Riv. Mat. Univ. Parma (2) 4 (1963), 149-158

T. Pati and Z. U. Ahmad (*)

A New Proof of a Theorem on the Absolute Summability Factors of Fourier Series. (**)

1.1. – Let $\sum a_n$ be a given infinite series. Let s_n^k and t_n^k denote the n-th Cesàro means of order k $(k > -1)$ of the sequences $\{s_n\}$ and $\{n\, a_n\}$ respectively, where $s_n = \sum_{\nu=0}^{n} a_\nu$. The series $\sum a_n$ is said to be absolutely summable (C, k), or summable $|C, k|$, if the sequence $\{s_n^k\}$ is of bounded variation, that is to say,

$$\sum |s_n^k - s_{n-1}^k| < \infty \text{ (1)}.$$

We shall require the following identities for $k > -1$:

(1.1.1) $$t_n^k = n\,(s_n^k - s_{n-1}^k) \text{ (2)},$$

(1.1.2) $$t_n^k = (1/A_n^k) \sum_{\nu=1}^{n} A_{n-\nu}^{k-1}\, \nu\, a_\nu ,$$

where A_n^k is given by the identity

(1.1.3) $$\sum_{n=0}^{\infty} A_n^k x^n = (1-x)^{-1-k} \qquad (|x| < 1);$$

(*) Address: Dept. of Mathematics, University of Jabalpur, Jabalpur, India.
(**) Ricevuto il 27-VI-1963.

(1) Cf. [3], [5]. Now onwards, we use $\sum$ to denote $\sum_{n=1}^{\infty}$.

(2) Cf. [5], [6].

and by definition (3)

$$(1.1.4) \qquad A_n^k = \begin{cases} (-1)^n \binom{-1-k}{n} = \binom{n+k}{n} & \text{for } n \geqslant 0 \\ 0 & \text{for } n < 0, \end{cases}$$

$$(1.1.5) \qquad A_n^{-1} = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0, \end{cases}$$

$$(1.1.6) \qquad A_n^k = \Gamma(n+k+1)/\{\Gamma(n+1)\,\Gamma(k+1)\} \sim n^k/\Gamma(k+1)$$

$$(k \neq -1,\ -2,\ \ldots).$$

We write

$$\Delta u_n = \Delta^1 u_n = u_n - u_{n+1},$$

and

$$\Delta^\sigma u_n = \sum_{\nu=n}^{\infty} A_{\nu-n}^{-\sigma-1}\, u_\nu\,,$$

provided the series on the right converges. In particular, when σ is a positive integer,

$$\Delta^\sigma u_n = \sum_{j=0}^{\sigma} (-1)^j \binom{\sigma}{j} u_{n+j}\,.$$

If σ and ϱ are positive integers, we also have the formulae

$$\Delta^\sigma \Delta^\varrho u_n = \Delta^{\sigma+\varrho} u_n;$$

and

$$(1.1.7) \qquad \Delta^\sigma(\delta_n u_n) = \sum_{\varrho=0}^{\sigma} \binom{\sigma}{\varrho} \Delta^\varrho \delta_n\, \Delta^{\sigma-\varrho} u_{\varrho+n}\,.$$

By repeated partial summation, we see that, for $h = 0, 1, 2, \ldots,$

$$\sum_{\mu=0}^{\nu} A_{n-\mu}^{\sigma-1}\, u_\mu\, a_\mu = \sum_{\mu=0}^{\nu} \Delta^{h+1}(A_{n-\mu}^{\sigma-1}\, u_\mu)\, C_\mu^h + \sum_{j=0}^{h} \Delta^j(A_{n-\nu-1}^{\sigma-1}\, u_{\nu+1})\, C_\nu^j\,,$$

(3) Cf. [4].

whence, by putting $\nu = n$,

$$\sum_{\mu=0}^{n} A_{n-\mu}^{\sigma-1} u_\mu a_\mu = \sum_{\mu=0}^{n} \Delta^{h+1}(A_{n-\mu}^{\sigma-1} u_\mu) C_\mu^h , \tag{1.1.8}$$

where C_n^h denotes the n-th CESÀRO-sum of $\sum_{\nu=0}^{\infty} a_\nu$ of order h.

A sequence $\{\lambda_n\}$ is said to be *convex* (4), if $\Delta^2 \lambda_n \geqslant 0$. It is said to be *hyper-convex* of order h (5), if $\Delta^{h+2}\lambda_n \geqslant 0$ $(h = 0, 1, 2, \ldots)$. By definition, hyper-convexity of order zero is the same as convexity.

1.2. – Let $f(t)$ be a periodic function with period 2π, and integrable (L), that is, integrable in the sense of LEBESGUE, over $(-\pi, \pi)$. Without any loss of generality, we may assume that the constant term in the FOURIER series of $f(t)$ is zero, that is,

$$\int_{-\pi}^{\pi} f(t)\, \mathrm{d}t = 0 , \tag{1.2.1}$$

and

$$f(t) \sim \sum (a_n \cos nt + b_n \sin nt) = \sum c_n(t). \tag{1.2.2}$$

We use the following notations:

$$\varphi(t) = \varphi_x(t) = \frac{1}{2}\{ f(x+t) + f(x-t) - 2f(x) \} ,$$

$$\begin{cases} \Phi_\alpha(t) = \{1/\Gamma(\alpha)\} \int_0^t (t-u)^{\alpha-1} \varphi(u)\, \mathrm{d}u & (\alpha > 0) \\ \Phi_0(t) = \varphi(t) & \\ \varphi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\, \Phi_\alpha(t) & (\alpha \geqslant 0), \end{cases} \tag{1.2.3}$$

$$K_n^\alpha(t) = (1/A_n^\alpha) \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \lambda_\nu\, \nu \cos \nu t, \tag{1.2.4}$$

$$\{F(t)\}_\varrho = (\partial/\partial t)^\varrho F(t) . \tag{1.2.5}$$

(4) Cf. [10], n. 3.7, p. 58.
(5) Cf. [9].

1.3. – *Introduction.* In a recent paper [6] on the absolute CESÀRO summability factors of FOURIER series, PATI and SINHA obtained a general result which covers a previous result of PATI [7]. They proved:

Theorem. *Let h be an integer $\geqslant 0$, and let $\{\lambda_n\}$ be a monotonic non-increasing sequence when $h = 0$, and a hyper-convex sequence of order $h-1$ when $h \geqslant 1$, such that*

$$(1.3.1) \qquad \begin{cases} \text{(i)} & \sum n^{-1} \lambda_n < \infty \\ \text{(ii)} & \sum n^h \Delta^{h+1}\lambda_n < \infty . \end{cases}$$

Then, if

$$(1.3.2) \qquad \int_0^t |\varphi_h(u)|\, du = O(t),$$

as $t \to 0$, $\sum \lambda_n c_n(x)$ is summable $|C, h+1+\delta|$ for every $\delta > 0$.

The object of the present paper is to give an alternative proof of this theorem, which is superior to the original proof.

2.1. – We require the following lemmas.

Lemma 1. *Let C_n^k denote the n-th Cesàro-sum of order k $(k \geqslant 0)$ corresponding to the series $\sum (\sin nt)_{h+1}$ $(h \geqslant 0)$. Then*

(i) $C_n^k = O(n^{k+h+2})$ *for* $0 < t \leqslant n^{-1}$,

(ii) $C_n^k = O(n^{h+1}t^{-k-1}) + O(n^k t^{-h-2})$ *for* $n^{-1} < t \leqslant \pi$,

(iii) $C_n^k = O(n^{h+1}\, t^{-k-1})$ *for* $k < h+1$, *and* $n^{-1} < t \leqslant \pi$.

(i) and (ii) of the lemma are known [8]. The proof of (iii) is easy.

Lemma 2 [9]. *If $0 < k \leqslant 1$ and $1 \leqslant \nu < n$, then*

$$\Big| \sum_{\mu=1}^{\nu} A_{n-\mu}^{k-1} c_\mu \Big| \leqslant \max_{1 \leq m \leq \nu} \Big| \sum_{\mu=1}^{m} A_{m-\mu}^{k-1} c_\mu \Big| .$$

---

(6) Cf. [9].
(7) Cf. [8].
(8) Cf. [7].
(9) Cf. [1], [2].

Lemma 3 (10). (I) *If h is an integer $\geqslant 1$ and the sequence $\{\lambda_n\}$ is hyper-convex of order $h-1$, such that*

$$\text{(i)} \quad \sum n^{-1}\lambda_n < \infty, \qquad \text{(ii)} \quad \sum n^h \Delta^{h+1}\lambda_n < \infty,$$

then $\{\lambda_n\}$ is hyper-convex of order $r-1$ ($r = 1, 2, \ldots, h-1$), and therefore $\{\lambda_n\}$ is a monotonic non-increasing positive sequence tending to zero.

(II) *If $\{\lambda_n\}$ is a monotonic non-increasing sequence such that $\sum n^{-1}\lambda_n < \infty$, then $\{\lambda_n\}$ is a positive sequence tending to zero.*

Lemma 4 (11). *If $\{\lambda_n\}$ is hyper-convex of order $h-1$ when $h \geqslant 1$, or monotonic non-increasing when $h = 0$, such that $\sum n^{-1}\lambda_n < \infty$, then*

$$\sum n^h \Delta^{h+1}\lambda_n < \infty$$

implies that

$$\sum n^h (\log \overline{n+1})\, \Delta^{h+1}\lambda_n < \infty .$$

Lemma 5 (12). *If $\alpha > h+1$, and (1.3.2) holds, then*

$$\int_{n^{-1}}^{\pi} t^{h-\alpha} \,|\, \varphi_h(t) \,|\, \mathrm{d}t = O(n^{\alpha-h-1}),$$

as $n \to \infty$.

Lemma 6 (13). *For $\alpha > r \geqslant 0$,*

$$\sum_{n \geq \nu} (n+1-\nu)^{\alpha-r-1}\, n^{-1-\alpha} = O(\nu^{-r-1}) .$$

2.2. - Proof of the Theorem. By virtue of the identity (1.1.1) and the consistency theorem for absolute CESÀRO summability, we need only to prove that, for $h+1 < \alpha < h+2$,

$$\sum n^{-1} \,|\, \zeta_n^{\alpha} \,| < \infty ,$$

(10) Cf. [9], Lemma 3.
(11) Cf. [9], Lemma 5.
(12) Cf. [9], Lemma 4.
(13) This is essentially the same as Lemma 8 of [9].

where

$$\zeta_n^\alpha = (2/\pi) \int_0^\pi \varphi(t)\, K_n^\alpha(t)\, dt\,.$$

Integrating by parts h times, and proceeding as in [**9**], we have

$$\zeta_n^\alpha = (2/\pi)\, \mathfrak{F} + (2/\pi) \left\{ (-1)^h / \Gamma(h+1) \right\} \int_0^\pi \varphi_h(t)\, \overline{K}_n^\alpha(t)\, dt\,,$$

where

$$\mathfrak{F} = \Big[\sum_{\varrho=1}^{h} (-1)^{\varrho-1}\, \Phi_\varrho(t)\, (K_n^\alpha(t))_{\varrho-1} \Big]_0^\pi\,, \qquad \overline{K}_n^\alpha(t) = t^h\, (K_n^\alpha(t))_h\,.$$

Hence, it suffices to show that

$$\text{(2.2.1)} \qquad \sum n^{-1} |\mathfrak{F}| < \infty\,,$$

and

$$\text{(2.2.2)} \qquad \sum n^{-1} \Big| \int_0^\pi \varphi_h(t)\, \overline{K}_n^\alpha(t)\, dt \Big| < \infty\,.$$

Proof of (2.2.1).
The proof is the same as the proof of (3.1) in [**9**].

Proof of (2.2.2).
We have

$$\overline{K}_n^\alpha(t) = (t^h / A_n^\alpha) \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}\, \lambda_\nu\, (\sin \nu t)_{h+1}\,,$$

or, by putting $\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}\, \lambda_\nu\, (\sin \nu t)_{h+1} = M$,

$$\overline{K}_n^\alpha(t) = (t^h / A_n^\alpha)\; M\,.$$

Now, applying the process of repeated partial summation, by (1.1.8) and (1.1.7), we obtain, in the notation of Lemma 1,

$$M = \sum_{\nu=1}^{n} \Delta^{h+1}(A_{n-\nu}^{\alpha-1}\lambda_\nu)\, C_\nu^h = \sum_{r=0}^{h+1} \binom{h+1}{r} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1}\, \Delta^{h+1-r}\lambda_{\nu+r}\, C_\nu^h =$$

$$= \sum_{r=0}^{h} \binom{h+1}{r} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1}\, \Delta^{h+1-r}\lambda_{\nu+r}\, C_\nu^h + \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-h-2}\, \lambda_{\nu+h+1}\, C_\nu^h\,,$$

or, by putting

$$\sum_{r=0}^{h}\binom{h+1}{r}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1}\,\Delta^{h+1-r}\lambda_{\nu+r}\,C_\nu^h = M_1 \quad\text{and}\quad \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-h-2}\,\lambda_{\nu+h+1}\,C_\nu^h = M_2,$$

we obtain

$$M = M_1 + M_2 .$$

Hence, we have to prove that

$$\sum n^{-1}\left|\int_0^\pi \varphi_h(t)\,(t^h/A_n^\alpha)\,(M_1+M_2)\,dt\right| < \infty$$

and it suffices to show that

$$\sum n^{-1-\alpha}\int_0^\pi |\varphi_h(t)|\,t^h\,|M_1|\,dt < \infty, \tag{2.2.3}$$

and

$$\sum n^{-1-\alpha}\int_0^\pi |\varphi_h(t)|\,t^h\,|M_2|\,dt < \infty . \tag{2.2.4}$$

Proof of (2.2.3).

It suffices to show that, for $r = 0, 1, 2, \dots, h$,

$$M_{1,1} = \sum n^{-1-\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1}\,\Delta^{h+1-r}\lambda_{\nu+r}\int_0^{\nu^{-1}} t^h\,|\varphi_h(t)|\,|C_\nu^h|\,dt < \infty,$$

and

$$M_{1,2} = \sum n^{-1-\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1}\,\Delta^{h+1-r}\lambda_{\nu+r}\int_{\nu^{-1}}^{\pi} t^h\,|\varphi_h(t)|\,|C_\nu^h|\,dt < \infty .$$

Now, for $r = 0, 1, 2, \dots, h$, we have

$$M_{1,1} \leqslant K\sum n^{-1-\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1}\,\Delta^{h+1-r}\lambda_{\nu+r}\,\nu^{2h+2}\int_0^{\nu^{-1}} t^h\,|\varphi_h(t)|\,dt \quad (^{14})$$

by Lemma 1 (i); then

$$\begin{aligned} M_{1,1} &\leqslant K\sum n^{-1-\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1}\Delta^{h+1-r}\,\lambda_{\nu+r}\,\nu^{2h+2}\nu^{-h-1} \\ &\leqslant K\sum n^{-1-\alpha}\sum_{\nu=1}^{n}\nu^{h+1}\,(n+1-\nu)^{\alpha-r-1}\,\Delta^{h+1-r}\,\lambda_{\nu+r} \\ &= K\sum_{\nu=1}^{\infty}\nu^{h+1}\,\Delta^{h+1-r}\,\lambda_{\nu+r}\sum_{n=\nu}^{\infty}(n+1-\nu)^{\alpha-r-1}n^{-1-\alpha} \\ &\leqslant K\sum_{\nu=1}^{\infty}\nu^{h-r}\Delta^{h-r+1}\lambda_{\nu+r} \leqslant K < \infty , \end{aligned}$$

(14) Throughout the paper K denotes an absolute positive constant, not necessarily the same at each occurrence.

by hypothesis and Lemmas 6 and 3;

$$M_{1,2} \leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1} \Delta^{h+1-r}\lambda_{\nu+r} \nu^{h+1} \int_{\nu^{-1}}^{\pi} t^{-1} |\varphi_h(t)| \, dt$$

by Lemma 1 (iii); then

$$M_{1,2} \leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-r-1} \Delta^{h+1-r}\lambda_{\nu+r} \nu^{h+1} \log \nu$$

$$\leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} \nu^{h+1} (\log \overline{\nu+1}) (n+1-\nu)^{\alpha-r-1} \Delta^{h+1-r}\lambda_{\nu+r}$$

$$= K \sum_{\nu=1}^{\infty} \nu^{h+1} (\log \overline{\nu+1}) \Delta^{h+1-r}\lambda_{\nu+r} \sum_{n=\nu}^{\infty} (n+1-\nu)^{\alpha-r-1} n^{-1-\alpha}$$

$$\leqslant K \sum_{\nu=1}^{\infty} \nu^{h-r} (\log \overline{\nu+1}) \Delta^{h-r+1}\lambda_{\nu+r} \leqslant K < \infty,$$

by hypothesis and Lemmas 6, 3 and 4. This completes the proof of (2.2.3).

Proof of (2.2.4).

By ABEL's transformation, we have

$$M_2 = \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-h-2} \lambda_{\nu+h+1} C_\nu^h = \sum_{\nu=1}^{n} \Delta\lambda_{\nu+h+1} \sum_{\mu=1}^{\nu} A_{n-\mu}^{\alpha-h-2} C_\mu^h + \lambda_{n+h+2} \sum_{\mu=1}^{n} A_{n-\mu}^{\alpha-h-2} C_\mu^h .$$

Now, using Lemma 2, we obtain

$$M_2 = O(\sum_{\nu=1}^{n} \Delta\lambda_{\nu+h+1} \max_{1 \leq m \leq \nu} |C_m^{\alpha-1}|) + O(\lambda_{n+h+2} |C_n^{\alpha-1}|) .$$

Hence, it is sufficient to show that

$$M_{2,1} = \sum n^{-1-\alpha} \sum_{\nu=1}^{n} \Delta\lambda_{\nu+h+1} \int_0^{\nu^{-1}} t^h |\varphi_h(t)| \max_{1 \leq m \leq \nu} |C_m^{\alpha-1}| \, dt < \infty,$$

$$M_{2,2} = \sum n^{-1-\alpha} \sum_{\nu=1}^{n} \Delta\lambda_{\nu+h+1} \int_{\nu^{-1}}^{\pi} t^h |\varphi_h(t)| \max_{1 \leq m \leq \nu} |C_m^{\alpha-1}| \, dt < \infty,$$

$$M_{2,3} = \sum n^{-1-\alpha} \lambda_{n+h+2} \int_0^{n^{-1}} t^h |\varphi_h(t)| \, |C_n^{\alpha-1}| \, dt < \infty,$$

$$M_{2,4} = \sum n^{-1-\alpha} \lambda_{n+h+2} \int_{n^{-1}}^{\pi} t^h |\varphi_h(t)| \, |C_n^{\alpha-1}| \, dt < \infty .$$

We proceed to prove these. We have

$$M_{2,1} \leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} \Delta\lambda_{\nu+h+1}\, \nu^{\alpha+h+1} \int_0^{\nu^{-1}} t^h \,|\, \varphi_h(t) \,|\, dt \quad \text{[by Lemma 1 (i)]}$$

$$\leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} \Delta\lambda_{\nu+h+1}\, \nu^{\alpha+h+1}\, \nu^{-h-1}$$

$$= K \sum_{\nu=1}^{\infty} \nu^{\alpha}\, \Delta\lambda_{\nu+h+1} \sum_{n=\nu}^{\infty} n^{-1-\alpha} \leqslant K < \infty,$$

by hypothesis;

$$M_{2,2} \leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} \Delta\lambda_{\nu+h+1}\, \nu^{h+1} \int_{\nu^{-1}}^{\pi} t^{h-\alpha} \,|\, \varphi_h(t) \,|\, dt \quad \text{[by Lemma 1 (iii)]}$$

$$\leqslant K \sum n^{-1-\alpha} \sum_{\nu=1}^{n} \Delta\lambda_{\nu+h+1}\, \nu^{h+1}\, \nu^{\alpha-h-1} \quad \text{[by Lemma 5]}$$

$$= K \sum_{\nu=1}^{\infty} \nu^{\alpha}\, \Delta\lambda_{\nu+h+1} \sum_{n=\nu}^{\infty} n^{-1-\alpha} \leqslant K < \infty,$$

by hypothesis;

$$M_{2,3} \leqslant K \sum n^{-1-\alpha}\, \lambda_{n+h+2}\, n^{\alpha+h+1} \int_0^{n^{-1}} t^h \,|\, \varphi_h(t) \,|\, dt \quad \text{[by Lemma 1 (i)]}$$

$$\leqslant K \sum n^{-1-\alpha}\, \lambda_{n+h+2}\, n^{\alpha+h+1}\, n^{-h-1} = K \sum n^{-1}\, \lambda_{n+h+2} \leqslant K < \infty,$$

by hypothesis;

$$M_{2,4} \leqslant K \sum n^{-1-\alpha}\, \lambda_{n+h+2}\, n^{h+1} \int_{n^{-1}}^{\pi} t^{h-\alpha} \,|\, \varphi_h(t) \,|\, dt \quad \text{[by Lemma 1 (iii)]}$$

$$\leqslant K \sum n^{-1-\alpha}\, \lambda_{n+h+2}\, n^{h+1}\, n^{\alpha-h-1} \quad \text{[by Lemma 5]}$$

$$= K \sum n^{-1}\, \lambda_{n+h+2} \leqslant K < \infty,$$

by hypothesis.

This completes the proof of (2.2.4). The proof of the Theorem is thus completed.

References.

[1] L. S. BOSANQUET, *A mean-value theorem*, J. London Math. Soc. **16** (1941), 146-148.

[2] L. S. BOSANQUET, *Note on convergence and summability factors*, III, Proc. London Math. Soc. (2) **50** (1949), 482-496.

[3] M. FEKETE, *Zur Theorie der divergenten Reihen*, Mat. és Term. Ért. (Budapest) **29** (1911), 719-726.

[4] G. H. HARDY, *Divergent Series*, Clarendon Press, Oxford 1949.

[5] E. KOGBETLIANTZ, *Sur les séries absolument sommable par la méthode des moyennes arithmétiques*, Bull. Sci. Math. (2) **49** (1925), 234-256.

[6] E. KOGBETLIANTZ, *Sommation des séries et intégrales divergentes par les moyennes arithmétiques et typiques*, Mémor. Sci. Math. **51** (1931).

[7] T. PATI, *On the absolute Riesz summability of Fourier series and its conjugate series*, Trans. Amer. Math. Soc. **76** (1954), 351-374.

[8] T. PATI, *The summability factors of infinite series*, Duke Math. J. **21** (1954), 271-284.

[9] T. PATI and S. R. SINHA, *On the absolute summability factors of Fourier series*, Indian J. Math. **1** (1958), 41-54.

[10] A. ZYGMUND, *Trigonometrical Series*, Warsaw 1935.

* * *