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Some Analogues of Certain Arithmetical Functions. (**)

1. - Introduction.

Let $p_1, p_2, \ldots$ denote the increasing sequence of the primes, and for positive integers n let $e_i(n)$ denote the multiplicity of p_i as a divisor of n for each $i \geqslant 1$. In particular, $e_i(n) = 0$ if $p_i \nmid n$. Given a positive integer k, we say that n is *k-free* if $e_i(n) < k$ for every $i \geqslant 1$, and that n is *k-full* if $e_i(n) \geqslant k$ for every i for which $e_i(n) \neq 0$. By the unique factorization law, it is evident that n has a unique decomposition of the form

$$(1.1) \qquad n = n_1 n_2 \quad (n_1, n_2) = 1, \qquad n_1 \in Q_k, \qquad n_2 \in L_k,$$

where Q_k and L_k denote the sets of k-free and k-full integers, respectively. In the representation (1.1) we call $n_1 = \alpha_{k-1}(n)$ the *k-free part* of n and $n_2 = \beta_{k-1}$ the *k-full part* of n.

In this paper we consider an analogue of the EULER totient function $\Phi(n)$ which arises when n is replaced by $\alpha_r(n)$, where r is an arbitrary positive integer. For this purpose, we write $\overline{\Phi}_r(n) = \beta_r(n)\ \Phi(\alpha_r(n))$, and note that $\overline{\Phi}_r(n)$ represents the number of integers (mod n) relatively prime to $\alpha_r(n)$.

It is recalled that

$$(1.2) \qquad \Phi(n) = n \sum_{d \mid n} \mu(d)/d = \sum_{d\delta = n} \mu(d)\delta\ ,$$

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where $\mu(n)$ is the inversion function of MÖBIUS. In (2.3) we deduce a corresponding representation for $\overline{\Phi}_r(n)$. The function which replaces $\mu(n)$ in this representation is denoted $\mu_r(n)$. In (2.4) we obtain an « r-analogue » of the characteristic property of $\mu(n)$,

$$\sum_{d \mid n} \mu(d) = \varepsilon(n) = \begin{cases} 1 & (n = 1) \\ 0 & (n < 0) . \end{cases} \tag{1.3}$$

In the familiar generalization (5.2) of (1.3), $\varepsilon(n)$ becomes the characteristic function $q_k(n)$ of Q_k. An r-analogue of this result is deduced in Lemma 7. The function corresponding to $q_k(n)$ in this analogue is denoted $\nu_{k,r}(n)$.

In n. **3** we obtain an estimate for the average order of $\overline{\Phi}_r(n)$, corresponding to the familiar asymptotic formula for $\Phi(n)$,

$$\Phi(x) \equiv \sum_{n \leq x} \Phi(n) = (3x^2/\pi^2) + O(x \log x) . \tag{1.4}$$

The proof of this result (Theorem 3 b) is based on an estimate essentially equivalent to (1.4). In Theorem 4 we prove an analogue of the well-known property of $q_k(n)$ $(k \geqslant 2)$,

$$Q_k(x) \equiv \sum_{n \leq x} q_k(n) = (x/\zeta(k)) + O(\sqrt[k]{x}) , \tag{1.5}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s > 1$.

The method employed in proving the asymptotic estimates of this paper makes no use of analysis beyond the classical properties of DIRICHLET series in the real domain (see n. **3**). A cruder argument of the same sort leads to the less refined estimates of n. **6**.

2. - Formal properties.

Let p denote an arbitrary prime and e an arbitrary positive integer. A multiplicative function f [(that is, a function satisfying $f(mn) = f(m)\, f(n)$ if $(m, n) = 1$, $f(1) = 1$] is evidently determined by the values assumed when n is of the form $n = p^e$. On this basis, by (1.2), $\Phi(n)$ is the multiplicative function for which

$$\Phi(p^e) = p^e(1 - 1/p) . \tag{2.1}$$

Define now $\mu_r(n)$ to be the multiplicative function determined by

$$\mu_r(p^e) = \begin{cases} p^r & (e = r+1) \\ -1 & (e = 1) \\ 0 & (\text{otherwise}) . \end{cases} \tag{2.2}$$

We prove

Theorem 1.

$$\overline{\Phi}_r(n) = n \sum_{d \mid n} \mu_r(d)/d = \sum_{d\delta = n} \mu_r(d)\delta . \tag{2.3}$$

Proof. There exists, by Möbius inversion, a unique function $g(n) = g_r(n)$ such that

$$\overline{\Phi}_r(n)/n = \sum_{d \mid n} g(d)/d ,$$

from which

$$g(n)/n = \sum_{d\delta = n} (\overline{\Phi}_r(d)/d)\, \mu(\delta) .$$

By (2.1) and the definition of $\overline{\Phi}_r(n)$, it follows then that $g(p^e) = \mu_r(p^e)$. But $g(n)$ is multiplicative, by the multiplicativity of $\overline{\Phi}_r(n)$, and hence $g(n) = \mu_r(n)$ for all n.

Let $\gamma(n)$ denote the largest square-free factor of n. We prove next

Theorem 2.

$$\sum_{d \mid n} \mu_r(d) = \nu_r(n) \equiv \begin{cases} \gamma^r(n) & \text{if} \quad n \in L_{r+1} \\ 0 & \text{otherwise} . \end{cases} \tag{2.4}$$

Remark 1. Clearly, the function $\nu_r(n)$ defined by the right of (2.4) is multiplicative.

Proof. Let $f_r(n)$ denote the necessarily multiplicative function defined by the left of (2.4). Then by (2.2)

$$f_r(p^e) = \begin{cases} p^r & \text{if} \quad e > r \\ 0 & \text{if} \quad e \leqslant r . \end{cases}$$

That is, $f_r(p^e) = \nu_r(p^e)$, and hence by Remark 1, $f_r(n) = \nu_n(n)$ for all n.

We also note for later use the identity,

$$\mu_r(n) = \sum_{d\delta = n} \nu_r(d)\ \mu(\delta)\ , \tag{2.5}$$

which is an immediate consequence of (2.4), by MÖBIUS inversion.

Remark 2. By definition, $\Phi(n)$ is the limiting case of $\overline{\Phi}_r(n)$ as $r \to \infty$. Moreover, by (2.2), $\mu(n)$ arises as the corresponding limiting case of $\mu_r(n)$. The relations (1.2) and (1.3) can therefore be regarded as limiting cases for (2.3) and (2.4), respectively.

3. - Lemmas based on Dirichlet series.

Define for real $s > 1$,

$$a_{s,r} = \prod_p \left(1 - \frac{1}{p^s} + \frac{p^r}{p^{s(r+1)}}\right), \tag{3.1}$$

where the product is over the primes p.

Lemma 1. *If $s > 1$, the Dirichlet series*

$$\sum_{n=1}^{\infty} \nu_r(n)/n^s \tag{3.2}$$

converges absolutely and its sum is equal to $\zeta(s) a_{s,r}$, where $a_{s,r}$ is defined by (3.1).

Proof. The product in (3.1) converges absolutely for $s > 1$. Hence by the EULER factorization of $\zeta(s)$, one obtains from (3.1),

$$\zeta(s) a_{s,r} = \prod_p \left\{1 + \frac{p^r}{p^{s(r+1)}}\left(1 - \frac{1}{p^s}\right)^{-1}\right\},$$

and it follows on the basis of Remark 1 that

$$\zeta(s) a_{s,r} = \prod_p \left(1 + \sum_{i=1}^{\infty} \frac{p^r}{p^{s(r+1)}}\right) = \sum_{n=1}^{\infty} \frac{\nu_r(n)}{n^s}\ . \tag{3.3}$$

Lemma 2. *If $s > 1$, then*

$$\sum_{n=1}^{\infty} \frac{\nu_r(n)}{n^s} = \frac{\zeta(s(r+1) - r)}{\zeta(2[s(r+1) - r])} \sum_{n=1}^{\infty} \frac{\lambda_r(n)}{n^s}\ , \tag{3.4}$$

where the series on the right converges absolutely for $s > (r+1)/(r+2)$, the coefficients $\lambda_r(n)$ being determined by (3.6) below.

Proof. By (3.3) we have for $s > 1$,

$$\sum_{n=1}^{\infty} \frac{\nu_r(n)}{n^s} = \left\{ \prod_p \left(1 + \frac{p^r}{p^{s(r+1)}}\right) \right\} \beta_{r,s}, \tag{3.5}$$

where $\beta_{r,s}$ is defined by

$$\beta_{r,s} \equiv \prod_p \left\{ 1 + \left(1 + \frac{p^r}{p^{s(r+1)}}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \frac{p^r}{p^{s(r+2)}} \right\} \equiv \sum_{n=1}^{\infty} \frac{\lambda_r(n)}{n^s}. \tag{3.6}$$

The product, and hence the series in (3.6), converges absolutely, provided the series

$$\sum_p f_{r,s}(p) \equiv \sum_p \left(1 + \frac{p^r}{p^{s(r+1)}}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \frac{p^r}{p^{s(r+2)}}, \tag{3.7}$$

converges. But

$$\sum_p f_{r,s}(p) \leqslant \left(1 - \frac{1}{2^s}\right)^{-1} \sum_p \frac{p^r}{p^{s(r+2)}} = \left(1 - \frac{1}{2^s}\right)^{-1} \sum_p p^{r - s(r+2)},$$

and the latter series converges if $s(r+2) - r > 1$. Hence the series in (3.7) converges for $s > (r+1)/(r+2)$, and the lemma results from (3.5), by the EULER factorization of the zeta-function.

Let $g_r(n)$ be the arithmetical function defined by

$$\sum_{n=1}^{\infty} \frac{g_r(n)}{n^s} = \frac{\zeta(s(r+1) - r)}{\zeta(2[s(r+1) - r])} \qquad (s > 1). \tag{3.8}$$

Then by Lemma 2 and the uniqueness theorem for DIRICHLET series, one deduces

Lemma 3.

$$\nu_r(n) = \sum_{d\delta = n} \lambda_r(d)\, g_r(\delta) \tag{3.9}$$

and

$$g_r(n) = \begin{cases} \displaystyle\sum_{d\delta^2 = m} d^r \delta^{2r} \mu(\delta) & \text{if } n = m^{r+1} \\ 0 \text{ if } n \text{ is not an } (r+1)\text{-th power}. \end{cases} \tag{3.10}$$

Remark 3. The O-constants in this paper are assumed to be dependent upon the parameters r and k, when they occur.

Lemma 4.

$$N_r(x) \equiv \sum_{n \leq x} \nu_r(n) = O(x). \tag{3.11}$$

Proof. By (3.10), for suitable positive numbers c and c', depending at most upon r,

$$\Big| \sum_{n \leq x} g_r(n) \Big| = \Big| \sum_{(d\delta^2)^{r+1} \leq x} d^r \delta^{2r} \mu(\delta) \Big| \leqslant \sum_{d\delta^2 \leq x^{1/(r+1)}} d^r \delta^{2r} =$$

$$= \sum_{\delta \leq x^{1/(2r+2)}} \delta^{2r} \sum_{d \leq \delta^{-2} x^{1/(r+1)}} d^r \leqslant \sum_{\delta \leq x^{1/(2r+2)}} \delta^{2r} \delta^{cx/(2r+2)} \leqslant$$

$$\leqslant cx \sum_{\delta \leq x} (1/\delta^2) \leqslant cx \sum_{\delta=1}^{\infty} (1/\delta^2) = c'x.$$

Hence $G_r(x) \equiv \sum_{n \leq x} g_r(n) = O(x)$, so that by (3.9) and Lemma 2,

$$N_r(x) = \sum_{d\delta \leq x} \lambda_r(d)\, g_r(\delta) = \sum_{d \leq x} \lambda_r(d)\, G_r(x/d) = O\Big(x \sum_{d \leq x} |\lambda_r(d)|/d\Big) = O(x).$$

The latter result leads to the following estimates involving $\nu_r(n)$.

Lemma 5.

$$\sum_{n \leq x} \nu_r(n)/n = O(\log x), \tag{3.12}$$

$$\sum_{n > x} \nu_r(n)/n^s = O(1/x^{s-1}) \quad \text{if} \quad s > 1. \tag{3.13}$$

Proof. Using partial summation, if follows as a consequence of (3.11) that

$$\sum_{n \leq x} \frac{\nu_r(n)}{n} = \sum_{n \leq x} N_r(n) \left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{N_r(x)}{[x]+1} =$$

$$= \sum_{n \leq x} O(n) \frac{1}{n(n+1)} = O\left(\sum_{n \leq x} \frac{1}{n}\right) = O(\log x).$$

This proves (3.12). Similarly, we have

$$\sum_{n > x} \frac{\nu_r(n)}{n^s} = \sum_{n > x} N_r(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) - \frac{N_r(x)}{([x]+1)^s} =$$

$$= \sum_{n > x} O(n) \frac{(1+(1/n)) - 1}{(n+1)^s} + \frac{O(x)}{x^s} = O\left(\sum_{n > x} \frac{1}{n^s}\right) + O(x^{1-s}) = O(x^{1-s}).$$

4. - Average order of $\overline{\Phi}_r(n)$.

The relation (2.3) suggests an indirect approach to $\overline{\Phi}_r(n)$ by way of the « mean-totient », $\overline{\Phi}_r(n)/n$. We adopt this point of view, noting in this connection the following analogue of (1.4):

(4.1) $$\Phi'(x) \equiv \sum_{n \leq x} \Phi(n)/n = (6x/\pi^2) + O(\log x), \qquad x \geqslant 2 .$$

The relation (1.4) is a direct consequence of (4.1) by partial summation [cf. Remark 4 and the derivation of (4.5) from (4.3) below]. We now deduce an estimate for the average of $\overline{\Phi}_r(n)/n$, based on the estimate (4.1) for the average $\Phi(n)/n$.

Lemma 6.

(4.2) $$\overline{\Phi}_r(n) = \sum_{d\delta = n} \nu_r(d)\, \Phi(\delta) .$$

Proof. Let the operation of DIRICHLET multiplication be denoted $\cdot$, and let j denote the identity ($j(n) = n$). Then by (2.3), (2.5), and (1.2), one finds that $\overline{\Phi}_r(n) = \mu_r \cdot j = \nu_r \cdot \mu \cdot j = \nu_r \cdot \Phi$, which is (4.2).

Theorem 3a. *If $x \geqslant 2$, then*

(4.3) $$\Phi_r'(x) \equiv \sum_{n \leq x} \overline{\Phi}_r(n)/n = a_r\, x + O(\log^2 x),$$

where

(4.4) $$a_r \equiv a_{2,r} = \prod_p \{1 - (1/p^2) + (1/p^{r+2})\} .$$

Proof. By (4.2) we have

$$\Phi_r'(x) = \sum_{d\delta \leq x} \{\nu_r(d)/d\}\{\Phi(\delta)/\delta\} = \sum_{n \leq x} \{\nu_r(n)/n\}\, \Phi'(x/n) .$$

Application of (4.1), Lemma 5, and Lemma 1 ($s = 2$) yields

$$\Phi_r'(x) = (6x/\pi^2) \sum_{n \leq x} \{\nu_r(n)/n^2\} + O(\sum_{n \leq (x/2)} \{\nu_r(n)/n\} \log (x/n)) + O(\sum_{(x/2) < n \leq x} \nu_r(n)/n) =$$

$$= (6x/\pi^2) \sum_{n=1}^{\infty} \nu_r(n)/n^2 + O(x \sum_{n > x} \nu_r(n)/n^2) + O(\log x \sum_{n \leq x} \nu_r(n)/n) =$$

$$= (6x/\pi^2)\zeta(2) a_{2,r} + O(1) + O(\log^2 x) = a_r\, x + O(\log^2 x) .$$

Teorem 3b. *If $x \geqslant 2$, then*

$$\Phi_r(x) \equiv \sum_{n \leq x} \Phi_r(n) = (a_r x^2/2) + O(x \log^2 x), \tag{4.5}$$

where a_r is defined by (4.4).

Proof. Applying partial summation, one deduces by (4.3)

$$\Phi_r(x) = -\sum_{n \leq x} \Phi_r'(n) + \Phi_r'(x)([x]+1) = -\sum_{n \leq x} \Phi_r'(n) + x\Phi_r'(x) + O(x) =$$

$$= -a_r \sum_{n \leq x} n + O\left(\sum_{n \leq x} \log^2 n\right) + a_r x^2 + O(x \log^2 x) =$$

$$= -a_r\big((x^2/2) + O(x)\big) + a_r x^2 + O(x \log^2 x) = (a_r x^2/2) + O(x \log^2 x).$$

Place $\overline{\Phi}(n) = \overline{\Phi}_1(n)$. We have

Corollary 1 ($r = 1$). *If $x \geqslant 2$, then*

$$\sum_{n \leq x} \overline{\Phi}(n) = (a x^2/2) + O(x \log^2 x), \tag{4.6}$$

where $a = \prod \left\{1 - (1/p^2) + (1/p^3)\right\}$.

Remark 4. The result in Theorem 3b can, of course, be proved directly on the basis of (1.4), just as Theorem 3a was deduced from (4.1). However, an attempt to deduce (4.3) from (4.5) by partial summation leads to a remainder term of order $O(\log^3 x)$. For proofs of (1.4) and (4.1) and generalizations, the reader is referred to ([3], § 7, p. 9); also cf. ([1], Lemma 2) and the references listed there.

5. - Average order of $\nu_{k,r}(n)$.

We can generalize $\nu_r(n)$ as follows. Let $L_{k,r+1}$ denote the set of those n for which $e_i(n)$ is not on the range, $k \leqslant e < (r+1)k$, $i \geqslant 1$. We define now (see n. **1**)

$$\nu_{k,r}(n) = \begin{cases} \gamma^r\big(\beta_k(n)\big) & \text{if} \quad n \in L_{k,r+1} \\ 0 & \text{otherwise}. \end{cases} \tag{5.1}$$

Evidently, $\nu_{1,r}(n) = \nu_r(n)$.

In this section we obtain an approximation to the average of $\nu_{k,r}(n)$, $k \geqslant 2$, based on the estimate (1.5) for $q_k(n)$. We note the familiar identity

(5.2)
$$q_k(n) = \sum_{d^k\delta=n} \mu(d) .$$

Analogous to (5.2), we have the following generalization of (2.4):

Lemma 7.

(5.3)
$$\nu_{k,r}(n) = \sum_{d^k\delta=n} \mu_r(d) .$$

Proof. By (2.2)

$$\sum_{d^k \mid p^e} \mu(d) = \begin{cases} 1 & \text{if } e < k \\ p^r & \text{if } e \geqslant (r+1)k \\ 0 & \text{if } k \leqslant e < (r+1)k \end{cases}$$

which is $\nu_{k,r}(p^e)$. The lemma follows, like Theorem 1, from the multiplicativity of the occurring functions.

Lemma 8.

(5.4)
$$\nu_{k,r}(n) = \sum_{d^k\delta=n} \nu_r(d)\ q_k(\delta) .$$

Proof. By (5.3), (2.5), and (5.2), we have

$$\nu_{k,r}(n) = \sum_{d^k\delta=n} \sum_{DE=d} \nu_r(D)\ \mu(E) = \sum_{D^kE^k\delta=n} \nu_r(D)\ \mu(E) = \sum_{D^k e=n} \nu_r(D) \sum_{E^k\delta=e} \mu(E) = \sum_{D^k e=n} \nu_r(D)\ q_k(e) .$$

Theorem 4. *If* $k > 1$, *then*

(5.5)
$$N_{k,r}(x) \equiv \sum_{n \leq x} \nu_{k,r}(n) = \alpha_{k,r}\, x + O(\sqrt[k]{x} \log x) ,$$

where $\alpha_{k,r}$ *is defined by* (3.1).

Proof. By (5.4), (1.5), and Lemmas 1 and 5, as in the proof of Theorem 3a we get

$$N_{k,r}(x) = \sum_{d^k\delta \le x} \nu_r(d)\, q_k(\delta) = \sum_{n \le x^{1/k}} \nu_r(n)\, Q_k(x/n^k) =$$

$$= (x/\zeta(k)) \sum_{n \le x^{1/k}}^{\infty} \nu_r(n)/n^k + O\big(\sqrt[k]{x} \sum_{n \le x^{1/k}} \nu_r(n)/n\big) =$$

$$= (x/\zeta(k)) \sum_{n=1}^{\infty} \nu_r(n)/n^k + O\big(x \sum_{n > x^{1/k}} \nu_r(n)/n^k\big) + O\big(\sqrt[k]{x} \sum_{n \le x} \nu_r(n)/n\big) =$$

$$= a_{k,r}\, x + O(\sqrt[k]{x}) + O(\sqrt[k]{x} \log x) = a_{k,r}\, x + O(\sqrt[k]{x} \log x)\,.$$

Place $h_r(n) = \nu_{2,r}(n)$, $h(n) = h_1(n)$, and let a_r and a be defined as in n. 4 [cf. (4.3) and (4.6)].

Corollary 1 $(k = 2)$.

$$\sum_{n \le x} h_r(n) = a_r(x) + O(\sqrt{x} \log x)\,. \tag{5.6}$$

Corollary 2 $(k = 2,\ r = 1)$.

$$\sum_{n \le x} h(n) = ax + O(\sqrt{x} \log x). \tag{5.7}$$

6. - A final remark.

We point out here that the arguments of n. 3 based on the otherwise superfluous functions $\lambda_r(n)$ and $g_r(n)$ can be avoided if one is content with estimates of a less precise order of magnitude. It is quite easy to prove the estimate

$$N_r(x) \equiv \sum_{n \le x} \nu_r(n) = O(x^{1+\varepsilon}) \tag{6.1}$$

for every $\varepsilon > 0$, as a replacement for the sharper appraisal of $N_r(x)$ contained in (3.11) (a proof is given below). On the basis of (6.1), the O-estimates in (3.12) and (3.13) are then replaced by $O(x^\varepsilon)$ and $O(1/x^{s-1-\varepsilon})$, respectively. The same approach as that employed in nn. **4** and **5** for the proofs of (4.3), (4.5), and (5.5) leads, as a consequence, to the corresponding asymptotic evaluations,

$$\sum_{n\leq x} \overline{\Phi}_r(n)/n = a_r\, x + O(x^{\varepsilon}) \,, \tag{6.2}$$

$$\sum_{n\leq x} \overline{\Phi}_r(n) = a_r\, x/2 + O(x^{1+\varepsilon}) \,, \tag{6.3}$$

$$\sum_{n\leq x} \nu_{k,r}(n) = a_{k,r}\, x + O(x^{(1/k)+\varepsilon}), \qquad k \geqslant 2 \,, \tag{6.4}$$

where ε is an arbitrary positive number.

Proof of (6.1). The series (3.2) plainly diverges for $s = 0$, and for $s > 1$ converges by Lemma 1. It follows that the abcissa of convergence α_r of (3.2) is $\leqslant 1$. Moreover, if γ is the lower limit of those β for which $N_r(x) = O(x^{\beta})$, then $\alpha_r = \gamma$, by a classical result on the partial sums of real DIRICHLET series ([2], pp. 122-123). Thus (6.1) results.

Remark 5. By the definition of $\nu_r(n)$, it is easily seen that (3.2) diverges at $x = 1$, and therefore that $\alpha_r = 1$.

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