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On a Generalization of Bernstein Polynomials. (**)

§ 1. - Introduction.

A recent theorem of Korovkin [1] gives a necessary and sufficient condition that a sequence of nonnegative linear operators \( L_n \) defined in \( C[a, b] \) have the property that \( L_nf \to f \) uniformly for all \( f \in C[a, b] \). The condition is simply that \( L_nf \to f \) for the particular functions \( f(x) = 1, x, \) and \( x^2 \). The proof is an adaptation of Bernstein’s proof that the Bernstein polynomials

\[
(B_n f)(x) = \sum_{y=0}^{n} \binom{n}{y} x^y (1-x)^{n-y}
\]

converge uniformly to \( f \) on \([0, 1] \). Korovkin’s theorem explains why the operators \( B_n \) defined by (1) originate in the identity

\[
1 = [x + (1-x)]^n = \sum_{y=0}^{n} \binom{n}{y} x^y (1-x)^{n-y},
\]

and indeed why all generalizations of the Bernstein polynomials seem to be based on some identity such as this. See for example Kac [2], Meyer-König and Zeller [3], Szász [4], Lorentz [5].


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(1) An operator \( L \) is nonnegative if \( f(x) > 0 \) (for all \( x \)) implies \( (Lf)(x) > 0 \) (for all \( x \)).
The object of this Note is to draw attention to an interesting generalization of the binomial theorem due to Jensen [6], and to show then that a generalization of the Bernstein polynomials may be based upon it. Jensen's formula is

\[(x + y + n\beta)^n = \sum_{v=0}^{n} \binom{n}{v} x^{v} y^{n-v} [1 + (n-v)\beta]^{n-v}.\]

The proof of (3) starts with Lagrange's formula

\[\frac{\Phi(z)}{1 - \frac{z f(z)}{f(z)}} = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{d^r}{dz^r} [(f(z))^r] x (x + y\beta)^{r-1} [1 + (n-r)\beta]^{n-r}.\]

and proceeds by setting \(\Phi(z) = e^{az}\) and \(f(z) = e^{bx}\).

\[\text{§ 2. - The operators and their convergence.}\]

In analogy with the Bernstein polynomials we set \(y = 1 - x\) in (3) to obtain the following extension of (2)

\[1 = (1 + n\beta)^{-n} \sum_{r=0}^{n} \binom{n}{r} x (x + r\beta)^{r-1} [1 - x + (n-r)\beta]^{n-r}.\]

The desired generalization of the Bernstein polynomials is, then, in analogy with (1)

\[\text{(Pnf)}(x) : = (1 + n\beta)^{-n} \sum_{r=0}^{n} \binom{n}{r} x (x + r\beta)^{r-1} [1 - x + (n-r)\beta]^{n-r}.\]

It is clear that the Bernstein polynomials form a special case of (5) obtained by setting \(\beta = 0\). The problem we propose to solve is the following: for what values of \(\beta\) will the operators \(P_n\) have the property that \(P_nf \to f\) uniformly on \([0, 1]\) for all \(f \in C[0, 1]\) ?

**Theorem 1.** If \(0 \leq \beta = o(n^{-1})\) then \(P_nf \to f\) (uniformly) for all \(f \in C[0, 1]\).

The proof can be made to depend on the following lemma.
Lemma 1. The functions

\[ S(k, n, x, y) := \sum_{r=0}^{n} \binom{n}{r} (x + r\beta)^{r+k-1} (y + (n-r)\beta)^{n-r} \]

satisfy the reduction formula

\[ S(k, n, x, y) = x S(k-1, n, x, y) + n\beta S(k, n-1, x + \beta, y). \]

The proof of the Lemma is a straightforward calculation and is therefore omitted. By repeated use of the reduction formula, noting from (3) that

\[ x S(0, n, x, y) = (x + y + n\beta)^n, \]

we may show that

\[ S(1, n, x, y) = \sum_{r=0}^{n} \binom{n}{r} r! \beta^r (x + y + n\beta)^{n-r}. \]

Replacing \( r! \) in this last expression by \( \int_0^1 r e^{-t} \, dt \) and using the binomial theorem we obtain

\[ S(1, n, x, y) = \int_0^\infty e^{-t}(x + y + n\beta + t\beta)^n \, dt. \]  \hspace{1cm} (6)

In a similar manner we may reduce \( S(2, n, x, y) \) to the following

\[ S(2, n, x, y) = \sum_{r=0}^{n} (x + r\beta) \binom{n}{r} r! \beta^r S(1, n-r, x + r\beta, y) \]

and thence to

\[ S(2, n, x, y) = \int_0^\infty e^{-t} \, dt \int_0^\infty e^{-s} \, ds [x(x + y + n\beta + t\beta + s\beta)^n + \]

\[ + n\beta^s(x + y + n\beta + t\beta + s\beta)^{n-s-1}]. \]  \hspace{1cm} (7)

These formulas will be of use presently. In order to prove Theorem 1 it is sufficient, by Korovkin's result, to verify that the operator \( P_n \) is nonnegative and that \( P_n f \to f \) for \( f(x) = 1, x, \) and \( x^2. \) From the definition it is clear that \( P_n \) is nonnegative when \( \beta \geq 0. \) It is also clear from (4) that \( P_n 1 = 1. \)
Going on to \( f(t) = t \) we have

\[
(P_n t)(x) = (1 + n\beta)^{-n} \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{\nu}{n^\nu} \nu (x + \nu \beta)^{\nu - 1} [1 - x + (n - \nu)\beta]^n =
\]

\[
= (1 + n\beta)^{-n} x \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} (x + \beta + \nu \beta)^\nu [1 - x + (n - 1 - \nu)\beta]^{n-1-\nu} =
\]

\[
= (1 + n\beta)^{-n} \beta \, S(1, n - 1, x + \beta, 1 - x).
\]

Using (6) this last expression becomes

\[
x(1 + n\beta)^{-n} \int_{0}^{\infty} \frac{x}{1 + n\beta} e^{-t}(1 + n\beta + t\beta)^{n-1} \, dt = \frac{x}{1 + n\beta} \int_{0}^{\infty} \frac{t\beta}{1 + n\beta} e^{-t} \left(1 + \frac{t\beta}{1 + n\beta}\right)^{n-1} \, dt =: A_n x.
\]

To show that \( A_n \) tends to \( 1 \), we make the change of variable \( u = t\beta/(1 + n\beta) \) to get

\[
A_{n+1} = \frac{1}{\beta} \int_{0}^{\infty} e^{-t}(1 + u)^n \, du.
\]

Using the estimate

\[
(8) \quad e^{u}(1 - nu) \leq (1 + u)^n \leq e^{nu},
\]

we have

\[
\beta^{-1} \int_{0}^{\infty} e^{-t} e^{nu}(1 - nu^2) \, du \leq A_{n+1} \leq \beta^{-1} \int_{0}^{\infty} e^{-t} e^{nu} \, du.
\]

Since \( -t + nu = -u/\beta \), the upper bound on \( A_{n+1} \) is \( 1 \) while the lower bound is \( 1 - 2n\beta^2 \). Hence, if \( \beta = o(u^{-1}) \) then \( A_n \to 1 \).

Proceeding to the function \( f(t) = t^2 \), we have

\[
(P_n t^2)(x) = (1 + n\beta)^{-n} \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{\nu^2}{n^\nu} \nu (x + \nu \beta)^{\nu - 1} [1 - x + (n - \nu)\beta]^{n-\nu} =
\]

\[
= (1 + n\beta)^{-n} \sum_{\nu=0}^{n} \frac{n-1}{n} \frac{\nu(n-\nu-1)}{n(n-1)} + \frac{\nu}{n^2} \binom{n}{\nu} \nu (x + \nu \beta)^{\nu-1} [1 - x + (n - \nu)\beta]^{n-\nu} =
\]

\[
= \frac{n-1}{n} (1 + n\beta)^{-n} \beta \, S(2, n - 2, x + 2\beta, 1 - x) + \frac{1}{n} (P_n t)(x).
\]
From the earlier work, $\frac{1}{n} (P_n(z))(x) \to 0$. The other term can be written with the aid of (7) as

$$\frac{n-1}{n} (1 + n\beta)^{-n} x(x + 2\beta) \int_0^\infty \int_0^\infty e^{-t} dt \int e^{-s} (1 + n\beta + t\beta + s\beta)^n ds +$$

$$+ (n-1)(1 + n\beta)^{-n} \alpha^2 \int_0^\infty \int_0^\infty e^{-t} dt \int e^{-s} (1 + n\beta + t\beta + s\beta)^{n-1} ds.$$  

The second term of (9) is positive and may be bounded above using (8) by

$$\frac{x(n-1)\beta^2}{1 + n\beta} \int_0^\infty e^{-t} dt \int e^{-s} \exp \left[ \frac{t\beta + s\beta}{1 + n\beta} (n-1) \right] ds = \frac{x(n-1)\beta^2}{1 + n\beta} \left( \frac{1 + n\beta}{1 + \beta} \right)^4,$$

which tends uniformly to zero if $\beta = o(n^{-1})$. The first term of (9) can be confined by (8) to an interval

$$\frac{n-1}{n} (1 + n\beta)^{x(x + 2\beta)(1 - n\beta^2)} < x < \frac{n-1}{n} w(x + 2\beta)(1 + n\beta)^x.$$

Thus if $\beta = o(n^{-1})$ this term tends uniformly to $x^2$.

§ 3. - Other polynomial operators.

Another generalization of Bernstein polynomials can be obtained from another formula of Jensen [6],

$$(x + y)(x + y + n\beta)^{n-1} = \sum_{\nu=0}^{n} \binom{n}{\nu} x(x + \nu\beta)^{n-1} y[y + (n - \nu)\beta]^{n-\nu-1}.$$

The corresponding operators $Q_n$ are now defined by the equation

$$(Q_n f)(x) := (1 + n\beta)^{1-n} \sum_{\nu=0}^{n} \binom{n}{\nu} f^{(\nu)}(\frac{1}{n}) x(x + \nu\beta)^{n-1} (1 - x)(1 - x + (n - \nu)\beta)^{n-\nu-1}.$$
It is clear that $Q_n 1 = 1$. Taking $f(t) = t$ we find that

\[
(Q_n t)(x) = x(1 + n \beta)^{-n} [S(1, n - 1, x + \beta, 1 - x) - \beta(n - 1) S(1, n - 2, x + \beta, 1 - x + \beta)].
\]

From the integral representation of $S$, formula (6), we see that $(Q_n t)(x) \to x$ uniformly if $\beta = o(n^{-1})$. Taking $f(t) = t^\epsilon$ we find that

\[
(Q_n t^\epsilon)(x) = x(1 + n \beta)^{-n} [S(2, n - 2, x + 2\beta, 1 - x) - (n - 2)\beta S(2, n - 3, x + 2\beta, 1 - x + \beta)],
\]

and again from the earlier work, this tends uniformly to $x^\epsilon$ if $\beta = o(n^{-1})$.

§ 4. **Further properties of the operators.**

For the operators $P_n$ of equation (5) it is possible to establish a generalization of Voronowskaia's result about Bernstein polynomials.

**Theorem 2.** If $f$ is bounded in $[0, 1]$ and possesses a second derivative at a point $x$, and if $\beta n^2 \to c$ then

\[
n[(P_n f)(x) - f(x)] \to \frac{1}{2} f''(x)[x - x^2 + 2cx^3].
\]

The proof proceeds from the equation

\[
f\left(\frac{x}{n}\right) - f(x) = \left(\frac{x}{n} - x\right) f'(x) + \left(\frac{x}{n} - x\right)^2 \left[\frac{1}{2} f''(x) + o\left(\frac{x}{n} - x\right)\right],
\]

from which it follows that

\[
n[(P_n f)(x) - f(x)] = n f'(x) [(P_n t)(x) - x] + \frac{n}{2} f''(x) [(P_n t^\epsilon)(x) - 2x(P_n t)(x) + x^2] +
\]

\[+ n(1 + n \beta)^{-n} \sum_{r=0}^n \begin{pmatrix} n \\ r \end{pmatrix} \left(\frac{x}{n} - x\right)^r \theta\left(\frac{x}{n} - x\right) \binom{n}{r} \binom{r}{\varepsilon} [1 - x + (n - r)\beta]^{n-r}.
\]

From earlier estimates, we know that $n[(P_n t)(x) - x] \to 0$. If $\beta n^2 \to c$ we can
show from the earlier work that

\[ n(P_n^2)(x) - 2x(P_n f)(x) + x^2 \to x - x^2 + ax^2. \]

The last term goes to zero by an argument similar to that given in [5, p. 22].

A result of Kantorovitch on Bernstein polynomials can also be proved for the operators \( P_n \).

**Theorem 3.** If \( f(z) \) is analytic in the interior of an ellipse \( E \) with foci 0, 1 and if \( 0 \leq \beta = o(n^{-1}) \), then \( (P_n f)(z) \to f(z) \) uniformly in any closed set interior to \( E \).

The proof will be exactly the same as in [5, p. 90] after establishing the following Lemma.

**Lemma 2.** If \( f \) is a polynomial of degree \( \leq k \), then so is \( P_n f \), for all \( n \).

**Proof.** We proceed by induction on \( k \). If \( f \) is of degree \( \leq 0 \), the lemma is true because \( P_n 0 = 0 \) and \( P_n 1 = 1 \). Now assume the lemma for polynomials of degree \( \leq k - 1 \). Since \( P_n \) is a linear operator it will be enough if we show that \( P_n f \) is of degree \( \leq k \) for the particular function

\[ f(t) = t \left( t - \frac{1}{n} \right) \left( t - \frac{2}{n} \right) \cdots \left( t - \frac{k - 1}{n} \right). \]

Computing in a straightforward way we find that

\[ (P_n f)(x) = x \frac{n(n - 1) \cdots (n - k + 1)}{n^k} (1 + n\beta)^{-n} S(k, n - k, x + k\beta, 1 - x). \]

The proof will be complete if we can show that \( S(k, n, x + a, 1 - x) \) is a polynomial of degree \( \leq k - 1 \) for all \( n \). That this is the case may be proved by induction on \( k \). For \( k = 1 \), equation (6) shows at once that \( S(1, n, x + a, 1 - x) \) is a constant. If our assertion is true for \( S(k, n, x + a, 1 - x) \) then we apply the reduction formula of Lemma 1 to \( S(k + 1, n, x + a, 1 - x) \) eventually obtaining a sum of terms of the form

\[ (A_x + B_x) S(k, n - \nu, x + a + \nu\beta, 1 - x) \]

each of which, by the induction hypothesis, is a polynomial of degree \( \leq k \).
References.


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