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On Some Dynamical Problems Arising in the Theory of Lubrication.

PART II. (**)

11. - Introduction.

In an earlier paper of the same title (1) some results were presented of a research on the dynamic behaviour of a rigid rotor housed in a lubricated bearing; we continue here the report and for ease of reference we do not interrupt the numbering of sections.

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(**) Received March 26, 1962.
(1) Riv. Mat. Univ. Parma (2) 1 (1960), 1-20. Some annoying errors crept in during the editing of that paper. The first at page 6: the second formula (3.8) should start with $3v_1^{(3)}$ instead of $2v_1^{(3)}$. The second at page 8, formula (4.1): the square bracket should be closed at the end of the first line and the factor $6R^2\eta c^{-3}\{1-b\sinh(\zeta/R)\cdot[2\zeta\sinh(b/2R)]^{-1}\}$ should multiply the quantity in the second line.

The third at p. 15: the sign should be changed in the right-hand side of the formula at the bottom of the page.

Finally, at line 15 of p. 16, C/A should be written instead of $1-(C/A)$; C is the moment of inertia of the journal around its axis, A the moment of inertia around a normal axis through O.

In Section 4 we have also failed to remark explicitly that the expressions (4.6), (4.8) for the force and moment of the couple due to the lubricant are correct to terms of the third order in a, $\dot{a}$, θ, $\dot{\theta}$. The terms of second order in a, $\dot{a}$ within the expression of p are, for instance,

$$3R^2\eta\, c^{-2}a\left\{[\dot{a}\cos 2(\vartheta-\beta)+(2\dot{\beta}-\omega)(a/2)\sin 2(\vartheta-\beta)]\cdot\right.$$
$$\left.\cdot\left[3-2\,\frac{\cosh(\zeta/R)}{\cosh(b/2R)}-\frac{\cosh(2\zeta/R)}{\cosh(b/R)}\right]+6\,\dot{a}\left[1-\frac{\cosh(\zeta/R)}{\cosh(b/2R)}\right]\right\};$$

when introduced in the integrals which express F_e, F_n, M_1, M_2 they give no contribution. Without this explicit remark the developments of Sect. 6 may appear unnecessary:

The systems we consider are: System 1, a rigid cylindrical shaft rotating at constant speed around its axis within a cylindrical bearing; System 2, as System 1 but with an additional mechanism providing a force of elastic type which tends to restore the journal axis onto the bearing axis; System 3, a massless resilient shaft running on two identical bearings and carrying a central massive disc.

In our analysis we will disregard the effect of weight on the rotor; hence we will take rotor and bearings to be coaxial during undisturbed rotation. We will assume the bearings to be of the flooded type (free side leakage and feed) or of the type without side leakage; full film lubrication will also be assumed (2). The last hypothesis is often seriously restrictive; there are reasons to believe, however, that, even during whirls of large amplitude, cavitation need not always occur (cf. Sect. **13**).

Instability in System 1 due to the onset of « parallel » or « conical » whirl was studied in Part 1. Here we pursue the study of « parallel » whirls at large amplitude and provide more satisfactory proofs of the results announced in Sect. **7** for both cases of very short or very long bearings (cf. Sect. **12**). It is confirmed that in both cases: the coaxial configuration of the journal is unstable; a whirl of increasing amplitude follows any disturbance; the eccentri-

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if the validity of the expressions for F_e, F_n were restricted to the cases when O is in the *immediate* neighbourhood of Ω, the equations of motion (5.1) should be completely linearised for consistency.

We note finally that higher-order approximations in θ for p are given in the paper by J. S. AUSMAN: *Torque produced by misaligment of hydrodynamic ga -lubricated journal bearings*, J. Basic Eng., **82** (1960), 335-341. AUSMAN made use of small parameter techniques for solving REYNOLDS equations also in earlier papers, for instance in a paper contributed at the *Conference on Lubrication and Wear* (London, 1957), cf. the Proceedings, pp. 39-45.

(2) These hypotheses are embodied in the expressions used for the force which acts on the journal and is due to the lubricant during « parallel » whirls (i.e. during motions, when journal and bearing remain aligned).

We take for the components F_e along the vector ΩO and F_n normal to ΩO the expressions:

$$F_e = -12\pi R^3 bc^{-2}\eta\dot{a}(1-a^2)^{-3/2}, \qquad F_n = 12\pi R^3 bc^{-2}\eta a(\omega - 2\dot{\beta})(2+a^2)^{-1}(1-a^2)^{-1/2}$$

(long bearings without side leakage; SOMMERFELD case), or

$$F_e = -\pi R b^3 c^{-2}\eta\dot{a}(1+2a^2)(1-a^2)^{-5/2}, \qquad F_n = (1/2)\pi R b^3 c^{-2}\eta a(\omega - 2\dot{\beta})(1-a^2)^{-3/2}$$

(short bearings with predominant effect of side leakage; OCVIRK case). In general, when $\dot{a} = 0$ we take

$$F_e = 0, \qquad F_n = Rb\eta a\ f(a;\ c/R;\ b/R)(\omega - 2\dot{\beta}),$$

and we assume that $f(a)$ is a non-decreasing function of a, tending to infinity when $a \to 1$.

city of the journal axis tends asymptotically to the clearance; the speed of the whirl tends to half the rotational speed. All these results are shown to apply also to System 2, when the rotational frequency exceeds twice the natural frequency of the restrained journal (cf. Sect. **13**).

The remaining part of the paper is devoted to a discussion of forced vibrations due to lack of balance in all three systems: it is shown that, owing to dissipation in the lubricant, System 2 can be run through its critical speed even when lack of balance is so severe that the displacement of the centre of gravity from the centre of the disc exceeds the clearance in the bearing; this result does *not* apply to System 3.

The paper ends with a discussion of the stability of these forced vibrations for System 1.

A list of the symbols which are used in more than one section is appended:

- a, eccentricity ratio: $a = e/c$.
- b, width of bearing.
- B, a bearing number: $B = 6\pi R^3 b\eta/(\sqrt{2}m\omega c^3)$.
- c, radial clearance of the bearing.
- C, disc centre in System 3.
- e, eccentricity of the journal: $e = |\Omega O|$.
- E, displacement of the centre of gravity of rotor from rotor centre: $E = |OG|$ in Systems 1 and 2; $E = |CG|$ in System 3.
- F_e, F_n, components along ΩO and normal to ΩO of the force due to the lubricant and acting on the journal.
- f, a function of a, see footnote (2).
- G, centre of gravity of rotor.
- k, stiffness of spring in System 2 (magnitude of spring force equal to $k|\Omega O|$).
- $2K$, stiffness of shaft in System 3 (magnitude of the force acting on C due to a shaft deflection σc equal to $2K\sigma c$).
- m, mass of rigid rotor in Systems 1 and 2.
- $2M$, mass of the disc in System 3.
- O, trace of journal axis in central plane.
- p, pressure in the lubricant.

R,	radius of bearing.
t,	time.
β,	angular co-ordinate of O in a fixed cylindrical system of reference with origin in Ω.
$\delta = 1 - a$, $\varepsilon = 1 - (2\dot{\beta}/\omega)$.	
η,	viscosity of lubricant.
ϑ, ζ,	polar angle and axial co-ordinate with reference to the system mentioned in the definition of β.
σ,	non-dimensional deflection of shaft in System 3: $\sigma C = \mid OC \mid$.
τ,	non-dimensional time: $\tau = \omega t$.
ω,	rotational speed of journal.
ω_c,	critical speed. In System 2: $\omega_c = (k/m)^{1/2}$; in System 3: $\omega_c = (K/M)^{1/2}$.
Ω,	trace of bearing axis in central plane.

Note: A dot indicates a time derivative; a prime a derivative towards τ.

12. - Complements to the study of whirls of large amplitude: (a) Proof of some statements made in section 7.

We proceed here to a closer study of the properties of the solutions of the differential system (7.2), when $a \sim 1$. These solutions describe parallel whirls of a rotor of type 1 within a very long bearing at high eccentricity ratios

$$(7.2)\qquad \begin{cases} m\ddot{a} = ma\dot{\beta}^2 - 2\mathfrak{F}_1\dot{a}(1-a^2)^{-3/2} \\ ma\ddot{\beta} = -2m\dot{a}\dot{\beta} + \mathfrak{F}_1 a(\omega - 2\dot{\beta})[1 + (a^2/2)]^{-1}(1-a^2)^{-1/2} \\ \mathfrak{F}_1 = 6\pi R^3 b c^{-3}\eta\,. \end{cases}$$

Let us put $a(t) = 1 - \delta(\tau)$, $\dot{\beta}(t) = \gamma[1 - \varepsilon(\tau)]$, $\tau = \omega t$ and see if the assumption that δ and ε are infinitesimal with increasing τ is consistent with equations (7.2). It turns out immediately that γ must be equal to $\omega/2$ for consistency; furthermore it must be:

$$(12.1)\qquad \begin{cases} \delta'' = -\dfrac{1}{4} - B\delta^{-3/2}\delta' \\ \varepsilon' = -2\delta' - \dfrac{4}{3}B\varepsilon\delta^{-1/2}\,, \end{cases} \qquad B = \frac{\mathfrak{F}_1}{\sqrt{2}m\omega}\,.$$

The first of these equations has the privilege of involving only δ, so that its study can be carried out independently of the second. The equation can, of course, be substituted by two equations of the first order, if the new variable $u = \delta'$ is introduced

$$u' = -\frac{1}{4} - B\delta^{-3/2}u, \qquad \delta' = u \,. \tag{12.2}$$

The right-hand sides of (12.2) are regular functions for all τ, u and all positive δ. (Note that for $\delta^{3/2}$ the positive determination must be taken). Hence there is one and only one solution of (12.2) that issues from any point (τ_0, δ_0, u_0), if $\delta_0 > 0$: say $\delta = \delta^{(0)}(\tau)$, $u = u^{(0)}(\tau)$.

We call (τ_1, τ_2) ($\tau_1 < \tau_0 < \tau_2$; τ_1, or τ_2, or both may be infinite) the interval of τ where the solution is defined. We propose to show first that $\delta^{(0)}(\tau) \to 0$ when $\tau \to \tau_2$, if δ_0, u_0 are appropriately small: for instance if $\delta_0^2 + u_0^2 < \frac{1}{16}$; the property is valid under less restrictive conditions, the special case will suffice here, however.

If we put $\varrho^2 = u^2 + \delta^2$ we have from (12.2)

$$\frac{1}{2}\varrho\varrho' = -u(Bu\delta^{-3/2} - \delta + 1/4) \,. \tag{12.3}$$

From this formula, it follows that the open arc C_1 of the circle $\varrho = 1/4$, which belongs to the quadrant $\delta > 0$, $u > 0$ of the phase plane of (12.2) is a line without contact for the paths of (12.2). Another line C_2 without contact is defined by $\delta = 1/4$, $u < 0$.

Let us call $\mathfrak{S}$ the open set of points which has the closed arc C_1, C_2 and the portion $u < 1/4$ of the u-axis as a boundary, and where $\delta < 1/4$; then it is evident from (12.2), (12.3) that any point $[\delta(\tau), u(\tau)]$ which enters $\mathfrak{S}$ moving along a path of (12.2) while τ increases, can never leave $\mathfrak{S}$ for $\tau \to \tau_2$.

Further, let us split $\mathfrak{S}$ into the four subsets: $\mathfrak{S}_1$ (where $u \geqslant 0$), $\mathfrak{S}_2$ [where $0 > u \geqslant -\delta^{3/2}(1-4\delta)/4B$], $\mathfrak{S}_3$ [where $-\delta^{3/2}(1-4\delta)/4B > u \geqslant -\delta^{3/2}/4B$], and $\mathfrak{S}_4$ [where $u < -\delta^{3/2}/4B$].

By considering the field of directions of (12.2) one can prove that the positive half-trajectories issuing from points of $\mathfrak{S}_1$ cross the segment ($0<\delta<1/4$, $u=0$) into $\mathfrak{S}_2$, and hence into $\mathfrak{S}_3$ (because ϱ cannot decrease along a trajectory in $\mathfrak{S}_2$); but all positive half-trajectories issuing from points of $\mathfrak{S}_3$ enter $\mathfrak{S}_4$. Therefore wherever we start in $\mathfrak{S}$, if we move along a path of (12.2) in the direction in which $[\delta(\tau), u(\tau)]$ moves with increasing time, we end up in $\mathfrak{S}_4$ and never leave it afterwards; also, if we start from a point of the circle where $\varrho <$

1/4, we remain in that circle. Finally, as $\tau \to \tau_2$, the point $[\delta(\tau),\ u(\tau)]$ must move towards a singular point of (12.2), hence it must be $\lim\limits_{\tau \to \tau_2} \delta(\tau) = 0$.

We pass on now to show that along the paths of (12.2) in $\mathcal{S}$ not only δ but also u tends to zero when τ tends to τ_2; this analytical property indicates that eqns. (12.2) cannot describe phenomena where a knocking of the journal against

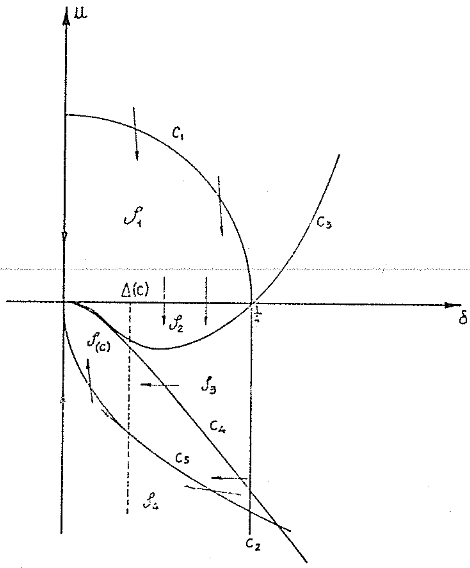

Fig. 1. - To illustrate the discussion of stability of Sect. 12 (a).

the bearing occurs. *The approach between journal and bearing must take place at a decreasing speed.*

Note first that a value $\varDelta(C)$ of δ can be found, such that the curve C_5: $u = -C\,\delta^{1/2}$ (C, a positive constant) is a line without contact for the paths of (12.2) when $0 < \delta < \varDelta(C)$. For $-(C/2)\delta^{-1/2}$ is larger than $[\delta^{3/2} - 4BC\delta^{1/2}]\,(4C\delta^2)^{-1}$ when $\delta < \varDelta(C) = 4BC(1 + 2C^2)^{-1}$. Hence all paths of (12.2) containing a point which falls into the sector $\mathcal{S}_{(c)}(0 < \delta < \varDelta,\ -C\delta^{1/2} < u < -\delta^{3/2}/4B)$ tend to the vertex (0, 0).

On the other hand, if a path $\mathcal{P}$ of (12.2) existed tending to a point of co-ordinates 0, u_0 ($u_0 < 0$) when $\tau \to \tau_2$, then $\mathcal{P}$ would also cross a sector $\mathcal{S}_{(c)}$; and this is a contradiction. In fact, under the absurd hypothesis just mentioned, $\mathcal{P}$ would possess points P (of co-ordinates, say, δ_1, u_1) belonging to $\mathcal{S}_4$. But P belongs also to $\mathcal{S}_{(c)}$ if C is chosen so that $C > |\,u_1\,|\,\delta_1^{-1/2}$ and $\delta_1 < 4BC(1 +$

$+ 2C^2)^{-1}$; such choice is always possible because there are C large enough to satisfy the inequality

$$\frac{|u_1|^2}{C^2} < \frac{4BC}{1+2C^2}.$$

It is possible to state the condition about the vanishing of $u(\tau)$ at the limit for $\tau \to \tau_2$ in a more precise form: along all paths of (12.2) issuing from a point of $\mathfrak{S}$ one has

$$\lim_{\tau \to \tau_2} \frac{u(\tau)}{[\delta(\tau)]^{3/2}} = -\frac{1}{4B}. \tag{12.4}$$

The proof of this statement requires a closer study of the behaviour of the paths of (12.2) near the singular point at the origin; we provide here only a sketch of proof.

Consider the differential equation

$$\frac{du}{d\delta} = -\frac{1}{4u} - \frac{B}{\delta^{3/2}}, \qquad \text{for} \quad \delta > 0; \tag{12.5}$$

we have proved above that all solutions $u = u(\delta)$ of this equation wh'ch are represented by paths in $\mathfrak{S}$ are infinitesimal with δ and we want now find out the order. Let us put in (12.5)

$$u(\delta) = -[v(\delta) + D]\delta^{\alpha},$$

where D and α are constant ($\alpha > 0$) and $v(\delta)$ is infinitesimal with δ. Disregarding quantities of higher order in δ, we find an identity which can be satisfied only if

$$\alpha = 3/2, \qquad D = (4B)^{-1}.$$

This result confirms our statement (12.4). However, if we proceed further, we put

$$u(\delta) = -[v(\delta) + (4B)^{-1}]\delta^{3/2}, \tag{12.6}$$

and attempt to solve formally eqn. (12.5) by taking in (12.6) for $v(\delta)$ a function which is analytic in a neighbourhood of $\delta = 0$ and vanishes with δ:

$$v = \sum_{1}^{\infty}{}_k\, a_k \delta^k, \tag{12.7}$$

we find a set of recurrent relations which determines the coefficients a_k completely: we must conclude that there is at most one solution of (12.5) of the indicated type. This conclusion seems at first to come into conflict with the statement which we want to prove; the contrast is only apparent, however. Although all solutions of (12.5) can be put into the form (12.6) (with $v(\delta)$ infinitesimal when $\delta \to 0$) in a neighbourhood of $\delta = 0$, one at most can be so represented using for $v(\delta)$ a function which is analytic at $\delta = 0$.

Some evidence of this behaviour of the solutions of (12.5) can be readily obtained. If a function of the type (12.6) has to satisfy eqn. (12.5), $v(\delta)$ must be a solution of the equation

$$\frac{\mathrm{d}v}{\mathrm{d}\delta} = -\frac{v + 6D\delta^2(v-D)^2}{4D\delta^3(v-D)}, \qquad D = (4B)^{-1}. \tag{12.8}$$

Near the singular point at the origin eqn. (12.8) is approximated by the simpler equation

$$\frac{\mathrm{d}v}{\mathrm{d}\delta} = \frac{v + 6D^3\delta^2}{4D^2\delta^3},$$

which is linear and has the general integral

$$v(\delta) = -\frac{3D}{4}\, e^{-\frac{1}{8D^2\delta^2}}\, E^*\left(\frac{1}{8D^2\delta^2}\right) + C\, e^{-\frac{1}{8D^2\delta^2}}; \tag{12.9}$$

here C is an arbitrary constant and E^* is the function

$$E^*(x) = \gamma + \log x + \sum_1^\infty{}_n \frac{x^n}{n!\,n} \qquad (\gamma,\ \text{Euler-Mascheroni constant}).$$

It is clear from (12.9) that though finite limits can be found for $v(\delta)$ and its derivatives when $\delta \to 0$, $v(\delta)$ itself is not in general analytic in a neighbourhood of $\delta = 0$.

12. - (b) Time dependence of the eccentricity ratio. Compatibility of the second equation (12.1). Case of very short bearings.

Formula (12.4) has an immediate consequence: τ_2 [the upper limit of the time interval over which the solution of the first eqn. (12.1) is defined] is infinite. In fact

$$\tau_2 - \tau_0 = \int_{\tau_0}^{\tau_2} \mathrm{d}\tau = \int_{\delta_0}^{0} \frac{\mathrm{d}\delta}{u};$$

and the last integral does not converge because u is a zero of order 3/2 when $\delta \to 0$. More precisely it follows from formula (12.4) that δ tends to zero as τ^{-2}, when $\tau \to +\infty$:

$$\lim_{\tau \to +\infty} \delta\, \tau^2 = 64 B^2. \tag{12.10}$$

In physical terms this means that *the contact between journal and bearing* (assuming, of course, that both are perfectly smooth) *cannot occur within a finite interval of time.* Actually, here and earlier in Sect. **12** (a), we ought not have drawn conclusions yet as to the physical significance of our analytical results, because these results follow from properties of the first eqn. (12.1) and the derivation of that equation can be considered sound only after proof is given that ε itself [i.e. any solution of the second eqn. (12.1)] tends to zero when $\tau \to +\infty$. However, such proof is fairly straight forward: we sketch it here.

Because the second eqn. (12.1) is linear, its general integral can be given explicitly

$$\begin{cases} \varepsilon(\tau) = e^{-\Delta}\left(\varepsilon_0 - 2\displaystyle\int_{\tau_0}^{\tau} \frac{d\delta}{d\tau}\, e^{\Delta}\, d\tau\right) \\[2ex] \Delta(\tau) = \dfrac{4}{3} B \displaystyle\int_{\tau_0}^{\tau} \frac{d\tau}{\delta^{1/2}}\,. \end{cases} \tag{12.11}$$

For large τ the integral in the right-hand side of the first eqn. (12.11) behaves as

$$-12\varepsilon\, B^2\, e^{-\frac{\tau_0^2}{12}} \int_{\tau_0}^{\tau} \frac{e^{\frac{\tau^2}{12}}}{\tau^3}\, d\tau\,,$$

or as

$$-\frac{16}{3} B^2\, e^{-\frac{\tau_0^2}{12}} \left\{ \frac{12\, e^{\frac{\tau_0^2}{12}}}{\tau_0^2} - \frac{12\, e^{\frac{\tau^2}{12}}}{\tau^2} + E^*\!\left(\frac{\tau^2}{12}\right) - E^*\!\left(\frac{\tau_0^2}{12}\right) \right\};$$

on the other hand the asymptotic behaviour of the function E^* is expressed by the formula (³)

$$E^*(x) = \frac{e^x}{x}\left[\sum_{m=0}^{M-1} \frac{m!}{x^m} + O(|x|^{-M})\right],$$

(³) See, for instance p. 144 in the second volume on *Higher Trascendental Functions* of the BATEMAN. Manuscript Project.

for $x \to +\infty$; hence the asymptotic behaviour of $\varepsilon(\tau)$ itself is given by

$$\varepsilon(\tau) \sim \frac{1536B^2}{\tau^4} \qquad (\text{for } \tau \to +\infty)\,. \tag{12.12}$$

From this result the desired property follows; furthermore the result shows that *the limit $\omega/2$ for the angular speed of the whirl is approached from below*, i.e. that this speed is always less than $\omega/2$ (when transient conditions are excluded).

It is appropriate at this point to remind the reader that our analysis concerns so far only a journal rotating in a very long bearing. As the general case of a bearing of any aspect ratio is difficult to approach, we have checked at least that parallel results obtain in a second special case: that of a very short bearing. For such a bearing, under conditions of perfect alignment, the components of the force due to the lubricant and acting on the journal are

$$\begin{cases} F_e = -\dfrac{\pi R b^3 \eta}{c^2}\,\dfrac{\dot a(1+2a^2)}{(1-a^2)^{5/2}} \\[2ex] F_n = \dfrac{\pi R b^3 \eta}{2c^2}\,\dfrac{a(\omega - 2\dot\beta)}{(1-a^2)^{3/2}}\,, \end{cases} \tag{12.13}$$

so that the differential system

$$\begin{cases} m\ddot a = m a \dot\beta^2 - 2\mathfrak{F}_2 \dot a(1+2a^2)(1-a^2)^{-5/2} \\[1ex] m a \ddot\beta = -2m\dot a\dot\beta + \mathfrak{F}_2 a(\omega - 2\dot\beta)(1-a^2)^{-3/2} \\[1ex] \qquad \mathfrak{F}_2 = \dfrac{\pi R b^3 \eta}{2c^3}\,, \end{cases} \tag{12.14}$$

takes the place of eqns. (7.2). At high eccentricity ratios eqns. (12.14) reduce to

$$\begin{cases} \delta'' = -(1/4) - (3/2)\,B_2 \delta^{-5/2}\delta' \\[1ex] \varepsilon' = -2\delta' - B_2 \varepsilon \delta^{-3/2} \\[1ex] B_2 = \mathfrak{F}_2/(\sqrt{2}\,m\omega)\,, \end{cases} \tag{12.15}$$

when the notation of Sect. **12** (a) is introduced. Along the lines of this and the previous section it is possible to prove that the solutions of system (12.15) behave asymptotically as follows

$$\delta \sim \left(\frac{4B_2}{\tau}\right)^{2/3}, \qquad \varepsilon \sim \frac{16}{3}(4B_2)^{2/3}\,\tau^{-8/3}, \quad (\text{for } \tau \to \infty), \tag{12.16}$$

and these formulae show that the results proved for the case of long bearings are substantially valid also in the present case.

13. - Discusssion over the assumption that the film of lubricant is complete. The case of an elastically restrained journal.

The expressions (7.1), (12.13) which we have used in our analysis for the lubricant force are obtained under the assumption that *the film of lubricant between journal and bearing is complete.* This assumption is often questioned because it is contrary to experimental evidence in those instances where it leads to the prediction of high negative pressures within the film. We want to show now that *the assumption is acceptable in our case.*

We consider explicitly only the case of very short bearings (though similar conclusions can be reached also in the alternative case). In general the pressure distribution is given by (4)

$$p = \frac{3\eta}{c^2}\left(\frac{b^2}{4} - \zeta^2\right)\frac{(2\dot\beta - \omega)a\sin(\vartheta-\beta) + 2\dot a\cos(\vartheta-\beta)}{[1 - a\cos(\vartheta-\beta)]^3}, \tag{13.1}$$

and when the eccentricity ratio a is large (~ 1) by

$$p = -\frac{3\eta}{c^2}\left(\frac{b^2}{4} - \zeta^2\right)\frac{\omega\varepsilon\sin(\vartheta-\beta) + 2\dot\delta\cos(\vartheta-\beta)}{[1-(1-\delta)\cos(\vartheta-\beta)]^3};$$

hence, for $\tau \to +\infty$,

$$p \sim \frac{4\eta}{c^2}\left(\frac{b^2}{4} - \zeta^2\right)(4B_2)^{2/3}\frac{\cos(\vartheta-\beta)\tau^{1/3}}{\{\tau^{2/3}[1-\cos(\vartheta-\beta)] + 4B_2\cos(\vartheta-\beta)\}^3}.$$

(4) See, for instance, formula (2.12) in the paper « *On the Vibrations of Shafts Rotating on Lubricated Bearings*, Ann. Mat. Pura Appl. (4) **50** (1960), 223-248 ».

The minimum of p (which occurs at the point of maximum film thickness)

$$p_{\min} \sim -\frac{\eta}{2\,c^2}\left(\frac{b^2}{4}-\zeta^2\right)(4B_2)^{2/3}\,\tau^{-5/3}$$

actually tends to zero, as $\tau \to +\infty$. On the contrary the maximum of p (which occurs at the point of minimum film thickness) tends to infinity

$$p_{\max} = \frac{\eta}{c^2}\left(\frac{b^2}{4}-\zeta^2\right)\frac{\tau^{1/3}}{2^{8/3}B_2^{7/3}}\,.$$

An essential ingredient in our proof is the circumstance that $(2\dot\beta/\omega)-1$ tends to zero more rapidly than $\dot a/\omega$. It is evident from formula (13.1) that if the whirl speed were not equal to one half the running speed at the limit, then there would be a tendency for the negative peak of the pressure to grow worse. There are cases (when the journal is a part of an elastic system: a resilient rotor, for instance, as in System 3) where the whirling speed tends to a resonant speed of the system (5) (different in general from $\omega/2$); in such cases it may be presumed that cavitation within the film will be caused by the movement of the journal.

The system considered in Sects. **5**, **6** (journal on which an elastic restoring force is acting; System of type 2) is a curious hybrid. On the one hand its rule of stability recalls the rule operating for resilient rotors; on the other hand *the frequency of the whirl*, when $\omega > 2\omega_c$, *tends towards* $\omega/4\pi$ *rather than towards the natural frequency* $\omega_c/4\pi$ *when the eccentricity ratio approaches unity*(and hence the onset of cavitation within the film of lubricant can be ruled out). To put this phenomenon in evidence we carry out a partial analysis of the differential system

$$(13.2)\qquad \begin{cases} m\ddot a = ma\dot\beta^2 - 2\mathfrak{F}_2\dot a(1+2a^2)(1-a^2)^{-5/2} - ka \\ ma\ddot\beta = -2m\dot a\dot\beta + \mathfrak{F}_2 a(\omega-2\dot\beta)(1-a^2)^{-3/2}\,, \end{cases}$$

where, as in Sect. **5**, k is the stiffness of the spring on which the journal is suspended. A general study of (13.2) being beyond the point here, we look for some reasonable simplifications. We intend to limit our analysis to cases of large whirls, when $\omega > 2\omega_c$; hence we will put $a = 1-\delta$, as before, keeping only

(5) See Sect. 5 of the paper quoted in footnote (4).

the terms with the lowest powers of δ. But as for $\dot{\beta}$ we want to leave now its choice free. We formulate instead an alternative simplifying hypothesis: that the radial component of the acceleration can be approximated by $a\dot{\beta}^2$, the ratio

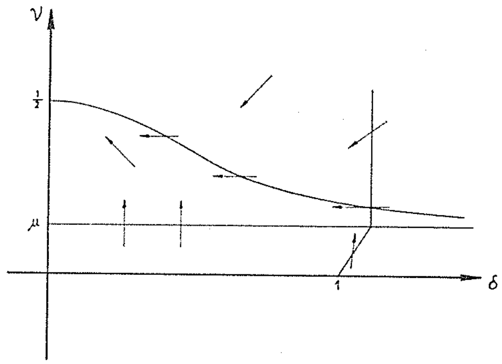

Fig. 2. - To illustrate the discussion of stability of Sect. 13.

$\ddot{a}/a\dot{\beta}^2$ being considered as very small. Note that, when a is nearly equal to 1, its second differential cannot be but negligible when compared with the centripetal acceleration.

Under these assumptions the system (13.2) is reduced to

$$\text{(13.3)} \qquad \begin{cases} \delta' = (2/3B_2)(\mu^2 - \nu^2)\delta^{5/2} \\[2ex] \nu' = \dfrac{3B_2^2(1-2\nu) - 8\delta^4\nu(\nu^2-\mu^2)}{6B_2\delta^{3/2}}, \end{cases}$$

if one sets $\nu = \dot{\beta}/\omega$, $\mu^2 = (k/\omega^2 m) = (\omega_c/\omega)^2$ to avoid dimensional variables. We are here concerned solely with the case $\mu^2 < 1/4$; in that case a study of the paths of (13.3) in the quadrant $\nu > 0$, $\delta > 0$ of the phase plane (δ, ν) leads to a proof that, along any path, $\delta(\tau) \to 0$ and $\nu(\tau) \to 1/2$ as $\tau \to +\infty$. In fact all paths converge toward the singular point (0, 1/2), which is an attractive fan.

These statements are based on the following remarks (See Fig. 2).

The derivative

$$\text{(13.4)} \qquad \frac{d\nu}{d\delta} = \frac{3B_2^2(1-2\nu) - 8\delta^4\nu(\nu^2-\mu^2)}{4\delta^4(\mu^2-\nu^2)},$$

is larger than $\dfrac{3B_2^2(1-2\mu)}{4\delta^4\mu^2}$ for any $\delta > 0$ when $0 \leqslant \nu \leqslant \mu$; in particular larger than $3B_2^2(1-2\mu)/(4\delta^4\mu^2)$ for $\delta \geqslant 1$, $0 \leqslant \nu \leqslant \mu$. Then, all paths of (13.3) issuing from points of the region $\mathfrak{R}$: $0 \leqslant \nu \leqslant \mu$, $0 < \delta \leqslant 1 + 3B_2^2(1-2\mu)\nu/(4\mu^2)$ must cross the segment $\nu = \mu$, $0 < \delta \leqslant 1 + 3B_2^2(1-2\mu\nu)/(4\mu)$ upwards (the upper limits imposed here upon δ are to some extent arbitrary; but we are not interested in what happens for large values of δ). For all $\nu > \mu$, δ' is negative; hence all paths cross any straight line $\delta =$ const., say $\delta = 1 + 3B_2^2(1-2\mu)/(4\mu)$ to the left, for $\nu > \mu$; and once they have entered the region $\nu > \mu$, all paths bear to the left, touching in the limit ($\delta \to 0$) the branch of the curve

$$3B_2^2(1-2\nu) = 8\delta^4\nu(\nu^2-\mu^2)\,,$$

which issues from the point (0, 1/2).

14. - Forced vibrations: (a) Resilient rotor.

Self-excited vibrations of resilient rotors are studied in the paper referred to in footnote (4), though the possibility that cavitation might set in within the lubricant is not considered there. In the previous section we have remarked, however, that the film of lubricant in the bearing will probably cavitate when the amplitude of vibration is sufficiently large. A fresh discussion of the problem is therefore necessary, but we will not pursue it for the moment. We want first to show in evidence another striking difference between the behaviour of a resilient rotor (System of type 3) and the behaviour of a rigid, but elastically restrained, rotor (System of type 2) in connection with their forced vibrations caused by unbalance.

Such vibrations are the subject-matter of Sect. **7** in the paper often mentioned; but the simplicity of the results is overshadowed there by the complexity of the developments due to having considered rotors with distributed mass. Hence it appears to be worthwhile to recast the problem in a simpler setting by examining here the behaviour of a resilient rotor consisting of a flexible massless shaft carrying one massive unbalanced disc; in other words, to repeat the elementary calculations which in current treatises (6) lead to the definition of critical speed; abandoning, however, the hypothesis of rigid supports and accounting for the forces due to the pressure generated in the film because of the eccentric position of the journals (see Fig. 3).

A preliminary remark is worthwhile. The radial and transverse components F_e, F_n of the force acting on the journal and due to the lubricant have exactly

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(6) As in TIMOSHENKO's treatise on vibrations, for instance, cfr. Sect. 17 (p. 92 in the second edition).

opposite values in these two cases: (i) when the position of the journal axis is steady; (ii) during a journal whirl of constant amplitude and constant speed equal to the rotational speed (i.e. during a whirl of the type which is caused by lack of balance). In fact REYNOLDS equation has the same form in the two cases except for the sign of the right-hand side (7) and also the boundary conditions remain the same. Even if cavitation had to be considered, the problem of solving REYNOLDS' equation would be the same in the two cases (8).

Under steady conditions, if the film of lubricant is complete, F_e is zero and F_n can be put in the form

$$F_n = \eta\omega R b f\,, \tag{14.1}$$

where f depends only on a and also on the geometry of the bearing (i.e. on the ratios c/R, b/R) (9); these properties of F_e, F_n must remain true then also during a synchronous whirl. For our present purposes we need not specify f; we need only assume that $f(a)$ is a positive non-decreasing function of a tending to infinity as $a \to 1$. Thereby we manage to avoid reference to special cases (such as the cases of very long or very short bearings), and we confer to our argument greater generality. We may mention nevertheless the asymptotic expressions

$$\begin{cases} f = \dfrac{6\pi R^2}{c^2}\left[1 - \dfrac{\tanh(b/2R)}{(b/2R)}\right] + O(a^2) & \text{when } a \sim 0 \\[2ex] f = \dfrac{\pi b^2}{2c^2}(1-a^2)^{-3/2} & \text{when } (b/R) \sim 0 \\[2ex] f = \dfrac{6\pi R^2}{c^2}\left(1+\dfrac{a^2}{2}\right)^{-1}(1-a^2)^{-1/2} & \text{when } (R/b) \sim 0\,. \end{cases} \tag{14.2}$$

(7) See eqn. (3.10); in this Part 2, θ is always taken to be zero and here we consider the special case $e =$ const. and $\dot\beta$ equal to either 0 or ω; furthermore h is unaltered, the angle $\vartheta - \beta$ being always measured from the direction pointing to the minimum of film-thickness.

(8) For an experimental verification see the paper: *Visual Study of Film Extent in Dynamically Loaded Complete Journal Bearings* by J. A. COLE and C. J. HUGHES in the « Prooceedings of the Conference on Lubrication and Wear (London, 1957; published by the Inst. Mech. Engrs.) » .

(9) The statement is correct for any choice of a and c/R, b/R. See: L. N. TAO, *General Solution of Reynolds Equation for a Journal Bearing of Finite Width*, Q. Appl. Math. **17** (1959), 129-136.

Returning now to our main problem, let us call G the centre of gravity of the disc, a point which is assumed to be slightly displaced from the geometric centre C: $|CG| = E$. During rotation, due to this eccentricity, a centrifugal force acts on the shaft causing deflection; under steady conditions the movement

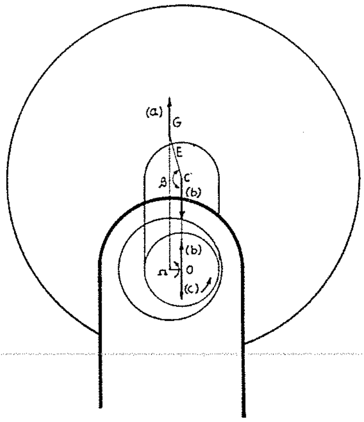

Fig. 3. - End View of Resilient Rotor.
(a) Centrifugal Force. - (b) Elastic Forces. - (c) Force due to the lubricant.

of the deflected shaft will appear to a fixed observer as a steady whirl of speed ω superimposed on the uniform rotation. The magnitude of the deflection, $\sigma c \equiv |OC|$, can be easily obtained from equilibrium conditions.

If the lubricating film is complete, the oil force, as we have mentioned, is normal to ΩO. The elastic force acting on C, which is generated by the bending of the shaft, has the direction of the vector OC. Finally, as the whirl is assumed to be steady, the motion of the bent shaft can be described as a rigid rotation around Ω; therefore the centrifugal force acting on G has the direction of ΩG.

The condition of equilibrium for the forces acting on O requires OC to be normal to ΩO and further the magnitude of the elastic force (which is proportional to σc) to equal the magnitude of the oil force

$$K\sigma c = \eta\omega Rba\, f(a); \tag{14.3}$$

here $2K$ is the stiffness of the shaft. The condition of equilibrium also requires that the elastic force on C be opposed to the centrifugal force acting on G (an appropriate couple being further supplied by the driving motor). As a consequence the relative position of the journal and the disc must be as shown in Fig. 3. This implies the relation

$$E \sin\varphi = ca, \tag{14.4}$$

if φ $(0 \leqslant \varphi \leqslant \pi)$ is the angle between the vectors CO and CG; furthermore the equality between centrifugal and elastic force can be expressed as

$$K\sigma c = M\omega^2(\sigma c - E \cos \varphi) , \tag{14.5}$$

where M is half the mass of the disc. Eqn. (14.5) can be put in a more convenient form by introducing the critical speed of the shaft when rotating on rigid supports $\omega_c = (K/M)^{1/2}$ as follows

$$[1 - (\omega_c/\omega)^2]\sigma c = E \cos \varphi . \tag{14.6}$$

Eqns. (14.3), (14.4), (14.6) determine σ, a and φ as functions of ω and we are interested in finding the values of ω for which σ and a are maxima. It is obvious from eqn. (14.4) that the maxima of a and $\sin\beta$ occur at the same speed. Let us then distinguish two cases: $E < c$, $E \geqslant c$. In the first instance the maximum of a is E/c and is reached when $\varphi = \pi/2$. But, for $a = E/c$, σ has a finite value given by eqn. (14.3); hence when a reaches its maximum, ω must be equal to ω_c. We conclude that, for $E < c$, *the absolute maximum of the amplitude of vibration in the bearings is reached at the critical speed* ω_c, as calculated in the hypothesis of rigid supports.

The absolute maximum of a is also its only maximum. The proof of this statement can be easily achieved: formula (14.3) to (14.6) above imply the following relationship between ω/ω_c and a

$$\begin{cases} \dfrac{\omega}{\omega_c} = \dfrac{\xi}{1 + \sqrt{1 + \xi^2}} & \text{for } \omega < \omega_c \\[2ex] \dfrac{\omega}{\omega_c} = \dfrac{\xi}{\sqrt{1 + \xi^2} - 1} & \text{for } \omega > \omega_c , \end{cases} \tag{14.7}$$

where

$$\xi = \frac{2Naf}{(E^2 - c^2a^2)^{1/2}} , \qquad N = \frac{Rb\omega_c\eta}{cK} .$$

ξ is a function of a defined within the interval $(0, E/c)$, increasing from 0 to $+\infty$ over that interval. On the other hand the first formula defines a function ω/ω_c of ξ increasing from 0 to 1 when ξ increases from 0 to $+\infty$; and the second formula defines another function of ξ increasing from 1 to $+\infty$ as ξ decreases from $+\infty$ to 0. Inverting now these monotonic relationships we find that a increases steadily from 0 to E/c when ω increases from 0 to ω_c and decreases henceforth to zero again while ω tends to infinity.

As for the maximum of σ, it is obvious from (14.3) that it is reached for $\omega > \omega_c$; in fact we have

$$\frac{d\sigma}{d\omega} = \frac{\eta Rb}{Kc}\left\{ af + \omega \frac{d(af)}{da}\frac{da}{d\omega} \right\},$$

and this formula shows that $\frac{d\sigma}{d\omega}$ remains positive when $\frac{da}{d\omega}$ is positive.

We may add that, because of eqn. (5.4), φ is betwen $\pi/2$ and π when $\omega < < \omega_c$ and between 0 and $\pi/2$ for $\omega > \omega_c$; this is equivalent to saying that for

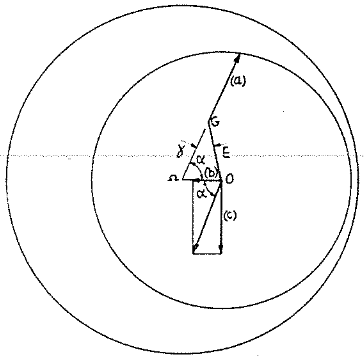

Fig. 4. - End View of Rigid Rotor.
(a) Centrifugal Force. - (b) Elastic Force. - (c) Force due to lubricant.

speeds less than the critical speed $|\Omega G|$ is greater than $|OC|$ and vice versa; this result recalls a similar one of the elementary theory (cf. loc. cit. footnote (6), p. 93).

Let us now consider the second instance which corresponds to conditions of severe lack of balance. *When $E > c$ the damping due to the lubricant is not sufficient to run the shaft through the critical speed.* In fact $\sin\varphi$ can then never be equal to 1 and $\cos\varphi$ never to zero; therefore for $\omega \to \omega_c$, σ tends to infinity and a to 1 .

14. - (b) Rigid rotor which is elastically restrained or free.

We go back now to consider, as in Sects. **5**, **6** and **13**, a rigid rotor which is elastically restrained (System 2). We assume here, however, that want of balance occurs and we intend to study whirls of steady amplitude which may ensue. With reference to Fig. 4 and the notation shown on it, the following equilibrium equations are self-explanatory

$$(14.8)\qquad \begin{cases} m(ca\cos\alpha + E\cos\gamma)\omega^2 = kca\sqrt{1 + N^2 f^2(\omega/\omega_c)^2} \\ \tan\alpha = Ng(\omega/\omega_c) \\ ca\sin\alpha = E\sin\gamma\,. \end{cases}$$

As in Sects. **5**, **6** and **13** we call now m the mass of the rotor and k the stiffness of the restraining spring; N is now the non-dimensional group $Rb\omega_c\eta/kc$.

Formulae (14.8) imply

$$(14.9)\qquad (E/ca)^2 = [1 - (\omega_c/\omega)^2]^2 + (\omega_c/\omega)^2 N^2 f^2,$$

whence it follows that a does not tend to 1 when $\omega \to \omega_c$ even when E is larger than c: when $\omega \to \omega_c$, a tends to the solution of the equation

$$E/c = Naf$$

which is unique and less than 1 for any value of the ratio E/c.

An elastically restrained rigid rotor can run through its critical speed even when lack of balance is such that the displacement of the centre of gravity from the rotor axis exceeds the clearance.

The dependence of a on ω can be defined for any special choice of the function f through a consequence of (14.9)

$$(\omega/\omega_c)^2 = \{1 - (N^2f^2/2) + [(N^4f^4/4) - N^2f^2 + E^2/(ca)^2]^{1/2}\}^{-1}.$$

When the ratio E/c is sufficiently small, for instance, so that a itself is small and f can be considered to be approximately constant, it turns out that a has a maximum when

$$\omega/\omega_c = [1 - (N^2f^2/2)]^{-1/2},$$

i.e. at a speed higher than ω_c: the maximum is

$$a_{\max} = (E/c)\{Nf[1 - (N^2f^2/4)]^{1/2}\}^{-1}.$$

It is interesting to remark at this point that, if the rotor is not elastically restrained (System 1), the relationship between eccentricity ratio a and the displacement of centre of gravity becomes

$$E/c = a\left[1 + \left(\frac{\eta bRf}{mc\omega}\right)^{2}\right]^{1/2} . \tag{14.10}$$

Again the eccentricity need not be less than the clearance at any speed. For any value of the left-hand side in (14.10) there is a solution in a, with $a < 1$.

15. - Stability of forced whirls.

Steady rotations of Systems of type 1 have been shown to be unstable; it is interesting to speculate now on the stability of the whirls which are due to lack of balance. A linear analysis, of which details are given below, suggests that the latter movements are also unstable.

Let us state again the results of Sect. 14 (b), before we approach the new problem:

(i) An unbalanced rigid rotor may move in a whirl of steady amplitude and frequency $\omega/2\pi$; hence, the positions of O and G during whirl (say, $\overline{O}$ and $\overline{G}$) remain stationary with reference to a system of co-ordinates $\mathbb{S}$: (x, y) centred at Ω and rotating at a speed ω;

(ii) if $\mathbb{S}$ is chosen so that the x-axis goes through $\overline{O}$, then the y-axis goes through $\overline{G}$ and the co-ordinates of $\overline{O}$ and $\overline{G}$ are respectively $(cA, 0)$ and $(cY, 0)$, where A is the solution of the equation

$$E/c = A\left\{1 + \left[\frac{\eta Rb}{mc\omega} f(A)\right]^{2}\right\}^{1/2} ,$$

or, in particular, for long bearings

$$E/c = A\left\{1 + 2B^{2}\left[1 + (A^{2}/2)\right]^{-2}(1 - A^{2})^{-1}\right\}^{1/2} ,$$

and Y is given by $\left\{(E/c)^{2} - A^{2}\right\}^{1/2}$.

If a disturbance displaces O from $\overline{O}$ to a new position of co-ordinates $\left\{cA(1+\mu),\ cA\nu\right\}$ then G moves to $\left\{cA\mu,\ cY + cA\nu\right\}$ because the vector OG has invariant orientation with respect to $\mathbb{S}$. If, to fix our ideas, we make reference to the case of long bearings we can calculate explicitly the value of the components of the force which will act on the journal in the new position; within a linear approximation it is

$$\begin{cases} F_n = -2\sqrt{2}mc\omega^2 BA(1-A^2)^{-3/2}\mu' \\ F_n = -\dfrac{\sqrt{2}mc\omega^2 BA}{[1+(A^2/2)](1-A^2)^{1/2}}\left\{1+2\nu' + \dfrac{1-(A^2/2)+A^4}{[1+(A^2/2)][1-A^2]}\xi\right\}. \end{cases}$$

Alternatively, making reference to the system $\mathcal{S}$:

$$\begin{cases} F_x = -\sqrt{2}mc\omega^2 BA(1-A^2)^{-3/2}\{2\mu' - (1-A^2)[1+(A^2/2)]^{-1}\nu\} \\ F_y = F_n. \end{cases}$$

The linearised equations of motion for G in a neighbourhood of $\overline{G}$ are therefore

$$(15.1)\quad \begin{cases} \mu'' - \mu - 2\nu' = -\sqrt{2}B(1-A^2)^{-3/2}\{2\mu' - (1-A^2)[1+(A^2/2)]^{-1}\eta\} \\ \nu'' - \nu + 2\mu' = \\ = -\sqrt{2}B[1+(A^2/2)]^{-1}(1-A^2)^{-2/2}\left\{2\nu' + \dfrac{1-(A^2/2)+A^4}{[1+(A^2/2)](1-A^2)}\mu\right\}, \end{cases}$$

where account is taken of the movement of the system $\mathcal{S}$. The characteristic equation is

$$Z^4 + \frac{4\sqrt{2}B[1-(A^2/4)]}{[1+(A^2/2)](1-A^2)^{3/2}}Z^3 + 2\{1+4B^2[1+(A^2/2)]^{-1}(1-A^2)^{-2}\}Z^2 +$$

$$-3\sqrt{2}BA^2(1-A^2)^{-3/2}[1+(A^2/2)]^{-2}[1-(A^2/2)]Z +$$

$$+1+2B^2[1-(A^2/2)+A^4][1+(A^2/2)]^{-3}(1-A^2)^{-2} = 0.$$

Routh' rule assures us that this equation has roots with positive real part, for any value of A less than 1; it follows that eqns. (15.1) have solutions which increase exponentially with time. Hence the conclusion that the whirls are always unstable. This conclusion does not seem to be fully borne out by experiments; rather, experiments seem to show that stability is achieved when A is sufficiently large. The failure of the theory in this respect is most probably due to having disregarded effects of cavitation, which may occur at high eccentricity ratios when the whirl speed, as happens here, is different from $\omega/2$. We propose to examine the question again in a later paper.

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