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On the order and type of integral functions. (**)

In this paper we have investigated certain relationship between two or more integral functions. The results obtained involve the coefficients in the Taylor expansion of integral functions, their orders and types. The results are given in three sections.

Section I.

1. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an integral function of order ϱ $(0 \leqslant \varrho \leqslant \infty)$ and lower order λ $(0 \leqslant \lambda \leqslant \infty)$. It is known [**1**, p. 9] that $f(z)$ is an integral function of finite order ϱ $(0 \leqslant \varrho < \infty)$, if and only if,

$$\limsup_{n \to \infty} \frac{n \log n}{\log\{1/|c_n|\}} = \varrho . \tag{1.1}$$

It is also known [**2**, p. 1046] that if $f(z)$ is an integral function of lower order λ $(0 \leqslant \lambda \leqslant \infty)$, then

$$\liminf_{n \to \infty} \frac{n \log n}{\log\{1/|c_n|\}} \leqslant \lambda . \tag{1.2}$$

Further, if $f(z)$ is an integral function of order ϱ and lower order λ $(0 \leqslant \lambda \leqslant \infty)$ and $|c_n/c_{n+1}|$ is a non-decreasing function of n, for $n > n_0$, then [**2**, p. 1047]

$$\liminf_{n \to \infty} \frac{\log n}{\log |c_n/c_{n+1}|} = \liminf_{n \to \infty} \frac{n \log n}{\log\{1/|c_n|\}} = \lambda , \tag{1.3}$$

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and

$$\limsup_{n\to\infty} \frac{\log n}{\log |c_n/c_{n+1}|} = \varrho . \tag{1.4}$$

1.1 THEOREM 1. – *If $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be integral functions of finite orders ϱ_1, ϱ_2 respectively, then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where $\log\{1/|c_n|\} \sim |\sqrt{\log\{1/|a_n|\}\log\{1/|b_n|\}}|$, is an integral function, such that*

$$\sqrt{\varrho_1 \varrho_2} \geqslant \varrho ,$$

where ϱ is the order of $f(z)$.

Proof. – Using (1.1) for $f_1(z)$ and $f_2(z)$, we have, for an arbitrary $\varepsilon > 0$,

$$\frac{\log\{1/|a_n|\}}{n \log n} > \left(\frac{1}{\varrho_1} - \varepsilon\right),$$

for $n > n_1$, and

$$\frac{\log\{1/|b_n|\}}{n \log n} > \left(\frac{1}{\varrho_2} - \varepsilon\right),$$

for $n > n_2$. Hence, for sufficiently large n,

$$\frac{\log\{1/|a_n|\}\log\{1/|b_n|\}}{(n \log n)^2} > \left(\frac{1}{\varrho_1} - \varepsilon\right)\left(\frac{1}{\varrho_2} - \varepsilon\right).$$

Thus, if $\log\{1/|c_n|\} \sim |\sqrt{\log\{1/|a_n|\}\log\{1/|b_n|\}}|$, we have

$$\liminf_{n\to\infty} \frac{\log\{1/|c_n|\}}{n \log n} \geqslant \frac{1}{\sqrt{\varrho_1 \varrho_2}} .$$

Therefore, $\limsup_{n\to\infty} \frac{n \log n}{\log\{1/|c_n|\}}$ is positive and finite. Hence from (1.1) $f(z)$ is an integral function and

$$\sqrt{\varrho_1 \varrho_2} \geqslant \varrho ,$$

where ϱ is the order of $f(z)$.

COROLLARY. – If $f_k(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n$, $(k = 1, 2, \ldots, m)$, be m integral functions of finite orders $\varrho_1, \varrho_2, \ldots, \varrho_m$ respectively, then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where $\log\{1/|c_n|\} \sim |[\log\{1/|a_n^{(1)}|\}\log\{1/|a_n^{(2)}|\} \ldots \log\{1/|a_n^{(m)}|\}]^{1/m}|$, is an integral function, such that

$$\{\varrho_1 \varrho_2 \cdots \varrho_m\}^{1/m} \geqslant \varrho \,,$$

where $f(z)$ is of order ϱ.

1.2 THEOREM 2. – *If $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be integral functions of regular growth and of finite orders ϱ_1, ϱ_2 respectively, then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where $\log\{1/|c_n|\} \sim |\sqrt{\log\{1/|a_n|\}\log\{1/|b_n|\}}|$, is an integral function of regular growth and order ϱ, such that*

$$\sqrt{\varrho_1 \varrho_2} = \varrho \,.$$

Proof. – Using (1.1), we have

$$\lim_{n\to\infty} \frac{\log\{1/|a_n|\}}{n \log n} = \frac{1}{\varrho_1} \,,$$

and

$$\lim_{n\to\infty} \frac{\log\{1/|b_n|\}}{n \log n} = \frac{1}{\varrho_2} \,.$$

Hence,

$$\frac{1}{\varrho} = \lim_{n\to\infty} \frac{\log\{1/|c_n|\}}{n \log n} = \lim_{n\to\infty} \frac{|\sqrt{\log\{1/|a_n|\}\log\{1/|b_n|\}}|}{n \log n} = \frac{1}{\sqrt{\varrho_1 \varrho_2}}$$

since

$$\log\{1/|c_n|\} \sim |\sqrt{\log\{1/|a_n|\}\log\{1/|b_n|\}}| \,.$$

COROLLARY. – If $f_k(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n$, $(k=1, 2, \ldots, m)$, be m integral functions of regular growth and of finite orders $\varrho_1, \varrho_2, \ldots, \varrho_m$ respectively, then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where

$$\log\{1/|c_n|\} \sim |[\log\{1/|a_n^{(1)}|\}\log\{1/|a_n^{(2)}|\} \ldots \log\{1/|a_n^{(m)}|\}]^{1/m}|,$$

is an integral function of regular growth and order ϱ, such that

$$\{\varrho_1 \varrho_2 \ldots \varrho_m\}^{1/m} = \varrho .$$

1.3 THEOREM 3. – *If $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be integral functions of lower orders λ_1 $(0 \leqslant \lambda_1 \leqslant \infty)$, λ_2 $(0 \leqslant \lambda_2 \leqslant \infty)$ respectively and $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ be non-decreasing functions for $n > n_0$, then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where $\log\{1/|c_n|\} \sim |\sqrt{\log\{1/|a_n|\}\log\{1/|b_n|\}}|$, is an integral function, such that*

$$\sqrt{\lambda_1 \lambda_2} \leqslant \lambda ,$$

where λ is the lower order of $f(z)$.

P r o o f. – Since $f_1(z)$ and $f_2(z)$ are integral functions, therefore,

$$\liminf_{n\to\infty} |a_n|^{-1/n} = \liminf_{n\to\infty} |b_n|^{-1/n} = \infty .$$

Also, we have, for any $\varepsilon > 0$ and $n > n_1$

$$1/|a_n| > (R - \varepsilon)^n$$

and

$$1/|b_n| > (R - \varepsilon)^n, \quad \text{for} \quad n > n_2 .$$

Therefore, for sufficiently large n,

$$\log\{1/|a_n|\}\log\{1/|b_n|\} > \{n \log(R - \varepsilon)\}^2$$

Thus, if $\log\{1/|c_n|\} \sim |\sqrt{\log\{1/|a_n|\}\log\{1/|b_n|\}}|$, we have

$$\log\{1/|c_n|\} > n\log(R-\varepsilon)\,.$$

Therefore $\liminf_{n\to\infty} |c_n|^{-1/n} \geqslant \infty$ and hence $f(z)$ is an integral function. Using (1.3) for $f_1(z)$ and $f_2(z)$, we have, for any $\varepsilon > 0$,

$$\frac{\log\{1/|a_n|\}}{n\log n} < \left(\frac{1}{\lambda_1} + \varepsilon\right)$$

for $n > n_1$, and

$$\frac{\log\{1/|b_n|\}}{n\log n} < \left(\frac{1}{\lambda_2} + \varepsilon\right)$$

for $n > n_2$. Therefore, for sufficiently large n,

$$\frac{\log\{1/|a_n|\}\log\{1/|b_n|\}}{(n\log n)^2} < \left(\frac{1}{\lambda_1} + \varepsilon\right)\left(\frac{1}{\lambda_2} + \varepsilon\right)$$

Hence, using (1.2), we get

$$\frac{1}{\lambda} \leqslant \limsup_{n\to\infty} \frac{\log\{1/|c_n|\}}{n\log n} \leqslant \frac{1}{\sqrt{\lambda_1\lambda_2}}\,,$$

since $\log\{1/|c_n|\} \sim |\sqrt{\log\{1/|a_n|\}\log\{1/|b_n|\}}|$.

COROLLARY. – If $f_k(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n$, $(k = 1, 2, \ldots, m)$, be m integral functions of lower orders λ_k $(0 \leqslant \lambda_k \leqslant \infty)$, $(k = 1, 2, \ldots, m)$ respectively and each of the functions $|a_n^{(k)}/a_{n+1}^{(k)}|$, $(k = 1, 2, \ldots, m)$, be non-decreasing for $n > n_0$, then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where

$$\log\{1/|c_n|\} \sim |[\log\{1/|a_n^{(1)}|\}\log\{1/|a_n^{(2)}|\}\ldots\log\{1/|a_n^{(m)}|\}]^{1/m}|\,,$$

is an integral function, such that

$$\{\lambda_1\lambda_2\ldots\lambda_m\}^{1/m} \leqslant \lambda\,,$$

where λ is the lower order of $f(z)$.

1.4 THEOREM 4. – *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be integral functions of orders* ϱ_1, ϱ_2 *and lower orders* λ_1 $(0 \leqslant \lambda_1 \leqslant \infty)$, λ_2 $(0 \leqslant \lambda_2 \leqslant \infty)$ *respectively, and* $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ *be non-decreasing functions for* $n > n_0$, *then the function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $\log|c_n/c_{n+1}| \sim |\sqrt{\log|a_n/a_{n+1}| \log|b_n/b_{n+1}|}|$, *is an integral function of order* ϱ *and lower order* λ, *such that*

$$\sqrt{\varrho_1 \varrho_2} \geqslant \varrho, \tag{1.5}$$

and

$$\sqrt{\lambda_1 \lambda_2} \leqslant \lambda. \tag{1.6}$$

Proof. – Since $|a_n/a_{n+1}|$ is a non-decreasing function, using the relations (1.3) and (1.4) for the function $f_1(z)$, we have, for any $\varepsilon > 0$,

$$\frac{\log|a_n/a_{n+1}|}{\log n} < \left(\frac{1}{\lambda_1} + \varepsilon\right), \tag{1.7}$$

for $n > n_1$ and

$$\frac{\log|a_n/a_{n+1}|}{\log n} > \left(\frac{1}{\varrho_1} - \varepsilon\right), \tag{1.8}$$

for $n > n_2$.

Similarly, for the function $f_2(z)$, we have, for any $\varepsilon > 0$,

$$\frac{\log|b_n/b_{n+1}|}{\log n} < \left(\frac{1}{\lambda_2} + \varepsilon\right), \tag{1.9}$$

for $n > n_3$ and

$$\frac{\log|b_n/b_{n+1}|}{\log n} > \left(\frac{1}{\varrho_2} - \varepsilon\right), \tag{1.10}$$

for $n > n_4$.

From (1.7) and (1.9), we have

$$\limsup \frac{\log|c_n/c_{n+1}|}{\log n} \leqslant \frac{1}{\sqrt{\lambda_1 \lambda_2}},$$

since

$$\log |c_n/c_{n+1}| \sim |\sqrt{\log |a_n/a_{n+1}| \log |b_n/b_{n+1}|}| .$$

Hence,

$$\sqrt{\lambda_1 \lambda_2} \leqslant \lambda .$$

Similarly, from (1.8) and (1.10), we get

$$\sqrt{\varrho_1 \varrho_2} \geqslant \varrho .$$

COROLLARY 1. – The results of Theorem 4 can be extended to m integral functions.

COROLLARY 2. – If $f_1(z)$ and $f_2(z)$ are of regular growth, then so is $f(z)$ and

$$\sqrt{\varrho_1 \varrho_2} = \varrho .$$

This follows from (1.5) and (1.6) since, for functions of regular growth, $\lambda_1 = \varrho_1$, $\lambda_2 = \varrho_2$, and so

$$\varrho \leqslant \sqrt{\varrho_1 \varrho_2} = \sqrt{\lambda_1 \lambda_2} \leqslant \lambda .$$

But $\lambda \leqslant \varrho$, therefore, $\lambda = \varrho$.

COROLLARY 3. – If $f_k(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n$, $(k = 1, 2, \ldots, m)$, be m integral functions of regular growth and of orders $\varrho_1, \varrho_2, \ldots, \varrho_m$ respectively, then so is the function $f(z)$ and

$$\varrho_1 \varrho_2 \cdots \varrho_m = \varrho^m .$$

Section II.

2. Let $f(z)$ be an integral function of order ϱ . Let T and t be the type and lower type respectively of $f(z)$, defined as

$$\limsup_{r\to\infty} \frac{\log M(r)}{r^\varrho} = T, \quad (0 \leqslant T \leqslant \infty) , \tag{2.1}$$

and

$$\liminf_{r\to\infty} \frac{\log M(r)}{r^{\varrho}} = t, \quad (0 \leqslant t \leqslant T \leqslant \infty). \tag{2.2}$$

2.1 THEOREM 1. – *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be integral functions of the same order* ϱ $(0 < \varrho < \infty)$, *types* T_1 $(0 < T_1 < \infty)$ *and* T_2 $(0 < T_2 < \infty)$ *and maximum moduli* $M_1(r)$, $M_2(r)$ *respectively for* $|z| = r$, *then the integral function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $\log M(r) \sim \log\{M_1(r) M_2(r)\}$, $M(r)$ *being the maximum modulus of* $f(z)$, *is of order* ϱ *and type* T, *such that*

$$T \leqslant T_1 + T_2. \tag{2.3}$$

Proof. – Since $f_1(z)$ and $f_2(z)$ are integral functions of the same order ϱ, we have [**1**, p. 8]

$$\log M_1(r) = 0\,(r^{\varrho+\varepsilon})$$

and

$$\log M_2(r) = 0\,(r^{\varrho+\varepsilon}),$$

for $\varepsilon > 0$ and large r.

Thus, if $\log M(r) \sim \log\{M_1(r) M_2(r)\}$, we get $\log M(r) = 0\,(r^{\varrho+\varepsilon})$, for large r.

Hence, the integral function $f(z)$ is of order ϱ.

Using (2.1) for $f_1(z)$ and $f_2(z)$, we have, for any $\varepsilon > 0$,

$$\frac{\log M_1(r)}{r^{\varrho}} < (T_1 + \varepsilon/2),$$

for $r > r_1$, and

$$\frac{\log M_2(r)}{r^{\varrho}} < (T_2 + \varepsilon/2),$$

for $r > r_2$. Hence for sufficiently large r,

$$\frac{\log\{M_1(r) M_2(r)\}}{r^{\varrho}} < (T_1 + T_2 + \varepsilon).$$

Thus, if $\log M(r) \sim \log\{M_1(r) M_2(r)\}$, we have

$$T = \limsup_{r\to\infty} \frac{\log M(r)}{r^\varrho} \leqslant T_1 + T_2 .$$

COROLLARY. – If $f_k(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n$, $(k = 1, 2, \ldots, m)$, be m integral functions of the same order ϱ $(0 < \varrho < \infty)$, types $T_1, T_2, \ldots, T_m$ $(0 < T_k < \infty)$, $(k = 1, 2, \ldots, m)$, and maximum moduli $M_1(r), M_2(r), \ldots, M_m(r)$ respectively for $|z| = r$, then the integral function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where $\log M(r) \sim \log\{M_1(r) M_2(r) \ldots M_m(r)\}$, $M(r)$ being the maximum modulus of $f(z)$, is of order ϱ and type T, such that

$$T \leqslant T_1 + T_2 + \ldots + T_m .$$

2.2 THEOREM 2. – *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be integral functions of the same order* ϱ $(0 < \varrho < \infty)$, *lower types* t_1 $(0 < t_1 < \infty)$ *and* t_2 $(0 < t_2 < \infty)$ *and maximum moduli* $M_1(r)$, $M_2(r)$ *respectively for* $|z| = r$, *then the integral function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $\log M(r) \sim \log\{M_1(r) M_2(r)\}$, $M(r)$ *being the maximum modulus of* $f(z)$, *is of order* ϱ *and lower type* t, *such that*

$$t \geqslant t_1 + t_2 . \tag{2.4}$$

If we follow the method of proof of Theorem 1 and use (2.2) instead of (2.1), the result follows.

COROLLARY 1. – The result of Theorem 2 can be extended to m integral functions.

COROLLARY 2. – If $f_1(z)$ and $f_2(z)$ are of perfectly regular growth, then so is the integral function $f(z)$ and

$$T = T_1 + T_2 .$$

This follows from (2.3) and (2.4), since, for functions of perfectly regular growth, $t_1 = T_1$, $t_2 = T_2$, and so

$$T \leqslant T_1 + T_2 = t_1 + t_2 \leqslant t$$

But $t \leqslant T$, therefore, $t = T$.

COROLLARY 3. – If $f_k(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n$, $(k = 1, 2, \ldots, m)$, be m integral functions of perfectly regular growth, same order ϱ $(0 < \varrho < \infty)$, types $T_1, T_2, \ldots, T_m$ $(0 < T_k < \infty)$, for $k = 1, 2, \ldots, m$ and maximum moduli $M_1(r), M_2(r), \ldots, M_m(r)$ respectively for $|z| = r$, so is the integral function $f(z)$ and

$$T = T_1 + T_2 + \ldots + T_m .$$

Section III.

3. It is known [1, p. 11] that the necessary and sufficient condition for an integral function $f(z)$ of order ϱ $(0 < \varrho < \infty)$, be of type T $(0 < T < \infty)$ is

$$\limsup_{n\to\infty} \left\{ \frac{1}{e\varrho}\, n\, |c_n|^{\varrho/n} \right\} = T . \tag{3.1}$$

Further, if $|c_n/c_{n+1}|$ is a non-decreasing function for all large n, then [3, p. 45]

$$\liminf_{n\to\infty} \left\{ \frac{1}{e\varrho}\, n\, |c_n|^{\varrho/n} \right\} = t , \tag{3.2}$$

where t denotes lower type of $f(z)$.

3.1 THEOREM 1. – *If $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be integral functions of the same order ϱ $(0 < \varrho < \infty)$, lower types t_1 $(0 < t_1 < \infty)$, t_2 $(0 < t_2 < \infty)$ respectively and $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ be non-decreasing functions for $n > n_0$, then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where $|c_n| \sim |\sqrt{|a_n|\,|b_n|}|$, is an integral function of order ϱ and lower type t, such that*

$$t \geqslant \sqrt{t_1 t_2} .$$

P r o o f. – Using (3.2) for functions $f_1(z)$ and $f_2(z)$, we have, for any $\varepsilon > 0$ and $n > n_1$,

$$\frac{n}{e\varrho} |a_n|^{\varrho/n} > (t_1 - \varepsilon)$$

and

$$\frac{n}{e\varrho} |b_n|^{\varrho/n} > (t_2 - \varepsilon)$$

for $n > n_2$. Hence, for sufficiently large n,

$$\left(\frac{n}{e\varrho}\right)^2 | a_n |^{\varrho/n} | b_n |^{\varrho/n} > (t_1 - \varepsilon)(t_2 - \varepsilon),$$

or

$$\frac{n}{e\varrho}\{ | \sqrt{| a_n || b_n |} | \}^{\varrho/n} > \sqrt{(t_1 - \varepsilon)(t_2 - \varepsilon)}.$$

Thus, if $| c_n | \sim | \sqrt{| a_n || b_n |} |$, we have

$$\liminf_{n\to\infty} \left\{ \frac{n}{e\varrho} | c_n |^{\varrho/n} \right\} = \liminf_{n\to\infty} \left[\frac{n}{e\varrho} \left\{ | \sqrt{| a_n || b_n |} | \right\}^{\varrho/n} \right] \geqslant \sqrt{t_1 t_2},$$

on using lemma 1. [4, p. 53].

Since $f_1(z)$ and $f_2(z)$ are integral functions, therefore,

$$\liminf_{n\to\infty} | a_n |^{-1/n} = \liminf_{n\to\infty} | b_n |^{-1/n} = \infty.$$

Also, we have, for any $\varepsilon > 0$ and $n > n_1$,

$$| a_n | < 1/(R - \varepsilon)^n$$

and

$$| b_n | < 1/(R - \varepsilon)^n,$$

for $n > n_2$.

Therefore, for sufficiently large n,

$$| a_n | | b_n | < 1/(R - \varepsilon)^{2n}.$$

Thus, if $| c_n | \sim | \sqrt{| a_n || b_n |} |$, we have

$$| c_n | < 1/(R - \varepsilon)^n$$

Therefore $\liminf_{n\to\infty} | c_n |^{-1/n} \geqslant \infty$, hence $f(z)$ is an integral function and

$$t \geqslant \sqrt{t_1 t_2},$$

where $f(z)$ is of lower type t.

Srivastava [**4**, p. 54] has proved the following result for type T of $f(z)$

$$T \leqslant \sqrt{T_1 T_2},$$

where $f_1(z)$ and $f_2(z)$ are of types T_1 and T_2 respectively.

COROLLARY. – The result of the above Theorem can be extended to m integral functions.

3.2 THEOREM 2. – *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be integral functions of finite orders* ϱ_1, ϱ_2 *and types* T_1 $(0 < T_1 < \infty)$, T_2 $(0 < T_2 < \infty)$ *respectively, then the function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $|c_n| \sim |a_n| |b_n|$, *is an integral function of order* ϱ *and type* T, *such that*

$$\left(\frac{T}{\sigma}\right)^{\sigma} \leqslant \left(\frac{T_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{T_2}{\sigma_2}\right)^{\sigma_2} \tag{3.3}$$

provided

$$\sigma = \varrho^{-1},\ \sigma_1 = \varrho_1^{-1},\ \sigma_2 = \varrho_2^{-1} \ \text{ and } \ \sigma = \sigma_1 + \sigma_2 .$$

Proof. – Since $f_1(z)$ and $f_2(z)$ are integral functions, therefore,

$$\liminf_{n\to\infty} |a_n|^{-1/n} = \liminf_{n\to\infty} |b_n|^{-1/n} = \infty .$$

Also $|c_n| \sim |a_n| |b_n|$, therefore,

$$\liminf_{n\to\infty} |c_n|^{-1/n} \geqslant \liminf_{n\to\infty} |a_n|^{-1/n} \times \liminf_{n\to\infty} |b_n|^{-1/n} = \infty$$

and hence $f(z)$ is an integral function.

Further, we have

$$n^{1/\varrho} |c_n|^{1/n} \sim n^{1/\varrho} |a_n|^{1/n} |b_n|^{1/n} <$$

$$< (1 + \varepsilon)\, n^{\sigma_1+\sigma_2} |a_n|^{1/n} |b_n|^{1/n}, \qquad \varepsilon > 0, \quad n > n_0 .$$

Taking limits of both the sides, we get

$$\left(\frac{T}{\sigma}\right)^{\sigma} \leqslant \left(\frac{T_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{T_2}{\sigma_2}\right)^{\sigma_2},$$

since

$$\limsup_{n\to\infty} n^{1/\varrho} \, | \, c_n \, |^{1/n} = \left(\frac{eT}{\sigma}\right)^{\sigma} ,$$

$$\limsup_{n\to\infty} n^{\sigma_1} \, | \, a_n \, |^{1/n} = \left(\frac{eT_1}{\sigma_1}\right)^{\sigma_1}, \quad \limsup_{n\to\infty} n^{\sigma_2} \, | \, b_n \, |^{1/n} = \left(\frac{eT_2}{\sigma_2}\right)^{\sigma_2}.$$

NOTE. – WILSON [**5**, p. 422] has proved this result by taking $f(z) = \sum_{n=0}^{\infty} a_n \, b_n \, z^n$.

COROLLARY. – The result of the above Theorem can be extended to m intergal functions.

3.3 THEOREM 3. – *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be integral functions of finite orders* ϱ_1, ϱ_2, *lower types* t_1 $(0 < t_1 < \infty)$, t_2 $(0 < t_2 < \infty)$ *respectively and* $| a_n / a_{n+1} |$, $| b_n / b_{n+1} |$ *be non-decreasing functions for* $n > n_0$, *then the function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $| c_n | \sim | a_n | \, | b_n |$, *is an integral function of order* ϱ *and lower type* t, *such that*

$$\left(\frac{t}{\sigma}\right)^{\sigma} \geqslant \left(\frac{t_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{t_2}{\sigma_2}\right)^{\sigma_2}, \tag{3.4}$$

provided $\sigma = \varrho^{-1}$, $\sigma_1 = \varrho_1^{-1}$ *and* $\sigma_2 = \varrho_2^{-1}$.

Proof. – It can be proved as in Theorem 2 that $f(z)$ is an integral function. From the asymptotic behaviour of the coefficients, we have

$$n^{1/\varrho} \, | \, c_n \, |^{1/n} \sim n^{1/\varrho} \, | \, a_n \, |^{1/n} \, | \, b_n \, |^{1/n}$$

$$> (1 - \varepsilon) \, n^{1/\varrho} \, | \, a_n \, |^{1/n} \, | \, b_n \, |^{1/n}, \qquad \varepsilon > 0, \quad n > n_0,$$

$$\geqslant (1 - \varepsilon) \left\{ n^{\sigma_1 + \sigma_2} \, | \, a_n \, |^{1/n} \, | \, b_n \, |^{1/n} \right\},$$

since $\sigma \geqslant \sigma_1 + \sigma_2$ [**6**, p. 25].

Taking limits of both the sides, we get

$$\left(\frac{t}{\sigma}\right)^{\sigma} \geqslant \left(\frac{t_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{t_2}{\sigma_2}\right)^{\sigma_2},$$

since

$$\liminf_{n\to\infty} n^{1/\varrho} \mid c_n \mid^{1/n} = \left(\frac{et}{\sigma}\right)^{\sigma},$$

$$\liminf_{n\to\infty} n^{\sigma_1} \mid a_n \mid^{1/n} = \left(\frac{et_1}{\sigma_1}\right)^{\sigma_1}, \quad \liminf_{n\to\infty} n^{\sigma_2} \mid b_n \mid^{1/n} = \left(\frac{et_2}{\sigma_2}\right)^{\sigma_2}.$$

COROLLARY 1. – The result of Theorem 3 can be extended to m integral functions.

COROLLARY 2. – If $f_1(z)$ and $f_2(z)$ are of perfectly regular growth, then so is $f(z)$ and

$$\left(\frac{T}{\sigma}\right)^{\sigma} = \left(\frac{T_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{T_2}{\sigma_2}\right)^{\sigma_2}.$$

This follows from (3.3) and (3.4), since, for functions of perfectly regular growth, $t_1 = T_1$, $t_2 = T_2$, and so

$$\left(\frac{T}{\sigma}\right)^{\sigma} \leqslant \left(\frac{T_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{T_2}{\sigma_2}\right)^{\sigma_2} = \left(\frac{t_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{t_2}{\sigma_2}\right)^{\sigma_2} \leqslant \left(\frac{t}{\sigma}\right)^{\sigma}$$

But $t \leqslant T$, therefore, $t = T$.

COROLLARY 3. – If $f_k(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n$, $(k = 1, 2, \dots, m)$, be m integral functions of perfectly regular growth, of finite orders ϱ_k and types T_k $(0 < T_k < \infty)$ for $k = 1, 2, \dots, m$ respectively, then so is the function $f(z)$ and

$$\left(\frac{T}{\sigma}\right)^{\sigma} = \left(\frac{T_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{T_2}{\sigma_2}\right)^{\sigma_2} \cdots \left(\frac{T_m}{\sigma_m}\right)^{\sigma_m}.$$

3.4 THEOREM 4. – *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be integral functions of finite orders* ϱ_1, ϱ_2, *types* T_1 $(0 < T_1 < \infty)$, T_2 $(0 < T_2 < \infty)$, *lower types* t_1 $(0 < t_1 < \infty)$, t_2 $(0 < t_2 < \infty)$ *respectively and* $\mid a_n/a_{n+1} \mid$, $\mid b_n/b_{n+1} \mid$ *be non-decreasing functions for* $n > n_0$, *then the function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where*

$| c_n | \sim | a_n | | b_n |$, *is an integral function of order* ϱ, *type* T *and lower type* t, *such that*

$$\left(\frac{T}{\sigma}\right)^{\sigma} \geqslant \frac{\left(\frac{T_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{t_2}{\sigma_2}\right)^{\sigma_2}}{\left(\frac{t_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{T_2}{\sigma_2}\right)^{\sigma_2}} \geqslant \left(\frac{t}{\sigma}\right)^{\sigma},$$

provided $\sigma = \varrho^{-1}$, $\sigma_1 = \varrho_1^{-1}$, $\sigma_2 = \varrho_2^{-1}$ *and* $\sigma = \sigma_1 + \sigma_2$.

Proof. – It is well known that for two non-negative functions $f(x)$ and $g(x)$

$$\text{(3.5)} \quad \limsup \{ f(x)\, g(x) \} \geqslant \frac{\limsup f(x) \times \liminf g(x)}{\liminf f(x) \times \limsup g(x)} \geqslant \liminf \{ f(x)\, g(x) \}.$$

Here we have,

$$n^{1/\varrho} | c_n |^{1/n} \sim n^{1/\varrho} | a_n |^{1/n} | b_n |^{1/n}.$$

Using (3.5), we get

$$\left(\frac{T}{\sigma}\right)^{\sigma} \geqslant \frac{\left(\frac{T_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{t_2}{\sigma_2}\right)^{\sigma_2}}{\left(\frac{t_1}{\sigma_1}\right)^{\sigma_1} \left(\frac{T_2}{\sigma_2}\right)^{\sigma_2}} \geqslant \left(\frac{t}{\sigma}\right)^{\sigma},$$

since

$$\lim_{n\to\infty} {\sup \atop \inf} \; n^{1/\varrho} \, | c_n |^{1/n} = \frac{\left(\frac{eT}{\sigma}\right)^{\sigma}}{\left(\frac{et}{\sigma}\right)^{\sigma}},$$

$$\lim_{n\to\infty} {\sup \atop \inf} \; n^{\sigma_1} | a_n |^{1/n} = \frac{\left(\frac{eT_1}{\sigma_1}\right)^{\sigma_1}}{\left(\frac{et_1}{\sigma_1}\right)^{\sigma_1}}, \quad \lim_{n\to\infty} {\sup \atop \inf} \; n^{\sigma_2} | b_n |^{1/n} = \frac{\left(\frac{eT_2}{\sigma_2}\right)^{\sigma_2}}{\left(\frac{et_2}{\sigma_2}\right)^{\sigma_2}}.$$

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