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# Laplace Transform and Self-reciprocal Functions. (**)

1. – The integral

$$(1.1) \qquad \Phi(p) = p \int_0^\infty e^{-pt} f(t) \, dt , \qquad R(p) > 0 ,$$

is known as the LAPLACE transform, provided the integral on the right converges, and is symbolically denotes as

$$(1.2) \qquad \Phi(p) \doteqdot f(t);$$

$\Phi(p)$ is called the image and $f(t)$ the original.

S. C. MITRA and B. N. BOSE [4] have investigated the behaviour of either of the functions $\Phi(x)$ or $f(x)$ when the other has a self-reciprocal property in a certain HANKEL transforms, or as particular cases in sine or cosine transforms. Recently V. P. MAINRA [3] has investigated the behaviour of these functions when either of them is a transform under the kernel $\Phi^{\lambda}_{\mu,\nu}(x)$, which he defines as

$$\Phi^{\lambda}_{\nu,\mu}(x) = \int_0^\infty \tilde{\omega}_{\mu,\nu}(xy) \, y^{\frac{1}{2}} J_\lambda(y) \, dy ,$$

where

$$(1.3) \qquad \tilde{\omega}_{\mu,\nu}(x) = \sqrt{x} \int_0^\infty J_\mu(t) \, J_\nu\left(\frac{x}{t}\right) \frac{dt}{t} , \qquad R(\mu) > -\frac{1}{2}, \qquad R(\nu) > -\frac{1}{2} ,$$

and plays the role of a transform (WATSON [5]).

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The object of this paper is to investigate the behaviour of $f(t)$ or $\Phi(p)$ when either of them is a transform under the kernel $\widetilde{\omega}_{n_1, n_2, \dots, n_n}(x)$ defined as (BHATNAGAR [1])

$$\widetilde{\omega}_{n_1, n_2, \dots, n_n}(x) = \sqrt{x} \int_0^\infty \dots \int_0^\infty J_{n_1}(t_1) J_{n_2}(t_2) \dots \tag{1.4}$$

$$\dots J_{n_{n-1}}(t_{n-1}) J_{n_n}\left(\frac{x}{t_1 t_2 \dots t_{n-1}}\right) \frac{dt_1 \dots dt_{n-1}}{t_1 t_2 \dots t_{n-1}} .$$

The relations obtained here give an additional method of finding the operational images or the original when the transform of either of them are known under the various kernels viz. $\widetilde{\omega}_{\mu, \nu}(x)$, $\widetilde{\omega}_{\mu,\nu,\lambda}(x)$ and in general the kernel in (1.4).

2. – Let $f(t) \doteqdot \Phi(p)$. We know that

$$\frac{1}{t} J_\nu(1/ct) \doteqdot 2p\, J_\nu(\sqrt{(2p/c)})\, K_\nu(\sqrt{2p/c}), \quad R(\nu) > -\frac{1}{2} . \tag{2.1}$$

Applying GOLDSTEIN's theorem [2], we get

$$2 \int_0^\infty J_\nu(\sqrt{2t/c})\, k_\nu(\sqrt{2t/c})\, f(t)\, dt = \int_0^\infty J_\nu\left(\frac{1}{c\,t}\right) \Phi(t) \frac{dt}{t^2} ,$$

provided $f(t)$ and $\Phi(t)$ are continuous and integrable in $(0, \infty)$. Let us put $c = \frac{1}{p}$, and interprete assuming that $\frac{1}{p} \doteqdot x$, we get

$$\frac{1}{x} \int_0^\infty f(t)\, J_\nu\left(\frac{t}{x}\right) dt \doteqdot p \int_0^\infty \Phi(t)\, J_\nu\left(\frac{p}{t}\right) \frac{dt}{t^2} . \tag{2.2}$$

Let us write (2.2) as

$$f_1(x) \doteqdot \Phi_1(p). \tag{2.3}$$

Applying GOLDSTEIN's theorem again with (2.1) and (2.3), with $\nu$ replaced by $\mu$, and repeating the process of interpretation, we get

$$(2.4)\qquad \frac{1}{x}\int_0^\infty f_1(t)\, J_\mu\left(\frac{t}{x}\right) dt \doteqdot p\int_0^\infty \Phi_1(t)\, J_\mu\left(\frac{p}{t}\right)\frac{dt}{t^2}\,.$$

Substituting for $f_1(t)$ and $\Phi_1(t)$ from (2.2), we get

$$\frac{1}{x}\int_0^\infty J_\mu\left(\frac{t}{x}\right)\frac{dt}{t}\int_0^\infty f(y)\, J_\nu\left(\frac{y}{t}\right) dy \doteqdot p\int_0^\infty J_\mu\left(\frac{p}{t}\right)\frac{dt}{t}\int_0^\infty \Phi(y)\, J_\nu\left(\frac{t}{y}\right)\frac{dy}{y^2}\,,$$

changing the order of integrations, assuming it to be permissible, we get, after a slight change in variables

$$\frac{1}{\sqrt{x}}\int_0^\infty \frac{f(y)}{\sqrt{y}}\, dy\, \sqrt{\frac{y}{x}}\int_0^\infty J_\mu(t)\, J_\nu\left(\frac{y}{xt}\right)\frac{dt}{t} \doteqdot \sqrt{p}\int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\, dy\, \sqrt{py}\int_0^\infty J_\mu\left(\frac{py}{t}\right) J_\nu(t)\,\frac{dt}{t}\,.$$

Hence by (1.3), we have

$$(2.5)\qquad \frac{1}{\sqrt{x}}\int_0^\infty \frac{f(y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu}\left(\frac{y}{x}\right) dy \doteqdot \sqrt{p}\int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu}(py)\, dy\,.$$

Now let $\frac{1}{\sqrt{t}}\, g(1/t)$ be the transform of $\frac{1}{\sqrt{t}}\, f(t)$ with respect to the Kernel $\tilde{\omega}_{\mu,\nu}(t)$ and let

$$g(t) \doteqdot \psi(p)\,,$$

then we have from (2.5)

$$g(x) \doteqdot \sqrt{p}\int_0^\infty \Phi\left(\frac{1}{y}\right)/\sqrt{y}\;\tilde{\omega}_{\mu,\nu}(py)\, dy\,,$$

$$\frac{\Psi(p)}{\sqrt{p}} = \int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu}(py)\, dy$$

showing that $\psi(t)/\sqrt{t}$ and $\Phi\left(\frac{1}{t}\right)/\sqrt{t}$ are $\tilde{\omega}_{\mu,\nu}$ transforms of each other. Hence

Theorem 1: If

$$f(t) \doteqdot \Phi(p),$$

$$g(t) \doteqdot \psi(p)$$

and if $f(t)/\sqrt{t}$ and $g(1/t)/\sqrt{t}$ be $\tilde{\omega}_{\mu,\nu}$ transform of each other, then $\Phi(1/t)/\sqrt{t}$ and $\psi(t)/\sqrt{t}$ are also transform of each other, provided the integrals involved converge absolutely and $R(\mu) > -\frac{1}{2}$, $R(\nu) > -\frac{1}{2}$.

If we start with the alternative assumption that $\Phi\left(\frac{1}{t}\right)/\sqrt{t}$ and $\psi(t)/\sqrt{t}$ are $\tilde{\omega}_{\mu,\nu}$ transforms of each other, the theorem will then read as

Theorem 1 (A): If $f(t) \doteqdot \Phi(p)$, $g(t) \doteqdot \psi(p)$ and if $\Phi\left(\frac{1}{t}\right)/\sqrt{t}$ and $\psi(t)/\sqrt{t}$ be $\tilde{\omega}_{\mu,\nu}$ transforms of each other, then $f(t)/\sqrt{t}$ and $g\left(\frac{1}{t}\right)/\sqrt{t}$ are also $\tilde{\omega}_{\mu,\nu}$ transforms of each other. Next let us suppose that $\Phi\left(\frac{1}{t}\right)/\sqrt{t}$ be self-reciprocal in the $\tilde{\omega}_{\mu,\nu}$ transform then (2.5) gives

$$\frac{1}{\sqrt{x}} \int_0^\infty \frac{f(y)}{\sqrt{y}} \tilde{\omega}_{\mu,\nu}\left(\frac{y}{x}\right) \mathrm{d}y \doteqdot \Phi\left(\frac{1}{p}\right).$$

Further, if $\Phi\left(\frac{1}{p}\right) \doteqdot h(t)$, then

$$\int_0^\infty \frac{f(y)}{\sqrt{y}} \tilde{\omega}_{\mu,\nu}(xy)\, \mathrm{d}y = \frac{1}{\sqrt{x}}\, h\left(\frac{1}{x}\right)$$

showing that $f(t)/\sqrt{t}$ and $\frac{1}{\sqrt{t}}\, h\left(\frac{1}{t}\right)$ are $\tilde{\omega}_{\mu,\nu}$ transforms of each other. Hence

Theorem 2: If

$$f(t) \doteqdot \Phi(p),$$

$$h(t) \doteqdot \Phi\left(\frac{1}{p}\right)$$

and if $\Phi\left(\frac{1}{t}\right)/\sqrt{t}$ be self-reciprocal in the $\tilde{\omega}_{\mu,\nu}$ transform, then $\frac{1}{\sqrt{t}}\, f(t)$ and $\frac{1}{\sqrt{t}}\, h\left(\frac{1}{t}\right)$ are transforms of each other provided the integrals involved converge absolutely and $R(\mu) > -\frac{1}{2}$, $R(\nu) > -\frac{1}{2}$.

If instead of the above, we assume that $\frac{1}{\sqrt{t}}\, f(t)$ and $\frac{1}{\sqrt{t}}\, h\left(\frac{1}{t}\right)$ are $\tilde{\omega}_{\mu,\nu}$ transforms of each other, where

$$h(t) \doteqdot \Phi\left(\frac{1}{p}\right)$$

we get from (2.5)

$$h(x) \doteqdot \sqrt{p} \int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\, \tilde{\omega}_{\mu,\nu}(py)\, dy\,,$$

whence

$$\frac{1}{\sqrt{p}}\, \Phi\left(\frac{1}{p}\right) = \int_0^\infty \Phi\left(\frac{1}{y}\right)/\sqrt{y} \cdot \tilde{\omega}_{\mu,\nu}(py)\, dy$$

showing that $\frac{1}{\sqrt{x}}\, \Phi\left(\frac{1}{x}\right)$ is self-reciprocal in the $\tilde{\omega}_{\mu,\nu}$ transform.

Hence converse of Theorem 2 is also true, i.e.

Theorem 2 (A): If

$$f(t) \doteqdot \Phi(p)\,,$$

$$h(t) \doteqdot \Phi(1/p)$$

and, if $\frac{1}{\sqrt{t}}\, f(t)$ and $\frac{1}{\sqrt{t}}\, h(1/t)$ be $\tilde{\omega}_{\mu,\nu}$ transforms of each other, then $\frac{1}{\sqrt{t}}\, \Phi(1/t)$ is self-reciprocal in the $\tilde{\omega}_{\mu,\nu}$ transform, provided conditions in Theorem 2 are satisfied.

If in the alternative we assume that $f(t)/\sqrt{t}$ is self-reciprocal in the $\tilde{\omega}_{\mu,\nu}$ transform and that

$$f\left(\frac{1}{x}\right) \doteqdot \chi(p)\,,$$

we get from (2.5)

$$f\left(\frac{1}{x}\right) \doteqdot \sqrt{p}\int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu}(py)\,dy\,.$$

Hence

$$\frac{\chi(p)}{\sqrt{p}} = \int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu}(py)\,dy$$

showing that $\chi(x)/\sqrt{x}$ and $\Phi\left(\frac{1}{x}\right)/\sqrt{x}$ are $\tilde{\omega}_{\mu,\nu}$ transforms of each other. Hence

Theorem 3: If

$$f(t) \doteqdot \Phi(p)\,,$$

$$f\left(\frac{1}{t}\right) \doteqdot X(p)$$

and if $\frac{1}{\sqrt{t}} f(t)$ be self-reciprocal in the the $\tilde{\omega}_{\mu,\nu}$ transform, then $\chi(t)/\sqrt{t}$ and $\Phi\left(\frac{1}{t}\right)/\sqrt{t}$ are $\tilde{\omega}_{\mu,\nu}$ transforms of each other provided the integrals involved converge absolutely and $R(\mu) > -\frac{1}{2}$, $R(\nu) \geqslant -\frac{1}{2}$.

The converse of this can also be proved in the manner of Theorem 2.

**3.** – We had obtained the operational relation

$$\frac{1}{\sqrt{x}}\int_0^\infty \frac{f(y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu}\left(\frac{y}{x}\right)dy \doteqdot \sqrt{p}\int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu}(py)\,dy\,. \tag{3.1}$$

Let us denote this as

$$f_2(x) \doteqdot \Phi_2(p)\,. \tag{3.2}$$

Also from (2.4), we have

$$\frac{1}{x}\int_0^\infty f_1(t)\, J_\mu\left(\frac{t}{x}\right) \mathrm{d}t \doteqdot p \int_0^\infty \Phi_1(t)\, J_\mu\left(\frac{p}{t}\right)\frac{\mathrm{d}t}{t^2}, \tag{3.3}$$

where $f_1(x) \doteqdot \Phi_1(p)$ from (2.3).

In (3.3) replacing $f_1(t)$ by $f_2(t)$, $\Phi_1(t)$ by $\Phi_2(t)$, $\mu$ by $\lambda$, we get

$$\frac{1}{x}\int_0^\infty f_2(t)\, J_\lambda\left(\frac{t}{x}\right) \mathrm{d}t \doteqdot p \int_0^\infty \Phi_2(t)\, J_\lambda\left(\frac{p}{t}\right)\frac{\mathrm{d}t}{t^2};$$

substituting for $f_2(t)$ and $\Phi_2(t)$ from (3.1), we get

$$\frac{1}{x}\int_0^\infty J_\lambda\left(\frac{t}{x}\right)\frac{\mathrm{d}t}{\sqrt{t}}\int_0^\infty \frac{f(y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu}(y/t)\,\mathrm{d}y \doteqdot p\int_0^\infty J_\lambda\left(\frac{p}{t}\right)\frac{\mathrm{d}t}{t^{3/2}}\int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu}(ty)\,\mathrm{d}y,$$

changing the order of integration, assuming it to be permissible, we get, after a slight change in variables,

$$\frac{1}{\sqrt{x}}\int_0^\infty \frac{f(y)}{\sqrt{y}}\,\mathrm{d}y\int_0^\infty J_\lambda(t)\,\tilde{\omega}_{\mu,\nu}\left(\frac{y}{xt}\right)\frac{\mathrm{d}t}{\sqrt{t}} \doteqdot \sqrt{p}\int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\mathrm{d}y\int_0^\infty J_\lambda(t)\,\tilde{\omega}_{\mu,\nu}\left(\frac{py}{t}\right)\frac{\mathrm{d}t}{\sqrt{t}}. \tag{3.4}$$

Now Bahtnagar has shown that

$$\tilde{\omega}_{\mu,\nu,\lambda}(xy) = \int_0^\infty J_\lambda(t)\,\tilde{\omega}_{\mu,\nu}\left(\frac{xy}{t}\right)\frac{\mathrm{d}t}{\sqrt{t}} \qquad \left(\mu+\frac{1}{2},\ \nu+\frac{1}{2},\ \lambda+\frac{1}{2}>0\right)$$

plays the role of a transform.

Hence we get from (3.4)

$$\frac{1}{\sqrt{x}}\int_0^\infty \frac{f(y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu,\lambda}(y/x)\,\mathrm{d}y \doteqdot \sqrt{p}\int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu,\lambda}(py)\,\mathrm{d}y \;. \tag{3.5}$$

Let us write (3.5) as

$$f_3(x) \doteqdot \Phi_3(p) \,.$$

In (3.3) replacing $f_1(t)$ by $f_3(t)$, $\Phi_1(t)$ by $\Phi_3(t)$ and $\mu$ by $\xi$, we get

$$\frac{1}{x}\int_0^\infty f_3(t)\,J_\xi\left(\frac{t}{x}\right)\mathrm{d}t \doteqdot p\int_0^\infty \Phi_3(t)\,J_\xi\left(\frac{p}{t}\right)\frac{\mathrm{d}t}{t^2}\,.$$

Substituting for $f_3(t)$ and $\Phi_3(t)$ from (3.5), inverting the order of integrations, we get after a slight readjustment in variables

$$\frac{1}{\sqrt{x}}\int_0^\infty \frac{f(y)}{\sqrt{y}}\int_0^\infty J_\xi(t)\,\tilde{\omega}_{\mu,\nu,\lambda}\left(\frac{y}{xt}\right)\frac{\mathrm{d}t}{\sqrt{t}} \doteqdot \sqrt{p}\int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\mathrm{d}y\int_0^\infty J_\xi(t)\,\tilde{\omega}_{\mu,\nu,\lambda}\left(\frac{py}{t}\right)\frac{\mathrm{d}t}{\sqrt{t}}\,. \tag{3.6}$$

Now by definition a kernel

$$\tilde{\omega}_{\mu_1,\mu_2,\ldots,\mu_n}(xy) = \int_0^\infty J_{\mu_n}(t)\,\tilde{\omega}_{\mu_1,\mu_2,\ldots,\mu_{n-1}}\left(\frac{xy}{t}\right)\frac{\mathrm{d}t}{\sqrt{t}}\left(\mu_n+\frac{1}{2}>0 \text{ for } n=1,2,\ldots,n\right).$$

Hence (3.6) yelds

$$\frac{1}{\sqrt{x}}\int_0^\infty \frac{f(y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu,\lambda,\xi}(y/x)\,\mathrm{d}y \doteqdot \sqrt{p}\int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu,\lambda,\xi}(py)\,\mathrm{d}y\,. \tag{3.7}$$

Denoting (3.7) as

$$f_4(x) \doteqdot \Phi_4(p)\,,$$

replacing $f_1(t)$ by $f_4(t)$, $\Phi_1(t)$ by $\Phi_4(t)$ in (3.3) and repeating the above process over and over again, we get finally

$$(3.8) \qquad \frac{1}{\sqrt{x}}\int_0^\infty \frac{f(y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu,\lambda,\xi\,\ldots,\,\eta}\,(y/x)\,\mathrm{d}y \doteqdot \sqrt{p}\int_0^\infty \frac{\Phi(1/y)}{\sqrt{y}}\,\tilde{\omega}_{\mu,\nu,\lambda,\xi\,\ldots,\,\eta}\,(py)\,\mathrm{d}y\,.$$

Now it can be noticed that the form of (2.5), (3.5), (3.7) and (3.8) is exactly similar where the kernels occurring are $\tilde{\omega}_{\mu,\nu}$, $\tilde{\omega}_{\mu,\nu,\lambda}$, $\tilde{\omega}_{\mu,\nu,\lambda,\xi}$ and $\tilde{\omega}_{\mu,\nu,\lambda,\xi,\,\ldots,\,\eta}$ respectively. Hence the theorems which have been stated for the kernel $\tilde{\omega}_{\mu,\nu}$ are true for the other higher kernels and in general true for the kernel $\tilde{\omega}_{\mu_1,\mu_2,\,\ldots,\,\mu_n}(x)$.

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