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On the Zeros of Entire Functions. (**)

1. – Let $n(r, f)$ denote the number of zeros of an entire function $f(z)$ for $|z| \leqslant r$ and σ the exponent of convergence of the zeros of $f(z)$. It is known ([1], p. 15) that if $f(z)$ has at least one zero in $|z| \leqslant r$, then

$$\limsup_{r\to\infty} \frac{\log n(r, f)}{\log r} = \sigma . \tag{1.1}$$

Further, if $M(r, f) = \max_{|z|=r} | f(z) |$ and $f(z)$ has at least one zero in $|z| \leqslant r$, $f(z) \not\equiv 0$, then ([1], p. 17):

$$\liminf_{r\to\infty} \frac{n(r, f)}{\log M(r, f)} \leqslant \liminf_{r\to\infty} \frac{\log n(r, f)}{\log r} = \delta \leqslant \sigma . \tag{1.2}$$

In analogy with the lower order of entire functions, we shall call δ to be the lower exponent of convergence of the zeros of $f(z)$. In this paper, we derive relations between the exponents of convergence of two or more entire functions and also prove a theorem concerning the distribution of zeros of two entire functions.

2. – Theorem 1. *Let $n(r, f_1)$, $n(r, f_2)$, $n(r, f)$ denote respectively the number of zeros of the entire functions $f_1(z)$, $f_2(z)$, $f(z)$ each having at least one zero in $|z| \leqslant r$. Further let δ_1, δ_2, δ denote the lower exponents of convergence and σ_1, σ_2, σ the exponents of convergence of the zeros of $f_1(z)$, $f_2(z)$, $f(z)$ respectively. Then, if*

$$\log n(r, f) \sim \log \{ n(r, f_1)\, n(r, f_2) \} , \tag{2.1}$$

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for $r \to \infty$, *we have*

$$\delta_1 + \delta_2 \leqslant \delta \leqslant \sigma \leqslant \sigma_1 + \sigma_2 . \tag{2.2}$$

Corollary 1. *If* $n(r, f_1)$, $n(r, f_2)$, ..., $n(r, f_m)$, $n(r, f)$ *denote respectively the number of zeros of the entire functions* $f_1(z)$, $f_2(z)$, ..., $f_m(z)$, $f(z)$, *each having at least one zero in* $|z| \leqslant r$, *and* δ_1, δ_2, ..., δ_m, δ *denote the lower exponents of convergence and* σ_1, σ_2, ..., σ_m, σ *the exponents of convergence of the zeros of* $f_1(z)$, $f_2(z)$, ..., $f_m(z)$, $f(z)$ *respectively; then, if*

$$\log n(r, f) \sim \log\{ n(r, f_1)\, n(r, f_2) \ldots n(r, f_m) \}, \tag{2.3}$$

we have

$$\delta_1 + \delta_2 + \ldots + \delta_m \leqslant \delta \leqslant \sigma \leqslant \sigma_1 + \sigma_2 + \ldots + \sigma_m . \tag{2.4}$$

Corollary 2. *Let* $f_1(z)$, ..., $f_m(z)$, $f(z)$ *be entire functions of regular growth, having non-integral orders* ϱ_1, ϱ_2, ..., ϱ_m, ϱ *respectively and* (2.3) *holds; then*

$$\varrho \leqslant \varrho_1 + \varrho_2 + \ldots + \varrho_m . \tag{2.5}$$

Proof. Using (1.1) for $f_1(z)$, we have, for any $\varepsilon > 0$ and $r > r_0 = r_0(f_1)$,

$$\log n(r, f_1) < (\sigma_1 + \varepsilon/2) \log r; \tag{2.6}$$

similarly, for the function $f_2(z)$, for any $\varepsilon > 0$ and $r > r_0' = r_0'(f_2)$,

$$\log n(r, f_2) < (\sigma_2 + \varepsilon/2) \log r . \tag{2.7}$$

Hence, for sufficiently large r, on adding the inequalities (2.6) and (2.7), we get

$$\log\{ n(r, f_1)\, n(r, f_2) \} < (\sigma_1 + \sigma_2 + \varepsilon) \log r ,$$

so that:

$$\limsup_{r\to\infty} \frac{\log \{ n(r, f_1)\, n(r, f_2) \}}{\log r} \leqslant \sigma_1 + \sigma_2 . \tag{2.8}$$

Therefore, if

$$\log\{n(r,\ f_1)\,n(r,\ f_2)\}\sim\log n(r,\ f)\ ,$$

we get, from (2.8),

$$\limsup_{r\to\infty}\frac{\log n(r,\ f)}{\log r}\leqslant\sigma_1+\sigma_2\ ,$$

or

(2.9) $$\sigma\leqslant\sigma_1+\sigma_2\,.$$

Next, using (1.2) for $f_1(z)$, we have, for any $\varepsilon>0$ and $r>r_0''=r_0''(f_1)$,

(2.10) $$(\delta_1-\varepsilon/2)\log r<\log n(r,\ f_1),$$

and similarly for $f_2(z)$, we have, for any $\varepsilon>0$ and $r>r_0'''=r_0'''(f_2)$,

(2.11) $$(\delta_2-\varepsilon/2)\log r<\log n(r,\ f_2)\,.$$

Hence, for sufficiently large r, we have, from (2.10) and (2.11) ,

$$\log\{n(r,\ f_1)\,n(r,\ f_2)\}>(\delta_1+\delta_2-\varepsilon)\log r\,.$$

Therefore

(2.12) $$\liminf_{r\to\infty}\frac{\log\{n(r,\ f_1)\,n(r,\ f_2)\}}{\log r}\geqslant\delta_1+\delta_2$$

and since

$$\log\{n(r,\ f_1)\,n(r,\ f_2)\}\sim\log n(r,\ f)\,.$$

(2.12) yields:

$$\liminf_{r\to\infty}\frac{\log n(r,\ f)}{\log r}\geqslant\delta_1+\delta_2\ ,$$

or

(2.13) $$\delta\geqslant\delta_1+\delta_2\,.$$

Combining (2.9) and (2.13), we get

$$\delta_1 + \delta_2 \leqslant \delta \leqslant \sigma \leqslant \sigma_1 + \sigma_2 ,$$

since $\delta \leqslant \sigma$ by (1.2). Hence the Theorem 1 is proved.

Corollary 1 follows as an immediate generalization of Theorem 1, while Corollary 2 follows as a direct consequence of Corollary 1 and the fact that for entire functions of regular growth and non-integral orders, the exponents of convergence of their zeros are equal to their orders.

3. - Theorem 2. *Let* $f(z) = \sum_0^\infty a_n z^n$, $\varphi(z) = \sum_0^\infty b_n z^n$ *be entire functions other than constants, being real for real* z *and* $\varphi(z)$ *having real zeros only. Further let* $\sum_0^\infty c_n z^n$ *represent another function* $\psi(z)$ *such that*

$$c_n = (a_1 b_n - b_1 a_n) + 2(a_2 b_{n-1} - b_2 a_{n-1}) + \ldots + (n+1)(a_{n+1} b_0 - b_{n+1} a_0) , \tag{3.1}$$

where the constants a_{n-p}, b_{n-p} *vanish for* $p > n$; *then, if* $\psi(z)$ *has the same sign for every real* z, *the zeros of* $f(z)$ *are also real and are separated by the zeros of* $\varphi(z)$.

Proof. We have $f(z) = \sum_0^\infty a_n z^n$ which is real for real z.

Therefore, its derivative $f'(z) = \sum_0^\infty (n+1) a_{n+1} z^n$ is also real for real z. Hence

$$f'(z)\,\varphi(z) = \Big\{ \sum_{n=0}^\infty (n+1)\, a_{n+1} z^n \Big\} \Big\{ \sum_{n=0}^\infty b_n z^n \Big\} . \tag{3.2}$$

As $\varphi(z)$, $f(z)$ [and therefore $f'(z)$] are entire functions, their TAYLOR series converge uniformly and absolutely for every finite $|z| = r$ and so multiplying the two series in (3.2) term by term, we get

$$f'(z)\,\varphi(z) = \sum_0^\infty [a_1 b_n + 2a_2 b_{n-1} + \ldots + (n+1)\, a_{n+1} b_0]\, z^n. \tag{3.3}$$

The series in (3.3) represents an entire function since the product of two entire functions is also an entire function. Similarly, we have

$$\varphi'(z)\, f(z) = \sum_0^\infty [b_1 a_n + 2b_2 a_{n-1} + \ldots + (n+1) b_{n+1} a_0]\, z^n, \tag{3.4}$$

which again represents an entire function.

Now

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{f(z)}{\varphi(z)} = \frac{f'(z)\,\varphi(z) - \varphi'(z)\,f(z)}{\{\,\varphi(z)\,\}^2} = \frac{\psi(z)}{\{\,\varphi(z)\,\}^2} = P(z),$$

in view of (3.1), (3.3) and (3.4). But $P(z)$ is continuous between the zeros of $\varphi(z)$ which are all real. Also, since $\psi(z)$ is of the same sign for every real z by hypothesis, it follows that $f(z)/\varphi(z)$ must vanish once and only once between the zeros of $\varphi(z)$, i.e., the zeros of $f(z)$ are also real and are separated by the zeros of $\varphi(z)$.

In conclusion, we give an application of Theorem 2 by taking the familiar example of the functions $\sin z$ and $\cos z$. Thus, let

$$f(z) = \sum_0^\infty a_n z^n = \sin z, \qquad \varphi(z) = \sum_0^\infty b_n z^n = \cos z,$$

then (for $m = 0, 1, 2, \ldots$)

$$a_{2m} = 0, \qquad a_{2m+1} = \frac{(-1)^m}{(2m+1)!}, \qquad b_{2m} = \frac{(-1)^m}{(2m)!}, \qquad b_{2m+1} = 0.$$

We find that

$$\psi(z) = \sum_0^\infty c_n z^n = 1,$$

where c_n is defined as in (3.1). Thus $\psi(z) > 0$ for every real z and so by Theorem 2 we conclude that zeros of $\sin z$ are separated by the zeros of $\cos z$.

References.

[1] R. P. Boas Jr., *Entire functions*, Academic Press Inc., New York 1954.

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