}\, \dot{x}^{k'}\,. \tag{1.14}$$

Thus we require the functions P^{i}_{hk} to satisfy the transformation law (1.14).

By direct calculation it is easily seen that the functions

$$P^{i}_{hk}(x,\, \dot{x}) = \Delta^{i}_{hk}(x,\, \dot{x}) - C^{i}_{hm}(x,\, \dot{x})\, \Delta^{m}_{kp}(x,\, \dot{x})\dot{x}^{p}\,, \tag{1.15}$$

where Δ^{i}_{hk} denote the generalised CHRISTOFFEL symbols, satisfy our requirement.

Thus equation (1.12) together with (1.15) defines the infinitesimal parallelism for any vector X^i. Moreover equation (1.15) can also be written as $\delta X^i/\delta t = 0$, where

$$\frac{\delta X^i}{\delta t} = \frac{\mathrm{d}X^i}{\mathrm{d}t} + P^{i}_{hk}(x,\, \dot{x})\, X^h\, \dot{x}^k\,. \tag{1.16}$$

This delta derivative is defined as the intrinsic derivative of the vector field X^i. This definition of the intrinsic derivative may easily be extended to a tensor of arbitrary order, e. g.,

$$\frac{\delta T^{i}_{jk}}{\delta t} = \frac{\mathrm{d}T^{i}_{jk}}{\mathrm{d}t} + P^{i}_{hm}\, T^{h}_{jk}\, \dot{x}^{m} - P^{h}_{jm} T^{i}_{hk}\, \dot{x}^{m} - \dot{P}^{h}_{km} \dot{T}^{i}_{jh} \dot{x}^{m}$$

can also be shown to be a tensor of the same type. It can further be shown that this derivative satisfies the usual laws of differentiation for sum, difference and product of tensors.

From the definition of a generalised CHRISTOFFEL symbol we deduce:

$$
(1.17)\quad
\begin{cases}
\Delta_{(ij)k} = \Delta_{k(ij)} = \dfrac{1}{2}\dfrac{\partial g_{(ij)}}{\partial x^k} \\[2ex]
\Delta_{[i|j|k]} = \Delta_{[j|k|i]} = \Delta_{[k|j|i]} = -\Delta_{[j|i|k]} = -\Delta_{[i|k|j]} = -\Delta_{[k|j|i]} \\
\qquad = \dfrac{1}{2}\left(\dfrac{\partial g_{[ij]}}{\partial x^k} + \dfrac{\partial g_{[jk]}}{\partial x^i} + \dfrac{\partial g_{[ki]}}{\partial x^j}\right) \\[2ex]
\Delta^i_{(hk)} = \left\{ {i \atop hk} \right\},
\end{cases}
$$

where $\left\{ {i \atop hk} \right\}$ are the CHRISTOFFEL symbols of the second type formed with respect to the symmetric part of the metric tensor.

Covariant differentiation. Let $X^i(x^k)$ be a vector field defined in a region R of F_n. Then, under a transformation of coordinates (1.1), we have

$$(1.18)\qquad X^i = A^i_{i'} X^{i'}.$$

Differentiating (1.18) with respect to x^k, we obtain

$$(1.19)\qquad \frac{\partial X^i}{\partial x^k} = A^i_{i'} \frac{\partial X^{i'}}{\partial x^{k'}} A^{k'}_k + X^{i'}(\partial_{k'} A^i_{i'}) A^{k'}_k .$$

Equation (1.19) shows that $\dfrac{\partial X^i}{\partial x^k}$ are not the components of a tensor. We therefore seek functions P^{*i}_{hk}, which are such that

$$(1.20)\qquad \delta_k X^i = \frac{\partial X^i}{\partial x^k} + P^{*i}_{hk} X^h$$

are the components of a mixed tensor of second order for an arbitrary vector X^i, i. e.,

$$(1.21)\qquad \delta_k X^i = A^i_{i'}\, A^{k'}_k \delta_{k'} X^{i'},$$

where

$$(1.22)\qquad \delta_{k'} X^{i'} = \frac{\partial X^{i'}}{\partial x^{k'}} + P^{*i'}_{h'k'} X^{h'},$$

$P^{*i'}_{h'k'}$ being the components of P^{*i}_{hk} in the new coordinate system.

From (1.20), (1.21) and (1.22) we obtain

$$X^{i'}\left(\frac{\partial^2 x^i}{\partial x^{i'}\partial x^{k'}}A_k^{k'}+P^{*i}_{hk}A^h_{i'}-P^{*h'}_{i'k'}A^{k'}_k A^i_{h'}\right)=0\,.$$

Due to the arbitrary character of $X^{i'}$, we have

$$\frac{\partial^2 x^i}{\partial x^{h'}\,\partial x^{k'}}+P^{*i}_{hk}A^h_{h'}A^k_{k'}=P^{*i'}_{h'k'}A^i_{i'}\,, \tag{1.23}$$

as the transformation law for P^{*i}_{hk}.

In analogy with other spaces [1], [3], we impose the condition

$$P^{*i}_{hk}\dot{x}^k=P^i_{hk}\dot{x}^k. \tag{1.24}$$

It is easily seen that the functions

$$P^{*i}_{hk}=\Delta^i_{hk}-G^{im}(C_{hml}P^l_{kp}+C_{kml}P^l_{hp}-C_{hkl}P^l_{mp})\dot{x}^p \tag{1.25}$$

satisfy our requrements.

Thus (1.20) together with (1.25) defines a process of covariant differentiation, which may again be extended to a tensor of arbitrary order. This process, as can be easily verified, satisfies the elementary laws of differentiation for the sum and product of tensors. From (1.25), we have

$$P^{*i}_{[hk]}=\Delta^i_{[hk]}\,. \tag{1.26}$$

We also define the associated quantities P^*_{ijk} by

$$P^*_{ijk}=g_{(jh)}P^{*h}_{ik}, \tag{1.27}$$

so that

$$\left\{\begin{aligned} P^*_{(ij)k}&=\frac{1}{2}\frac{\partial g_{(ij)}}{\partial x^k}-C_{ijm}P^m_{kp}\dot{x}^p\\ P^*_{[ij]k}&=\Delta_{[ij]k}+(C_{jkl}P^l_{im}-C_{ikl}P^l_{jm})\dot{x}^m\,. \end{aligned}\right. \tag{1.28}$$

Evidently, much of the development in the subject is not possible without the covariant derivative being defined for tensors involving the directional argument also.

For this purpose we suppose that the field of the directional argument $\dot{x}_i$ is given in the neighbourhood of the point where the covariant derivative is to be evaluated. Then we define the covariant derivative of a tensor $T_{ij}(x, \dot{x})$ by

$$\delta_k T_{ij} = \frac{\partial T_{ij}}{\partial x^k} + \frac{\partial T_{ij}}{\partial \dot{x}^h} \frac{\partial \dot{x}^h}{\partial x^k} - P^{*h}_{ik} T_{hj} - P^{*h}_{jk} T_{ih} . \tag{1.29}$$

It can be shown that these quantities form the components of a tensor with one order of covariance more.

Towards the conclusion of this section, we quote a few special cases in which covariant derivatives have been found out with a view to further development of the subject:

$$\delta_k g_{(ij)} = 2 C_{ijh} \left(\frac{\partial \dot{x}^h}{\partial x^k} + P^h_{km} \dot{x}^m \right), \tag{1.30}$$

$$\delta_k \dot{x}^h = \frac{\partial \dot{x}^h}{\partial x^k} + P^h_{km} \dot{x}^m - 2 \Delta^l_{[km]} \dot{x}^m , \tag{1.31}$$

$$\delta_k g_{(ij)} = 2 \ C_{ijh}(\delta_k \dot{x}^h + 2 \Delta^h_{[km]} \dot{x}^m). \tag{1.32}$$

From (1.32), we deduce

$$\dot{x}^i \delta_k g_{(ij)} = \dot{x}^j \delta_k g_{(ij)} = 0 . \tag{1.33}$$

We observe that the covariant derivative of the symmetric part of the metric tensor is different from the one obtained by RUND [1] for FINSLER spaces, though of course the principal advantage drawn out of it, viz. (1.33) is maintained.

$$\delta_k \delta^i_j = 0, \tag{1.34}$$

$$\delta_k G^{ij} = - G^{ip} G^{jq} \delta_k g_{(pq)} , \tag{1.35}$$

and

$$\dot{x}^i \delta_k C_{ijh} = - \frac{1}{2} \delta_k g_{(jh)} = - C_{ljh}(\delta_k \dot{x}^l + 2\Delta^l_{[km]} \dot{x}^m) . \tag{1.36}$$

Section II.

Consider a manifold F_n endowed with a local coordinate system $\dot{x}^i$ $(i = 1, \ldots, n)$. With each point $P(x^i)$ we associate a direction $\dot{x}^i$ so that if $\dot{x}^{i'}$ be the components of that direction in a new coordinate system, then we have

$$(2.1) \qquad \dot{x}^{i'} = A_i^{i'} \dot{x}^i.$$

The set of $2n$ quantities $(x^i, \dot{x}^i)$ is defined to be the element of support at each point $P(x^i)$. F_n will be called a generalised FINSLER space of the second kind if:

(i) A metric tensor with non-symmetric components $g_{ij}(x, \dot{x})$ (in general) is given such that the square of the distance between the centres x^i and $x^i + dx^i$ of two neighbouring elements $(x, \dot{x})$ and $(x + dx, \dot{x} + d\dot{x})$ is given by the expression

$$(2.2) \qquad ds^2 = g_{ij}(x, \dot{x})\, dx^i\, dx^j = g_{(ij)}(x, \dot{x})\, dx^i\, dx^j.$$

(No relation between $\dot{x}^i$ and dx^i is implied).

(ii) An analytical expression is given for the variation of the vector X^i when its element of support $(x, \dot{x})$ changes to $(x + dx, \dot{x} + d\dot{x})$. This variation called the geometrical variation is represented by means of an absolute differential

$$(2.3) \qquad DX^i = dX^i + C^i_{hk} X^h\, d\dot{x}^k + \Gamma^i_{hk} X^h\, dx^k.$$

The magnitude of a vector X^i is defined by $\sqrt{g_{ij}(x, \dot{x}) X^i X^j} = \sqrt{g_{(ij)}(x, \dot{x}) X^i X^j}$. It is of course assumed here that the quadratic form

$$(2.4) \qquad g_{ij}(x, \dot{x})\, X^i X^j$$

is positive-definite for all X^i and for an arbitrary element of support $(x, \dot{x})$. $g_{ij}(x, \dot{x})$ is supposed to be positively homogeneous of degree zero in the $\dot{x}^i$. $C^i_{hk}(x, \dot{x})$ and $\Gamma^i_{hk}(x, \dot{x})$ occurring in (2.3) are to be determined as functions of the element of support, and are not absolutely arbitrary. In fact they satisfy certain transformation laws which may be obtained from the fact that DX^i form the components of a vector:

$$(2.5) \qquad A^i_{h'k'} + \Gamma^i_{hk} A^h_{h'} A^k_{k'} + C^i_{hk} A^h_{h'} A^k_{k'l'} \dot{x}^{l} = \Gamma^{i'}_{h'k'} A^i_{i'} \qquad (A^i_{h'k'} \equiv \partial_{k'} A^i_{h'})$$

and

$$C^i_{hk}\, A^h_{h'}\, A^k_{k'} = C^{i'}_{h'k'}\, A^i_{i'} \,. \tag{2.6}$$

Following CARTAN [2], in analogy with FINSLER spaces, we make the following assumptions of an intrinsic nature:

a) If the direction of a vector X^i coincides with that of its element of support, then its magnitude is equal to $F(x, \dot{x})$.

b) The magnitude of a vector undergoing infinitesimal parallel displacement (i. e. when its geometrical variation is zero) is constant.

c) Let $\vec{X}$ and $\vec{Y}$ be two vectors with the same element of support $(x, \dot{x})$; Let $\mathrm{D}\vec{X}$ and $\mathrm{D}\vec{Y}$ be their absolute differentials when these vectors conserve the fixed contravariant components X^i and Y^i and when their common element of support undergoes the same infinitesimal rotation about its centre. Then $\vec{X}\cdot\mathrm{D}\vec{Y} = \vec{Y}\cdot\mathrm{D}\vec{X}$.

d) The absolute differential of a vector with fixed components X^i and whose direction coincides with that of its element of support, corresponding to an infinitesimal rotation of its element of support about the centre is null.

e) The components Γ^{*k}_{ij} which enter the expression for the absolute differential of a vector when its element of support is displaced parallel to itself, satisfy the equation

$$\Gamma^*_{[i|j|k]} = \Delta_{[i|j|k]} = g_{(jh)}\, \Gamma^{*h}_{[ik]} \,. \tag{2.7}$$

As in Section I, here also we assume that $\partial g_{[ij]}/\partial x^k = 0$.
Conditions (a) to (d) give us

$$\Gamma_{ijh} + \Gamma_{jih} = \frac{\partial g_{(ij)}}{\partial x^k} \,, \tag{2.8}$$

$$C_{ijk} = \frac{1}{4}\, \frac{\partial^3 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k} \,, \tag{2.9}$$

and

$$g_{(ij)}\,(x, \dot{x}) = \frac{1}{2}\, \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i\, \partial \dot{x}^j} \,. \tag{2.10}$$

In addition $F(x, x)$ is positively homogeneous of the first degree in the x^i.

Condition (e) gives us

$$\Gamma^*_{hjk} - \Gamma^*_{kjh} = \Delta_{hjk} - \Delta_{kjh} \,, \tag{2.11}$$

where

$$\Gamma^{*i}_{hk} = \Gamma^i_{hk} - C^i_{hl}\, \Gamma^l_{jk}\, x^j \,. \tag{2.12}$$

From (2.11) and (2.12) we have

$$\Gamma_{hjk} - \Gamma_{kjh} = \Delta_{hjk} - \Delta_{kjh} + (C_{hjl}\, \Gamma^l_{ik} - C_{kjl}\, \Gamma^l_{ih})\dot{x}^i \,. \tag{2.13}$$

Also, from (1.17) and (2.8) we obtain

$$\Gamma_{hjk} + \Gamma_{jhk} = \Delta_{hjk} + \Delta_{jhk} \,. \tag{2.14}$$

Equations (2.13) and (2.14) fix all the components of Γ_{ijh}. We now proceed to find an explicit expression for Γ_{ijh}.

Introducing the quantities

$$S_{ijh} = \Gamma_{ijh} - \Delta_{ijh} \,, \tag{2.15}$$

we obtain from (2.13) and (2.14),

$$S_{ijh} + S_{jih} = 0 \tag{2.16}$$

and

$$S_{ijh} - S_{hji} = (C_{ijl}\, \Gamma^l_{kh} - C_{hjl}\, \Gamma^l_{ki})\, \dot{x}^k \,. \tag{2.17}$$

This type of equations were solved by CARTAN [2]. Proceeding according to his technique we obtain

$$S_{hij} = C_{ijl}\, \Gamma^l_{kh}\, \dot{x}^k - C_{hjl}\, \Gamma^l_{ki}\, \dot{x}^k \,, \tag{2.18}$$

giving us

$$\Gamma_{ihj} = \Delta_{ihj} + C_{ijl}\, \Gamma^l_{kh}\, \dot{x}^k - C_{hjl}\, \Gamma^l_{ki}\, \dot{x}^k \tag{2.19}$$

and

$$\Gamma^*_{ihj} = \Delta_{ihj} + (C_{ijl}\, \Gamma^l_{kh} - C_{hjl}\, \Gamma^l_{ki} - C_{ihl}\, \Gamma^l_{kj})\, \dot{x}^k \,. \tag{2.20}$$

Angle between two vectors. The angle θ between two vectors with contravariant components X^i and Y^i is defined by

$$\cos\theta = \frac{g_{(ij)}(x, \dot{x})\, X^i Y^j}{\sqrt{g_{hk} X^h X^k}\,\sqrt{g_{pq} Y^p Y^q}}\,.$$

We notice that if l^i and l_i be the unit vectors along the element of support, we have

$$\text{(2.21)}\qquad \begin{cases} l^i = \dfrac{\dot{x}^i}{F(x, \dot{x})} \\[2ex] l_i = \dfrac{\partial F}{\partial \dot{x}^i}\,. \end{cases}$$

Therefore

$$\mathrm{D} l^i \equiv \overline{\omega}^i = \frac{1}{F}\,\mathrm{d}\dot{x}^i - \frac{\mathrm{d}F}{F^2}\,\dot{x}^i + \frac{1}{F}\,\Gamma^i_{hk}\,\dot{x}^h\,\mathrm{d}x^k\,,$$

giving us

$$\mathrm{d}\dot{x}^i = F\,\overline{\omega}^i + \frac{\mathrm{d}F}{F}\,\dot{x}^i - \Gamma^i_{hk}\,\dot{x}^h\,\mathrm{d}x^k\,.$$

Therefore if we consider a tensor, $T_{ij}(x, \dot{x})$, covariant of second order and homogeneous in the $\dot{x}^i$, of zero degree, then an easy manipulation gives us

$$\text{(2.22)}\qquad \mathrm{D}T_{ij} = \left(\frac{\partial T_{ij}}{\partial x^k} - \frac{\partial T_{ij}}{\partial \dot{x}^h}\,\Gamma^h_{sk}\,\dot{x}^s - T_{ih}\,\Gamma^{*h}_{jk} - T_{hj}\,\Gamma^{*h}_{ik}\right)\mathrm{d}x^k + \\ + \left(F\,\frac{\partial T_{ij}}{\partial \dot{x}^k} - T_{ih}\,A^h_{jk} - T_{hj}\,A^h_{ik}\right)\overline{\omega}^k.$$

The coefficient of $\mathrm{d}x^k$ in this expression is defined to be the covariant derivative of X^i with respect to x^k, so that if we adopt the notation $\overset{0}{\nabla}_k T_{ij}$ for it, then

$$\text{(2.23)}\qquad \overset{0}{\nabla}_k T_{ij} = \frac{\partial T_{ij}}{\partial x^k} - \frac{\partial T_{ij}}{\partial \dot{x}^h}\,\Gamma^h_{sk}\,\dot{x}^s - T_{ih}\,\Gamma^{*h}_{jk} - T_{hj}\,\Gamma^{*h}_{ik}\,.$$

The coefficient of $\overline{\omega}^k$ in (2.22) is defined as the covariant derivative of X^i with respect to $\dot{x}^k$. It will be denoted by

$$\text{(2.24)}\qquad \overset{1}{\nabla}{}^k T_{ij} = F\,\frac{\partial T_{ij}}{\partial \dot{x}^k} - T_{ih}\,A^h_{jk} - T_{hj}\,A^h_{ik}$$

where $A^i_{hk} = F C^i_{hk}$.

It can be shown that both these processes satisfy the elementary laws of differentiation for the sum and product of tensors.

In the end, we quote the following results obtainable by simple direct calculation and which are likely to be useful at a later stage:

$$\overset{0}{\nabla}_k g_{(ij)} = 0, \quad \overset{0}{\nabla}_k \delta^i_j = 0, \quad \overset{0}{\nabla}_k G^{ij} = 0, \quad \overset{0}{\nabla}_k F = 0, \quad \overset{0}{\nabla}_k \dot{x}^i = 0, \tag{2.25}$$

$$\dot{x}^i \overset{0}{\nabla}_k A_{ihj} = \dot{x}^h \overset{0}{\nabla}_k A_{ihj} = \dot{x}^j \overset{0}{\nabla}_k A_{ihj} = 0\,, \tag{2.26}$$

$$\frac{\partial \Gamma^{*l}_{ij}}{\partial \dot{x}^r} \dot{x}^i \dot{x}^j = 0, \tag{2.27}$$

$$\frac{\partial \Gamma^{*l}_{ij}}{\partial \dot{x}^r} \dot{x}^i = \dot{x}^j \overset{0}{\nabla}_j C^l_{jr} + 2\,(C^h_{jr} \Delta^l_{[hk]} - C^l_{hr} \Delta^k_{[jk]})\, \dot{x}^k, \tag{2.28}$$

$$\frac{\partial \Gamma^{*l}_{ij}}{\partial \dot{x}^r} \dot{x}^j = \dot{x}^j \overset{0}{\nabla}_j C^l_{ir} + 2(C^l_{hr} \Delta^h_{[ik]} + C^h_{ir} \Delta^l_{[hk]})\, \dot{x}^k\,, \tag{2.29}$$

$$\begin{aligned} \frac{\partial \Gamma^{*l}_{ij}}{\partial \dot{x}^r} = {} & \overset{0}{\nabla}_i C^l_{jr} + \overset{0}{\nabla}_j C^l_{ir} - G^{kl} \overset{0}{\nabla}_k C_{ijr} - \\ & - x^k (C^l_{ih} \overset{0}{\nabla}_k C^h_{jr} + C^l_{jh} \overset{0}{\nabla}_k C^h_{ir} - C^h_{ij} \overset{0}{\nabla}_k C^l_{hr}) - 2\, \Delta^m_{[hk]} \dot{x}^k (C^l_{im} C^h_{jr} + C^l_{jm} C^h_{ir}) + \\ & + 2\, C^h_{mr} (C^l_{ih} \Delta^m_{[jk]} + C^l_{jh} \Delta^m_{[ik]})\, \dot{x}^k + 2\, C^h_{ij}\, C(C^m_{hr} \Delta^l_{[mk]} - C^l_{mr} \Delta^m_{[hk]})\, \dot{x}^k - \\ & - 2\,(C^l_{hr} \Delta^h_{[ji]} + C^h_{ir} \Delta^l_{[jh]} + C^h_{jr} \Delta^l_{[ih]})\,. \end{aligned} \tag{2.30}$$

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