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On the Means of Products of Integral Functions. (**)

1. – Let $f_1(z)$, $f_2(z)$, ..., $f_s(z)$ be s integral functions of orders ϱ_1, ϱ_2, ..., ϱ_s respectively and let

$$\mu_\delta(r,\ f_1 f_2 \dots f_s) = \frac{1}{2\pi}\int_0^{2\pi} |\ f_1(re^{i\theta})\ f_2(re^{i\theta}) \dots f_s(re^{i\theta})\ |^\delta\ d\theta . \tag{1.1}$$

$$\mu_\delta(r,\ f_1^{(n)} f_2^{(n)} \dots f_s^{(n)}) = \frac{1}{2\pi}\int_0^{2\pi} |f_1^{(n)}(re^{i\theta})\ f_2^{(n)}(re^{i\theta}) \dots f_s^{(n)}(re^{i\theta})\ |^\delta\ d\theta . \tag{1.2}$$

$$m_{\delta,k}(r,\ f_1 f_2 \dots f_s) = \frac{1}{\pi\, r^{k+1}}\int_0^r\int_0^{2\pi} |f_1(xe^{i\theta}) f_2(xe^{i\theta}) \dots f_s(xe^{i\theta})\ |^\delta\ x^k\ dx\ d\theta . \tag{1.3}$$

$$m_{\delta,k}(r,\ f_1^{(n)} f_2^{(n)} \dots f_s^{(n)}) = \frac{1}{\pi\, r^{k+1}}\int_0^r\int_0^{2\pi} |f_1^{(n)}(xe^{i\theta})\ f_2^{(n)}(xe^{i\theta}) \dots f_s^{(n)}(xe^{i\theta})\ |^\delta\ x^k\ dx\ d\theta , \tag{1.4}$$

where $f_1^{(n)}(z)$, $f_2^{(n)}(z)$, ..., $f_s^{(n)}(z)$ are the n-th derivatives of $f_1(z)$, $f_2(z)$, ..., $f_s(z)$ respectively, and δ and k are any positive numbers.

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Also let

$$L_{\delta,k} = \limsup_{r\to\infty} \left\{ \frac{\mu_\delta (r,\, f_1 f_2 \dots f_s)}{m_{\delta,k}(r,\, f_1 f_2 \dots f_s)} \right\}^{1/\log r} .$$

Here we have considered the mean values of products of two integral functions and have obtained some of their properties. The results can easily be extended to s integral functions.

2. - Theorem 1. *For integral functions $f_1(z)$ and $f_2(z)$ of orders ϱ_1 and ϱ_2 respectively, both not polynomials,*

$$\log L_{\delta,k} = \max(\varrho_1,\ \varrho_2) .$$

We shall first prove the following Lemmas.

Lemma 1. *Let*

$$\alpha = \limsup_{r\to\infty} \frac{\log\log \mu_\delta (r,\, f_1 f_2)}{\log r}\,; \qquad \beta = \limsup_{r\to\infty} \frac{\log\log m_{\delta,k} (r,\, f_1 f_2)}{\log r} .$$

Then $\alpha = \beta = \max(\varrho_1,\ \varrho_2)$.

Proof. If $M(r,\, f_1)$ and $M(r,\, f_2)$ denote the maximum moduli of $f_1(z)$ and $f_2(z)$ respectively for $|z| = r$, then

$$\mu_\delta(r,\, f_1 f_2) = \frac{1}{2\pi}\int_0^{2\pi} |f_1(re^{i\theta})\, f_2(re^{i\theta})\,|^\delta\, d\theta \leqslant \{ M(r,\, f_1)\ M(r,\, f_2)\, \}^\delta . \tag{2.1}$$

Further, if $f(z)$ is regular in $|z| \leqslant R$, and if $z = re^{i\theta}$, $0 \leqslant r < R$, $\delta > 0$, then [**1**, p. 192]

$$|f(z)\,|^\delta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, |f(Re^{i\varphi})\,|^\delta\, d\varphi}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} .$$

Let $f(z) = f_1(z)\ f_2(z)$. Then we get

$$|f_1(z) f_2(z)\,|^\delta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, |f_1(Re^{i\varphi})\, f_2(Re^{i\varphi})\,|^\delta\, d\varphi}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} .$$

Choose z, such that

$$\begin{matrix} \{ M(r, f_1) \ | f_2(re^{i\theta}) |\}^\delta \\ \{ | f_1(re^{i\theta}) | \ M(r, f_2) \}^\delta \end{matrix} \leqslant \frac{R+r}{R-r} \mu_\delta(R, f_1 f_2) .$$

according as $\varrho_1 \geqslant \varrho_2$ or $\varrho_1 \leqslant \varrho_2$. Taking $R = 2r$, leads to

$$\mu_\delta(2r, f_1 f_2) \geqslant \begin{matrix} 1/3 \{ M(r, f_1) \ | f_2(re^{i\theta}) | \}^\delta \\ 1/3 \{ | f_1(re^{i\theta}) | \ M(r, f_2) \}^\delta . \end{matrix} \tag{2.2}$$

From (2.1) and (2.2), it follows that $\alpha = \max(\varrho_1, \varrho_2)$. Since $\mu_\delta(x, f_1 f_2)$ is an increasing function of x,

$$m_{\delta,k}(r, f_1 f_2) = \frac{2}{r^{k+1}} \int_0^r \mu_\delta(x, f_1 f_2)\, x^k \, dx \leqslant \frac{2\mu_\delta(r, f_1 f_2)}{r^{k+1}} \int_0^r x^k \, dx = \frac{2\mu_\delta(r, f_1 f_2)}{k+1}$$

and from this follows $\beta \leqslant \alpha$.

Further,

$$m_{\delta,k}(2r, f_1 f_2) = \frac{2}{(2r)^{k+1}} \int_0^{2r} \mu_\delta(x, f_1 f_2)\, x^k \, dx \geqslant \frac{2}{(2r)^{k+1}} \int_r^{3r} \mu_\delta(x, f_1 f_2)\, x^k \, dx \geqslant$$

$$\geqslant \frac{2\mu_\delta(r, f_1 f_2)}{(2r)^{k+1}} \, \frac{(2r)^{k+1} - r^{k+1}}{k+1} ,$$

which leads to $\beta \geqslant \alpha$. Hence

$$\alpha = \beta = \max(\varrho_1, \varrho_2) .$$

Lemma 2. *$\log \mu_\delta(r, f_1 f_2)$ is a convex function of $\log r$.*

Proof. We have

$$\begin{matrix} | f_1(z) |^\delta \\ | f_2(z) |^\delta \end{matrix} \leqslant |f_1(z) f_2(z)|^\delta \leqslant |f(z)|^{2\delta},$$

where $|f| =$ greater of $(|f_1|, |f_2|)$ for $|z| \leqslant r$.

From this, it follows

$$\begin{matrix} \mu_\delta(r, f_1) \\ \mu_\delta(r, f_2) \end{matrix} \leqslant \mu_\delta(r, f_1 f_2) \leqslant \mu_{2\delta}(r, f) . \tag{2.3}$$

Since $\log \mu_\delta(r, f_1)$, $\log \mu_\delta(r, f_2)$ and $\log \mu_{2\delta}(r, f)$ are convex functions of $\log r$, therefore, it follows from (2.3) that $\log \mu_\delta(r, f_1 f_2)$ is also a convex function of $\log r$. Since we are considering $f_1(z)$ and $f_2(z)$ to be integral functions and $\log \mu_\delta(r, f_1)$, $\log \mu_\delta(r, f_2)$ and $\log \mu_{2\delta}(r, f)$ are increasing convex functions of $\log r$, therefore, $\log \mu_\delta(r, f_1 f_2)$ is an *increasing convex function of* $\log r$ for $r > r_0 = r_0(f_1, f_2)$.

Lemma 3. *If δ and k are any positive numbers, $r^{k+1}\mu_\delta(r, f_1 f_2)$ is a convex function of $r^{k+1} m_{\delta,k}(r, f_1 f_2)$.*

Proof. Since

$$\frac{\mathrm{d}}{\mathrm{d}r}\{ r^{k+1} m_{\delta,k}(r, f_1 f_2) \} = \frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{1}{\pi}\int_0^r\int_0^{2\pi} |(f_1 xe^{i\theta})\, f_2(xe^{i\theta})\,|^\delta\, x^k\, \mathrm{d}x\, \mathrm{d}\theta\right\} =$$

$$= 2r^k \frac{1}{2\pi}\int_0^{2\pi} |f_1(re^{i\theta})\, f_2(re^{i\theta})\,|^\delta\, \mathrm{d}\theta = 2r^k \mu_\delta(r, f_1 f_2),$$

we have, therefore,

$$\frac{\mathrm{d}\{r^{k+1}\mu_\delta(r, f_1 f_2)\}}{\mathrm{d}\{r^{k+1} m_{\delta,k}(r, f_1 f_2)\}} = \frac{\frac{\mathrm{d}}{\mathrm{d}r}\{r^{k+1}\mu_\delta(r, f_1 f_2)\}}{\frac{\mathrm{d}}{\mathrm{d}r}\{r^{k+1} m_{\delta,k}(r, f_1 f_2)\}} =$$

$$= \frac{(k+1)r^k \mu_\delta(r, f_1 f_2) + r^{k+1}\mu'_\delta(r, f_1 f_2)}{2r^k \mu_\delta(r, f_1 f_2)} = \frac{1}{2}\left\{(k+1) + r\frac{\mu'_\delta(r, f_1 f_2)}{\mu_\delta(r, f_1 f_2)}\right\},$$

which increases with r for $r > r_0$, because, by Lemma 2, $\log \mu_\delta(r, f_1 f_2)$ is a convex function of $\log r$. In fact it is an *increasing convex function* of $\log r$ for $r > r_0$.

Lemma 4. *$\log m_{\delta,k}(r, f_1 f_2)$ is an increasing convex function of $\log r$ for $r > r_0 = r_0(f_1, f_2)$.*

Proof. Since

$$\frac{\mathrm{d}}{\mathrm{d}r}\{ \log m_{\delta,k}(r, f_1 f_2) \} = \frac{1}{m_{\delta,k}(r, f_1 f_2)}\left\{\frac{1}{\pi\, r^{k+1}}\, r^k \int_0^{2\pi} |\, f_1(re^{i\theta})\, f_2(re^{i\theta})\,|^\delta\, \mathrm{d}\theta \,-\right.$$

$$\left. - \frac{k+1}{\pi\, r^{k+2}}\int_0^r\int_0^{2\pi} |f_1(xe^{i\theta})\, f_2(xe^{i\theta})\,|^\delta\, x^k\, \mathrm{d}x\, \mathrm{d}\theta\right\} =$$

$$= \frac{1}{m_{\delta,k}(r, f_1 f_2)}\left\{\frac{2}{r}\mu_\delta(r, f_1 f_2) - \frac{k+1}{r} m_{\delta,k}(r, f_1 f_2)\right\},$$

hence

$$\frac{\mathrm{d}\{\log m_{\delta,k}(r,\ f_1 f_2)\}}{\mathrm{d}\{\log r\}} = \left\{\frac{2\mu_\delta\ (r,\ f_1 f_2)}{m_{\delta,k}(r,\ f_1 f_2)} - (k+1)\right\},$$

which increases with r for $r > r_0$, because, by Lemma 3, $r^{k+1}\mu_\delta(r,\ f_1 f_2)$ is an increasing convex function of $r^{k+1}\ m_{\delta,k}(r,\ f_1 f_2)$ for $r > r_0$.

P r o o f o f T h e o r e m 1. If $L_{\delta,k} < \infty$, then, for a positive ε and a suitable constant a,

$$\log\{r^{k+1}\ m_{\delta,k}(r,\ f_1 f_2)\} = 2\int_0^r \frac{x^k\ \mu_\delta\ (x,\ f_1 f_2)}{x^{k+1}\ m_{\delta,k}(x,\ f_1 f_2)}\, \mathrm{d}x =$$

(2.4)

$$= O(1) + 2\int_a^r \frac{\mu_\delta\ (x,\ f_1 f_2)}{m_{\delta,k}(x,\ f_1 f_2)} \frac{dx}{x} < O(1) + 2\int_a^r (L_{\delta,k} + \varepsilon)^{\log x} \frac{dx}{x} = O(1) + \frac{2(L_{\delta,k} + \varepsilon)^{\log r}}{\log\ (L_{\delta,k} + \varepsilon)}.$$

Since by Lemma 4, $\log m_{\delta,k}(r,\ f_1 f_2)$ is an increasing convex function of $\log r$, therefore,

$$\lim_{r \to \infty} \frac{\log m_{\delta,k}(r,\ f_1 f_2)}{\log r} = \infty,$$

if at least one of the functions is not a polynomial. Hence, from (2.4), we have $\log L_{\delta,k} \geqslant \max\ (\varrho_1,\ \varrho_2)$. The above inequality obviously holds when $L_{\delta,k} = \infty$.

It follows from Lemma 3 that $\dfrac{\mu_\delta\ (x,\ f_1 f_2)}{m_{\delta,k}(x,\ f_1 f_2)}$ is an increasing function of x for $x > x_0$ and therefore, for $0 < L_{\delta,k} < \infty$,

$$\log\{(2r)^{k+1}\ m_{\delta,k}(2r,\ f_1 f_2)\} \geqslant 2\int_r^{2r} \frac{x^k\ \mu_\delta\ (x,\ f_1 f_2)}{x^{k+1}\ m_{\delta,k}(x,\ f_1 f_2)}\, \mathrm{d}x \geqslant$$

$$\geqslant \frac{2\mu_\delta\ (r,\ f_1 f_2)}{m_{\delta,k}(r,\ f_1 f_2)} \int_r^{2r} \frac{\mathrm{d}x}{x} = \frac{2\mu_\delta\ (r,\ f_1 f_2)}{m_{\delta,k}(r,\ f_1 f_2)} \log 2 > (L_{\delta,k} - \varepsilon)^{\log r}\, 2 \log 2,$$

for an infinite sequence of values of r tending to infinity.

Consequently $\log L_{\delta,k} \leqslant \max\ (\varrho_1,\ \varrho_2)$, which also holds for $L_{\delta,k} = 0$. If $L_{\delta,k} = \infty$, the above argument gives $\max\ (\varrho_1,\ \varrho_2) = \infty$.

This proves that $\log L_{\delta,k} = \max(\varrho_1, \varrho_2)$.

3. - Theorem 2. *Let $f_1(z)$ and $f_2(z)$ be integral functions, both not polynomials, of orders ϱ_1 and ϱ_2, $n(r, f_1)$ and $n(r, f_2)$ denote the number of zeros of $f_1(z)$ and $f_2(z)$ respectively in $|z| \leqslant r$ and $f_1(0) \neq 0$, $f_2(0) \neq 0$. Further,*

if

(i) $$\liminf_{r\to\infty} \frac{n(r, f_1) + n(r, f_2)}{r^\varrho} = \nu, \qquad \text{where } \varrho = \max(\varrho_1, \varrho_2),$$

then

(3.1) $$\liminf_{r\to\infty} \frac{\log \mu_\delta(r, f_1 f_2)}{r^\varrho} \geqslant \frac{\nu\delta}{\varrho};$$

if

(ii) $$\liminf_{r\to\infty} \frac{n(r, f_1) + n(r, f_2)}{r \log r} > 1,$$

then

(3.2) $$\liminf_{r\to\infty} \frac{\log \mu_\delta(r, f_1 f_2)}{r \log r} > \delta.$$

Proof. (i) We have

$$\log \mu_\delta(r, f_1 f_2) = \log \left[\frac{1}{2\pi} \int_0^{2\pi} |f_1(re^{i\theta})\ f_2(re^{i\theta})|^\delta \, d\theta\right],$$

and using lemma [3, p. 311], we get

$$\log \mu_\delta(r, f_1 f_2) \geqslant \frac{\delta}{2\pi} \int_0^{2\pi} \log |f_1(re^{i\theta})\, f_2(re^{i\theta})| \, d\theta.$$

From JENSEN's formula, we have

(3.3) $$\begin{aligned} \log \mu(r, f_1 f_2) &\geqslant \delta \left[\int_0^r \frac{n(x, f_1) + n(x, f_2)}{x} \, dx + \log |f_1(0)\, f_2(0)|\right] \\ &\geqslant \delta \left[\int_{r_0}^r \frac{(n(x, f_1) + n(x, f_2)}{x} \, dx + \log |f_1(0)\, f_2(0)|\right]. \end{aligned}$$

Since $\liminf_{r\to\infty} \dfrac{n(r, f_1) + n(r, f_2)}{r^\varrho} = \nu$, we have for any $\varepsilon > 0$ and $r > r_1$,

$n(r, f_1) + n(r, f_2) > (\nu - \varepsilon)\, r^{\varrho}$, therefore,

$$\log \mu_\delta(r, f_1 f_2) > \delta\left[(\nu - \varepsilon)\frac{r^{\varrho} - r_0^{\varrho}}{\varrho} + \log|f_1(0)\, f_2(0)|\right], \quad r > r_0 \geqslant r_1 + 1 .$$

Proceeding to limits, we get

$$\liminf_{r\to\infty} \frac{\log \mu_\delta(r, f_1 f_2)}{r^{\varrho}} \geqslant \frac{\nu\delta}{\varrho} .$$

(ii) Again, if $\liminf_{r\to\infty} \dfrac{n(r, f_1) + n(r, f_2)}{r \log r} > 1$, we have for any $\varepsilon > 0$ and $r > r_1$, $n(r, f_1) + n(r, f_2) > (1 - \varepsilon) r \log, r$, and therefore, from (3.3), we get

$$\log \mu_\delta(r, f_1 f_2) > \delta\left[(1 - \varepsilon)\int_{r_0}^{r} \log x\, dx + \log|f_1(0) f_2(0)|\right] \qquad (r_0 \geqslant r_1 + 1)$$

$$= \delta[(1 - \varepsilon)\{(r \log r - r) - (r_0 \log r_0 - r_0)\} + \log|f_1(0)\ f_2(0)|] .$$

Taking limits leads to $\liminf_{r\to\infty} \dfrac{\log \mu_\delta(r, f_1 f_2)}{r \log r} > \delta$.

4. - Theorem 3. *Let $f_1(z)$ and $f_2(z)$ be integral functions, both not polynomials, of orders ϱ_1 and ϱ_2, $n(r, f_1)$ and $n(r, f_2)$ denote the number of zeros of $f_1(z)$ and $f_2(z)$ respectively in $|z| \leqslant r$ and $f_1(0) \neq 0$, $f_2(0) \neq 0$. Further, if*

(i) $$\liminf_{r\to\infty} \frac{n(r, f_1) + n(r, f_2)}{r^{\varrho}} = \nu, \qquad \textit{where } \varrho = \max(\varrho_1, \varrho_2),$$

then

(4.1) $$\liminf_{r\to\infty} \frac{\log m_{\delta,k}(r, f_1 f_2)}{r^{\varrho}} \geqslant \frac{\nu\delta}{\varrho(\varrho + 1)} ;$$

if

(ii) $$\liminf_{r\to\infty} \frac{n(r, f_1) + n(r, f_2)}{r \log r} > 1 ,$$

then

(4.2) $$\liminf_{r\to\infty} \frac{\log m_{\delta,k}(r, f_1 f_2)}{r \log r} > \frac{\delta}{2} .$$

Proof. (i) We have from (3.3)

$$\log \mu_\delta(x,\ f_1 f_2) \geqslant \delta\left[\int_{x_1}^{x} \frac{n(t,\ f_1) + n(t,\ f_2)}{t}\, dt + \log |f_1(0)\, f_2(0)|\right],\ r \geqslant x. \tag{4.3}$$

Hence, if $\liminf\limits_{r\to\infty} \dfrac{n(r,\ f_1) + n(r,\ f_2)}{r^\varrho} = \nu$, we have for any $\varepsilon > 0$ and $r > r_1$, $n(r, f_1) + n(r, f_2) > (\nu - \varepsilon) r^\varrho$, therefore,

$$\log \mu_\delta(x,\ f_1 f_2) > \delta(\nu - \varepsilon)\frac{x^\varrho}{\varrho} + \delta\left\{\log |f_1(0)\, f_2(0)| - (\nu - \varepsilon)\frac{x_1^\varrho}{\varrho}\right\},\ x_1 \geqslant r_1 + 1 .$$

Using the above inequality, we get

$$\frac{1}{r}\int_0^r \log\{\mu_\delta(x,\ f_1 f_2)\, x^k\}\, dx = \frac{1}{r}\int_{r_0}^r \log\{\mu_\delta(x, f_1 f_2)\, x^k\}\, dx + \frac{1}{r}\int_0^{r_0} \log\{\mu_\delta(x, f_1 f_2)\, x^k\}\, dx >$$

$$> \frac{(\nu - \varepsilon)\delta}{\varrho(\varrho+1)}\left(r^\varrho - \frac{r_0^\varrho}{r}\right) + \delta\left\{\log |f_1(0)\ f_2(0)| - (\nu - \varepsilon)\frac{x_1^\varrho}{\varrho}\right\}\left(1 - \frac{r_0}{r}\right) +$$

$$+ k\left\{(\log\ r - 1) - \frac{r_0 \log r_0 - r_0}{r}\right\} + \frac{A}{r},$$

where $r_0 \geqslant x_1 + 1$ and A is independent of r.

Since $\log\left\{\dfrac{1}{r}\displaystyle\int_0^r \mu_\delta(x,\ f_1 f_2)\, x^k\, dx\right\} \geqslant \dfrac{1}{r}\displaystyle\int_0^r \log\{\mu_\delta(x,\ f_1 f_2)\, x^k\}\, dx$, therefore,

$$\log m_{\delta,k}\ (r,\ f_1 f_2) > \frac{(\nu - \varepsilon)\delta r^\varrho}{\varrho(\varrho + 1)} + O(1) .$$

Hence

$$\liminf_{r\to\infty} \frac{\log m_{\delta,k}(r,\ f_1 f_2)}{r^\varrho} \geqslant \frac{\nu\delta}{\varrho(\varrho + 1)} .$$

(ii) Again, if $\liminf\limits_{r\to\infty} \dfrac{n(r,\ f_1) + n(r,\ f_2)}{r \log r} > 1$, we have for any $\varepsilon > 0$ and

$r > r_1$, $n(r, f_1) + n(r, f_2) > (1-\varepsilon) r \log r$, and from (4.3), we get

$$\log \mu_\delta(x, f_1 f_2) > \delta[(1-\varepsilon)\int_{x_1}^{x} \log t \, dt + \log |f_1((0)\, f_2(0)|] =$$

$$= \delta(1-\varepsilon)(x \log x - x) + \delta\{\log |f_1(0)\, f_2(0)| - (1-\varepsilon)(x_1 \log x_1 - x_1)\},$$

$$(x_1 \geqslant r_1 + 1),$$

or

$$\log\{\mu_\delta(x, f_1 f_2)\, x^k\} > \delta(1-\varepsilon)(x \log x - x) +$$

$$+ \delta\{\log |f_1(0)\, f_2(0)| - (1-\varepsilon)(x_1 \log x_1 - x_1)\} + \log x^k.$$

Therefore,

$$\frac{1}{r}\int_0^r \log\{\mu_\delta(x, f_1 f_2)\, x^k\} \, dx = \frac{1}{r}\int_{r_0}^r \log\{\mu_\delta(x, f_1 f_2)\, x^k\} \, dx + \frac{A}{r} >$$

$$\delta(1-\varepsilon)\left\{\frac{1}{2}\left(r \log r - \frac{r_0^2 \log r_0}{r}\right) - \frac{1}{4}\left(r - \frac{r_0^2}{r}\right) - \frac{1}{2}\left(r - \frac{r_0^2}{r}\right)\right\} +$$

$$+ \delta\{\log|f_1(0) f_2(0)| - (1-\varepsilon)(x_1 \log x_1 - x_1)\}\left(1 - \frac{r_0}{r}\right) + k\left\{(\log r - 1) - \frac{r_0 \log r_0 - r_0}{r}\right\} + \frac{A}{r},$$

$$(r_0 \geqslant x_1 + 1).$$

Since $\log\left\{\frac{1}{r}\int_0^r \mu_\delta(x, f_1 f_2)\, x^k \, dx\right\} \geqslant \frac{1}{r}\int_0^r \log\{\mu_\delta(x, f_1 f_2)\, x^k\} \, dx$, therefore,

$$\log m_{\delta,k}(r, f_1 f_2) > \frac{\delta}{2}(1-\varepsilon)\left(r \log r - \frac{3}{2} r\right) + O(1).$$

Proceeding to limits, we get

$$\liminf_{r \to \infty} \frac{\log m_{\delta,k}(r, f_1 f_2)}{r \log r} > \frac{\delta}{2}.$$

5. Theorem 4. *For integral functions $f_1(z)$ and $f_2(z)$ of finite orders ϱ_1 and ϱ_2 respectively and for every positive value of ε,*

(5.1) $$\mu_\delta(r, f_1^{(1)} f_2^{(1)}) < A\, \mu_\delta(r, f_1 f_2)\, r^{(\varrho_1 + \varrho_2 - 2 + \varepsilon)\delta},$$

for large r, where A is independent of r.

Proof. We have

$$\mu_\delta(r,\ f_1^{(1)}f_2^{(1)}) = \frac{1}{2\pi}\int_0^{2\pi} |f_1^{(1)}(re^{i\theta})\ f_2^{(1)}(re^{i\theta})\ |^\delta\ d\theta\ =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f_1^{(1)}(re^{i\theta})}{f_1(re^{i\theta})}\right|^\delta \left|\frac{f_2^{(1)}(re^{i\theta})}{f_2(re^{i\theta})}\right|^\delta |f_1(re^{i\theta})\ f_2(re^{i\theta})\ |^\delta\ d\theta < A\,\mu_\delta(r, f_1f_2)\ r^{(\varrho_1+\varrho_2-2+\varepsilon)\delta}\ ,$$

on using the result [2, p. 363], $\left|\dfrac{f^{(1)}(re^{i\theta})}{f(re^{i\theta})}\right| \leqslant O(r^{\varrho-1+\varepsilon})$ for large r and $\varepsilon > 0$.

Corollary 1.

$$\limsup_{r\to\infty} \left\{\left[\log r^2\left(\frac{\mu_\delta\ (r,\ f_1^{(1)}f_2^{(1)})}{\mu_\delta\ (r,\ f_1f_2)}\right)^{1/\delta}\right]/\log r\right\} \leqslant \varrho_1 + \varrho_2\ .$$

Corollary 2 *For integral functions $f_1(z)$ and $f_2(z)$ of finite orders ϱ_1 and ϱ_2 respectively and for every positive value of ε*

$$\mu_\delta(r,\ f_1^{(n)}f_2^{(n)}) < A^n\ \mu_\delta(r,\ f_1f_2)\ r^{(\varrho_1+\varrho_2-2+\varepsilon)\delta n}.$$

Writing (5.1) for k-th derivatives of $f_1(z)$ and $f_2(z)$, we get

$$\frac{\mu_\delta\ (r,\ f_1^{(k)}f_2^{(k)})}{\mu_\delta\ (r,\ f_1^{(k-1)}f_2^{(k-1)})} < A_k\ r^{(\varrho_1+\varrho_2-2+\varepsilon)\delta}\ .$$

Giving k the values $k = 1, 2, \ldots, n$, multiplying and replacing $A_1, A_2, \ldots, A^n$ by A, where $A =$ (greater of $A_1, A_2, \ldots, A_n$), the result follows.

Corollary 3. *From Corollary 2 follows*

$$\limsup_{r\to\infty} \left\{\left[\ \log\ r^2\left(\frac{\mu_\delta\ (r,\ f_1^{(n)}f_2^{(n)})}{\mu_\delta(r,\ f_1f_2)}\right)^{1/\delta n}\right]/\log r\ \right\} \leqslant \varrho_1 + \varrho_2\ .$$

6. Theorem 5. *If $f_1(z)$ and $f(z)_2$ are integral functions of finite orders ϱ_1 and ϱ_2 respectively, then*

$$\limsup_{r\to\infty}\left\{\left[\log r^2\left(\frac{m_{\delta,\varepsilon}(r,\ f_1^{(1)}f_2^{(2)})}{m_{\delta,k}(r,\ f_1f_2)}\right)^{1/\delta}\right]/\log r\right\} \leqslant \varrho_1+\varrho_2 .$$

We shall first prove the following Lemma:

Lemma. *For integral functions $f_1(z)$ and $f_2(z)$ of finite orders ϱ_1 and ϱ_2 respectively and for every positive value of ε*

$$(6.1)\qquad m_{\delta,k}(r,\ f_1^{(1)}f_2^{(1)}) - \left(\frac{r_0}{r}\right)^{k+1} m_{\delta,k}(r_0,\ f_1^{(1)}f_2^{(1)}) < A\ m_{\delta,k}(r,\ f_1f_2)\ r^{(\varrho_1+\varrho_2-2+\varepsilon)\delta},$$

for larger r, where $\varrho_1+\varrho_2>2$ and A is independent of r.

Proof. We have

$$m_{\delta,k}(r, f_1^{(1)}f_2^{(1)}) - r_0^{k+1}\frac{m_{\delta,k}(r_0,\ f_1^{(1)}f_2^{(1)})}{r^{k+1}} =$$

$$= \frac{1}{\pi r^{k+1}}\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{f_1^{(1)}(xe^{i\theta})}{f_1(xe^{i\theta})}\right|^{\delta}\left|\frac{f_2^{(1)}(xe^{i\theta})}{f_2(xe^{i\theta})}\right|^{\delta}|f_1(xe^{i\theta})\ f_2(xe^{i\theta})|^{\delta}\ x^k\,\mathrm{d}x\,\mathrm{d}\theta <$$

$$< \frac{A}{\pi r^{k+1}}\int_{r_0}^{r}\int_0^{2\pi} x^{(\varrho_1+\varrho_2-2+\varepsilon)\delta}\,|f_1(xe^{i\theta})\ f_2(xe^{i\theta})|^{\delta}\ x^k\,\mathrm{d}x\,\mathrm{d}\theta \leqslant$$

$$\leqslant \frac{A}{\pi r^{k+1}}\ r^{(\varrho_1+\varrho_2-2+\varepsilon)\delta}\int_{r_0}^{r}\int_0^{2\pi}|\,f_1(xe^{i\theta})\ f_2(xe^{i\theta})\,|^{\delta}\ x^k\,\mathrm{d}x\,\mathrm{d}\theta \leqslant$$

$$\frac{A}{\pi r^{k+1}}\ r^{(\varrho_1+\varrho_2-2+\varepsilon)\delta}\int_0^{r}\int_0^{2\pi}|f_1(xe^{i\theta})\ f_2(xe^{i\theta})\,|^{\delta}\ x^k\,\mathrm{d}x\,\mathrm{d}\theta = A\ m_{\delta,k}(r,\ f_1f_2)\ r^{(\varrho_1+\varrho_2-2+\varepsilon)\delta},$$

on using the result [2, p. 363], $\dfrac{f^{(1)}(re^{i\theta})}{f(re^{i\theta})} < Ar^{\varrho-1+\varepsilon}$ for large r and $\varepsilon>0$.

Proof of Theorem 5. (6.1) can be put in the form

$$r^2 \left\{\frac{m_{\delta,k}(r,\ f_1^{(1)} f_2^{(1)})}{m_{\delta,k}(r,\ f_1 f_2)}\right\}^{1/\delta} \left\{1 - \frac{r_0^{k+1}\, m_{\delta,k}(r_0,\ f_1^{(1)} f_2^{(1)})}{r^{k+1}\, m_{\delta,k}(r,\ f_1^{(1)} f_2^{(1)})}\right\}^{1/\delta} < A^{1/\delta}\, r^{\varrho_1 + \varrho_2 + \varepsilon} .$$

Taking logarithm and then proceeding to limits as $r \to \infty$, the result follows.

References.

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