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On the Foundations of Macroscopic Theories of Creep. (**)

1. - Generalities.

The hypotheses wich form the basis of the macroscopic theories of creep are suggested by the results of experiments which are simple in principle: for instance experiments of simple pull of a rod-shaped specimen under constant stress P and at constant and uniform temperature θ. The application of a tensile load induces an instantaneous elongation in a test piece; if the stress P is not too high this elongation is purely elastic and would disappear on removal of the load. But, if the temperature θ is sufficiently high and the load is not removed, a slow *creep* follows the elastic deformation. Let us call L the original length of the specimen and $\Delta^{(e)} L$ the elastic elongation; if $\Delta^{(e)} L$ is much smaller than L the ratio $\Delta^{(e)} L/L$ is a convenient measure of the elastic strain $\varepsilon^{(e)}$, which is related to the stress by Hooke's law

$$\varepsilon^{(e)} = P/E_\theta, \tag{1.1}$$

where E_θ is the modulus of elasticity of the specimen at the temperature θ.

A measure $\varepsilon^{(p)}$ for the permanent strain can be defined in a manner similar to $\varepsilon^{(e)}$. $\varepsilon^{(p)}$ is found to increase with time at first rapidly (first stage creep); then, for a period which is often very long, at an almost constant minimum rate (second stage creep), and finally rapidly again (third stage creep). Qualitatively this behaviour is common to many metals and alloys in relatively wide ranges of temperature and stress; but quantitatively the dependence of $\varepsilon^{(p)}$

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on time is extremely sensitive both to changes of composition in the material and to P and θ. If one could duplicate the experiments on specimens of exactly defined composition and structure one would undoubtedly find a uniquely defined dependence of $\varepsilon^{(p)}$ on θ, P and t; but in practice the experimental results show often a considerable amount of scatter. However, no allowance is made for this fact in the present analysis; rather, for any structural member under constant uniaxial tension P and at constant uniform temperature θ, $\varepsilon^{(p)}$ is taken as a well defined function of time: explicitly

$$\varepsilon^{(p)} = \frac{P}{E_\theta}\,\alpha_\theta\,(P;\ t)\,. \tag{1.2}$$

Here the factor P/E_θ is introduced so that the *creep function* α_θ represents the fractional increase of strain over and above the elastic strain.

Even when the difficulties associated with the scatter of experimental results are disregarded, formula (1.2) cannot be by any means considered as significant for creep phenomena as Hooke's law (1.1) is for elastic ones. In the elastic range the proportionality between stress and strain has general validity; a direct (non-linear) relation between stress and creep strain, as expressed by eqn (1.2), is true only on first loading: for instance, formula (1.2) predicts the value zero for $\varepsilon^{(p)}$ when $P = 0$, but the creep strain does not vanish on unloading.

Analogy with phenomena of plastic deformation leads one to assume that the *creep strain rate* $\dfrac{d\varepsilon^{(p)}}{dt}$, not $\varepsilon^{(p)}$, is directly defined by the instantaneous mechanical condition of the specimen; say

$$\frac{d\varepsilon^{(p)}}{dt} = \frac{P}{E_\theta T_\theta}\,,$$

where T_θ can be called the *time function* of the material, as its dimension is that of a time.

Such an assumption is plausible; but it is difficult to decide on the status of the parameters which define the mechanical condition of the specimen. The time function T_θ certainly depends on the stress, but seems to be also governed by the history of the specimen (cfr. for instance, [1], [2], [3]); there is no contradiction between this statement and what has been said above (as to the dependence of the creep rate on the instantaneous mechanical condition of the specimen only) if we assume that previous mechanical history has effect only through the total creep strain $\varepsilon^{(p)}$. On the contrary the consequences of thermal history may be more complex.

The effects of creep strain, time and temperature on the creep rate may be to some extent explained if the microscopic mechanism of creep is explored. There is firstly the thermal-action factor: creep resistance may be influenced by metallurgical changes induced when keeping the specimen at high temperature, as for instance by carbide spheroidization in carbon steels [4]. If the changes arise from a metastable condition and do not involve a constitutional modification, T_θ does not depend on time and temperature separately but rather on the combination $t \exp(-A/k\theta)$ (t time at temperature, A activation energy of the change, k Boltzmann constant) if θ is constant ([5], Appendix I and [6], Appendix 2), or else on the integral $\int_0^t \exp(-A/k\theta)\, dt$ if θ varies with time [7]. In complex alloys, however, the effect of temperature is more involved [8]. Nevertheless, when studying structures under isothermal conditions, it will be enough to account for the fact that T_θ may depend, coeteris paribus, on the total time at temperature, being possibly influenced by heating preliminary to loading.

Actually it was believed for some time that thermal action was the overriding factor in creep problems [5], [9]; more recently, however, the effects of strain-hardening have been emphasized [10], [11]. As we have already remarked above, in a macroscopic theory these effects can be accounted for by assuming that T_θ depends on the total creep strain $\varepsilon^{(p)}$.

In conclusion we will assume here that T_θ depends on P, $\varepsilon^{(p)}$ and t only, if the temperature θ is kept constant, and we will write the fundamental relation between creep-strain rate and stress in the form

$$\frac{d\varepsilon^{(p)}}{dt} = \frac{P}{E_\theta T_\theta(P;\, \varepsilon^{(p)};\, t)}\,. \tag{1.3}$$

2. — Significance of tension and relaxation tests.

The qualitative conclusions of Sect. **1**, summarized by eqn (1.3) must be made more definite quantitatively before applications are possible. The problem is that of giving explicitly T_θ as a function of P, $\varepsilon^{(p)}$ and t. To this end sufficient information is not yet available, unfortunately: most experiments on creep are carried out at constant stress and therefore lead to a more or less complete knowledge of the function α_θ (see eqn (1.2)). But this knowledge does not allow a unique determination of T_θ; this fact can be realized most easily if a mathematical point of view is taken.

Records of the relative elongation of a specimen under constant stress (i.e. records of the function $P\alpha_\theta/E_\theta$) versus time can be interpreted as records of solutions of the differential equation (1.3), precisely of solutions which correspond to the initial condition

$$\varepsilon^{(p)}(0) = 0 .$$

But in general to reconstruct the right-hand side of a differential equation (i.e., in the present instance, to determine T_θ) knowledge of a set of solutions, which correspond to different initial conditions, is necessary. If only α_θ is known, $T_\theta(P, \varepsilon^{(p)}, t)$ is determined only on the points of the octant $(P > 0, \varepsilon^{(p)} > 0, t > 0)$ of the space $(P, \varepsilon^{(p)}, t)$, which belong to the surface $\sum_1$

$$\varepsilon^{(p)} = (P/E_\theta)\,\alpha_\theta(P, t)$$

and over that surface one has

$$T_\theta[P, (P/E_\theta)\alpha, t] = \left[\frac{\partial\alpha_\theta}{\partial t}\right]^{-1} . \tag{2.1}$$

For a complete determination of T_θ experiments under stepwise varying stress are necessary. Precisely, to determine T_θ at a point $(\overline{P}, \overline{\varepsilon}^{(p)}, \overline{t})$ one should first reach through an ordinary tension test the point Q, which belongs to $\sum_1$ and whose last two co-ordinates are $\overline{\varepsilon}^{(p)}$, $\overline{t}$; i.e., if $\overline{P}_1$ is the first co-ordinate of Q, in the first stage of the experiment a test specimen should be stressed at a constant level $\overline{P}_1$ for the time $\overline{t}$. Then the stress should be rapidly changed to $\overline{P}$ and the ensuing creep rate recorded; T_θ could be finally calculated using formula (1.3). Of course, if $\overline{P}$ is less than $\overline{P}_1$, in particular if $\overline{P}$ is zero, phenomena of creep recovery may be involved (see, for instance, [**1**], Ch. 6).

Very little information is at the present available on experiments of the type just described [**12**]; nor is it possible to overcome all difficulties caused by lack of data, using the results of relaxation tests, that is tests where the total strain of a rod-shaped specimen in tensions is kept constant and the stress allowed to vary. Calling ε the total strain

$$\varepsilon = \varepsilon^{(e)} + \varepsilon^{(p)}$$

one gets from eqns (1.1), (1.3)

$$E_\theta \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\mathrm{d}P}{\mathrm{d}t} + \frac{P}{T_\theta(P;\, \varepsilon^{(p)};\, t)} . \tag{2.2}$$

When ε is constant, equal, say, to ε^*, the creep strain can be expressed by the difference

$$\varepsilon^{(p)} = \varepsilon^* - (P/E_\theta) \tag{2.3}$$

and eqn (2.2) becomes

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{P}{T_\theta[P;\, \varepsilon^* - (P/E_\theta);\, t]}\,. \tag{2.4}$$

Therefore records of experiments of stress relaxation can be interpreted as records of solutions of the differential equation (2.4) in P. These records are usually expressed in the form

$$P = E_\theta\, \varepsilon^* \left\{ 1 - \beta_\theta\, (\varepsilon^*,\, t) \right\}, \tag{2.5}$$

where β_θ is a positive function of t (increasing from 0 to 1 when t increases from 0 to ∞) which is called the *relaxation function* ([**13**], Art. 38).

It is obvious from eqns (2.3), (2.5) that all characteristic lines of relaxation tests in the space $(P,\, \varepsilon^{(p)},\, t)$ belong to the surface $\sum_2$ defined by the equation

$$P = (E_\theta\, \varepsilon^{(p)} + P) \left\{ 1 - \beta_\theta\, [\varepsilon^{(p)} + (P/E_\theta),\, t] \right\}. \tag{2.6}$$

It follows that relaxation tests can determine T_θ only for values of its arguments which are co-ordinates of points belonging to $\sum_2$. For such values of P, $\varepsilon^{(p)}$ and t, T_θ is given by the formula

$$T_\theta = (P/E_\theta) \left\{ [\varepsilon^{(p)} + (P/E_\theta)] \cdot \frac{\partial}{\partial t}\, \beta[\varepsilon^{(p)} + (P/E_\theta),\, t] \right\}^{-1}. \tag{2.7}$$

It may be noted incidentally from eqn (2.4) that the relaxation phenomena explored here are of the type shown by viscoelastic materials (See, for instance, [**13**], Ch. 9). Here, however, the coefficient which multiplies P on the right-hand side is not a constant but depends on stress and time (as well as on ϑ). If T_θ were constant its significance would be that of a relaxation time, i.e. the time taken by the stress in a specimen under constant strain to be reduced to $1/e$ of its initial value. As it is, the qualitative character of eqn (2.4) remains unaltered but the decrease of stress with time does not necessarily follow an exponential law.

3. – Simplifying hypotheses: time hardening, strain hardening, constant creep rate.

In Section **2** we have shown that the usual tension and relaxation tests at constant stress and constant strain respectively determine the time function $T_\theta(P; \varepsilon^{(p)}; t)$ only on the two surfaces $\sum_1$ and $\sum_2$ of the octant ($P > 0$, $\varepsilon^{(p)} > 0$, $t > 0$) of the space (P, $\varepsilon^{(p)}$, t). Similar compression tests give the values of T_θ on analogous surfaces lying in the octant ($P < 0$, $\varepsilon^{(p)} < 0$, $t > 0$). On the other hand the information available at present on creep behaviour originates almost exclusively from tests of the type just mentioned; it must be thus considered as far from complete.

Fortunately complete information on the time function is not necessary for the usual applications; in most pratical cases the initial loading is followed only by a limited variation of stress or conversely the initial straining is followed only by small changes in strain. For this reason only the values of T_θ in a neighbourhood of $\sum_1$ or $\sum_2$ are needed and these values can be obtained with some confidence by extrapolation.

Extrapolation may be achieved in many ways: for instance, trivially, by fitting a function of the type

$$A P^{\alpha} (\varepsilon^{(p)})^{\beta} t^{\gamma}$$

to the values of T_θ on $\sum_1$ (or $\sum_2$) and then assuming that the range of validity of the approximation is extended to a neighbourhood of $\sum_1$ (or $\sum_2$). However, one can take better advantage of the fact that there is a certain degree of arbitrariness in the choice of the method of extrapolation; having in mind future calculations one can adopt a method which will make these calculations simpler.

In this respect it is important to note that on $\sum_1$ (and $\sum_2$) T_θ can be considered as a function of two variables only [P, t or P, $\varepsilon^{(p)}$ or t, $\varepsilon^{(p)}$]. The remark suggests a simple process of extrapolation, which is no more arbitrary than other processes and leads to quite easier developments: it consists of assuming that the dependence of T_θ on two variables only remains approximately true in a neighbourhood of $\sum_1$ (and $\sum_2$). Although any one of the three possible couples of variables may appear in principle equally good, it is usual to choose t, P or P, $\varepsilon^{(p)}$ because T_θ is found to vary more rapidly with P than with t or $\varepsilon^{(p)}$.

It might be remarked at this point that in most technical applications factors of safety are chosen so that tertiary creep is avoided during the lifetime of structures; hence, in the periods of time of interest, creep rates are found to decrease steadily. This effect is described as a time-hardening or as a strain-hardening phenomenon according to whether greatest importance is ascribed to the influence of time or to the influence of strain. It is then natural to call *time hardening hypothesis* the assumption that T_θ depends on t and P alone and to call *strain hardening hypothesis* the alternative assumption, although their acceptance may not be restricted to the range of primary and secondary creep.

In some instances secondary creep is by far the most important feature of the phenomenon of flow, both because the deformation associated with primary creep is relatively small and because, within the limited interval of time under investigation, the onset of tertiary creep is avoided; then it is customary to reduce further the complexity of the developments (to some extent at the expense of accuracy) by assuming that T_θ depends on P only. The assumption is referred to usually as the *hypothesis of constant creep rate.*

Some of the implications of the hypotheses listed above may be appreciated by comparing the ensuing predictions on the behaviour of a rod under varying load. The simplest case of a specimen under a stress P_1 for $0 \leqslant t < t_1$ and a stress P_2 for $t_1 < t < t_2$ (P_1 and P_2 having the same order of magnitude and, in particular, having the same sign) will be considered, assuming that the functions $\alpha_\theta\,(P_1,\ t)$ and $\alpha_\theta\,(P_2,\ t)$ are known for $0 \leqslant t < t_2$.

Under the time-hardening hypothesis the differential equation (1.3) degenerates in the form

$$\frac{\mathrm{d}\varepsilon^{(p)}}{\mathrm{d}t} = \frac{P}{E_\theta T_\theta(P;\,t)}; \tag{3.1}$$

if $E(t)$ is a solution of this equation so is also $E\,(t) + C$, where C is a constant. In our case, where P changes from P_1 to P_2 at $t = t_1$, $\varepsilon^{(p)}$ is given by

$$\varepsilon^{(p)}\,(t) = (P_1/E_\theta)\,\alpha_\theta\,(P_1;\ t), \quad \text{for} \quad 0 \leqslant t \leqslant t_1, \tag{3.2}$$

and by

$$\varepsilon^{(p)}\,(t) = (P_2/E_\theta)\,[\alpha_\theta\,(P_2;\ t) - \alpha_\theta\,(P_2;\ t_1)] + \\ + (P_1/E_\theta)\,\alpha_\theta\,(P_1;\ t_1), \quad \text{for} \quad t_1 \leqslant t < t_2\,. \tag{3.3}$$

In fact formula (3.2) is true by definition; on the other hand the expression on the right-hand side of formula (3.3) is a solution of eqn (3.1) by the remark just made and reduces to $(P_1/E_\theta)\,\alpha_\theta\,(P_1;\, t_1)$ for $t = t_1$, thus ensuring continuity.

Under the strain-hardening hypothesis eqn (1.3) becomes

$$\frac{\mathrm{d}\varepsilon^{(p)}}{\mathrm{d}t} = \frac{P}{E_\theta T_\theta(P;\,\varepsilon^{(p)})};$$

because the right-hand side of this equation does not depend on time, if $E(t)$ is one of its solutions so is also $E(t+\tau)$, where τ is a constant.

In our case $\varepsilon^{(p)}$ is given by

$$\varepsilon^{(p)}(t) = (P_1/E_\theta)\,\alpha_\theta(P_1;\ t), \quad \text{for} \quad 0 \leqslant t \leqslant t_1,$$

and by

$$\varepsilon^{(p)}(t) = (P_2/E_\theta)\,\alpha_\theta(P_2;\ t - t_1 + t_*), \text{ for } t_1 \leqslant t \leqslant t_2,$$

provided t_* is chosen in such a manner that

$$P_2\,\alpha_\theta(P_2;\ t_*) = P_1\,\alpha_\theta(P_1;\ t_1)\,.$$

As $P\alpha_\theta$ is a monotonically increasing function of both P and t, this equation always admits of a solution if $P_2 > P_1$, in which case t_* is less than t_1. If P_2 is smaller than P_1 a solution exists only if t_2 is sufficiently large and $P_1 - P_2$ sufficiently small so that

$$P_2\,\alpha_\theta(P_2;\ t_2) \geqslant P_1\,\alpha_\theta(P_1;\ t_1)\,.$$

It is obvious that within the ranges of primary and secondary creep (where the creep rate decreases) the creep predicted for $t > t_1$ on the basis of the strain-hardening hypothesis is in excess of that predicted on the basis of the time-hardening hypothesis if $P_2 > P_1$ and vice versa.

In some instances it is possible to treat the problem of a rod under constant stress but at stepwise varying temperature in the same way as that of a rod at constant temperature but under stepwise varying load; we do not enter in details, however, because we intend to concentrate our attention to isothermal conditions. To mark this point we will drop henceforth the subscript θ.

4. — Further comments on the simplifying hypotheses. A similarity parameter.

If it is assumed that the strain-hardening hypothesis is valid for values of P, $\varepsilon^{(p)}$, t representing co-ordinates of points near the surface $\sum_2$ (i.e. for conditions near those found in experiments of stress relaxation at constant strain), knowledge of the function β (defined in Sect. **2**) is sufficient to predict the change of stress with time in a rod under stepwise varying strain. This is so because, under the strain-hardening hypothesis, eqn (2.4) reduces to

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{P}{T[P;\ \varepsilon^* - (P/E)]}; \tag{4.1}$$

hence, if $S(t)$ is one of its solutions, so is also $S(t+\tau)$ where τ is an arbitrary constant. It follows in particular that if we know, for instance, β for two different values of ε^*, say ε_1^* and ε_2^*, and for t in the interval $(0, t_2)$, we can give the explicit expression of P when ε is kept equal to ε_1^* for $0 \leqslant t < t_1$ $(t_1 < t_2)$ and to ε_2^* for $t_1 < t < t_2$. The statement needs some qualifications; they will be mentioned presently. For $0 \leqslant t < t_1$ we have, of course,

$$P = E\varepsilon_1^* \{ 1 - \beta(\varepsilon_1^*, t) \},$$

and for $t_1 < t < t_2$, instead,

$$P = E\varepsilon_2^* \{ 1 - \beta(\varepsilon_2^*, t + \tau) \},$$

provided τ satisfies the equation

$$\varepsilon_1^* \{ 1 - \beta(\varepsilon_1^*, t_1) \} = \varepsilon_2^* \{ 1 - \beta(\varepsilon_2^*, t_1 + \tau) \}. \tag{4.2}$$

Because $\varepsilon^* \{ 1 - \beta(\varepsilon^*, t) \}$ is a monotonically decreasing function of both ε^* and t, eqn (7.2) has a solution only if ε_2^* differs little from ε_1^*. Precisely, if ε_2^* is less than ε_1^*, it must be

$$\varepsilon_1^* \{ 1 - \beta(\varepsilon_1^*, t_1) \} \leqslant \varepsilon_2^* \{ 1 - \beta(\varepsilon_2^*, 0) \},$$

and, if ε_2^* is greater than ε_1^*, it must be

$$\varepsilon_1^* \{ 1 - \beta(\varepsilon_1^*, t_1) \} \geqslant \varepsilon_2^* \{ 1 - \beta(\varepsilon_2^*, t_2) \}.$$

It is remarkable that, in problems of relaxation, the time-hardening hypothesis does not lead to similar simple developments; in fact both the dependent and the independent variables appear in the right-hand side of the equation

$$\frac{dP}{dt} = -\frac{P}{T(P; t)}.$$

So far we have spoken here of both the strain-hardening and the time-hardening hypotheses as means of extrapolating the values of the function T from the surfaces $\sum_1$ and $\sum_2$ to their respective neighbourhoods. These assumptions, however, embody more than a mathematical artifice: they make reference to possibile atomic mechanisms of creep. Their validity may not be restricted to a neighbourhood of the surfaces $\sum_1$ and $\sum_2$; where such is the case experiments of simple pull and relaxation are necessarily related: more precisely an equation can be written which involves the functions α and β.

Under the time-hardening hypothesis the function T can be obtained from eqn (2.1)

$$T(P;\ t) = \left[\frac{\partial\alpha}{\partial t}\right]^{-1},$$

and also from eqn (2.7). The relation follows

$$\text{(4.3)} \qquad \frac{\partial\alpha}{\partial t} = -\frac{E}{P}\left\{[\varepsilon^{(p)} + (P/E)]\frac{\partial}{\partial t}\beta[\varepsilon^{(p)} + (P/E),\ t]\right\},$$

where $\varepsilon^{(p)}$ must be eliminated using eqn (2.6).

Vice versa in the strain-hardening hypothesis $T(P;\ \varepsilon^{(p)})$ can be first obtained from eqn (2.1) (where t must be eliminated using eqn (1.2)) and then from eqn (2.7). Eqn (4.3) follows again; but here t must be eliminated using eqn (2.6).

Systematic experimental work designed to relate, if possibile, creep and relaxation phenomena (*on general lines*, as indicated above) has not been carried out so far. Partial result have led authors to contradictory statements; to some extent the disagreement can be justified with the extreme variability of response in specimens prepared in accordance with the same nominal specification and the consequent difficulties in correlating experimental results. Comparisons are also made difficult by extreme differences in time scales (and lives) for parts designed to work at not greatly different temperatures. Here some help may come from the introduction of a similarity parameter. Note that the equation (2.2) wich relates the total strain to stress and time can be put in a non-dimensional form by introducing a typical strain-rate D; D could be, for instance, the ratio of the maximum permissible strain over the expected lifespan. Replacing then P and T with the quantities

$$\text{(4.4)} \qquad P = E\pi, \qquad T = \mathfrak{T}/D$$

and the variable t with the ratio τ/D, eqn (2.2) becomes

$$\text{(4.5)} \qquad \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} = \frac{\mathrm{d}\pi}{\mathrm{d}\tau} + \frac{\pi}{\mathfrak{T}},$$

an equation which involves only non-dimensional variables: the strain ε, the non-dimensional stress π, the non-dimensional time τ and the parameter $\mathfrak{T}$. We call $\mathfrak{T}$ the *Truesdell number*, because it was first introduced by C. TRUESDELL in a general analysis of the mechanics of bodies possessed of a time function, such as T [14].

As T depends on P, $\varepsilon^{(p)}$ and t it will be impossible to speak of a Truesdell number as characteristic of a certain problem; comparisons may be stated, however, between ranges of values for $\mathfrak{T}$ in different problems.

We may note here incidentally that dimensional considerations restrict greatly the type of the function T, if the material does not posess, beside the modulus of elasticity E, also at least one time modulus. To be physically acceptable T must have the form

$$T = t \cdot F[(P/E);\ \varepsilon^{(p)}] \tag{4.6}$$

for materials without time moduli, as immediately follows from a comparison of the dimensions of the variables concerned. The hypothesis of absence of time moduli, leading to the specification (4.6) for T, would allow, if accepted, simplifications of the type permitted by the strain-hardening hypothesis, because the time could be effectively reduced to the rôle of an auxiliary variable.

In particular it is immediately obvious that, under the circumstances, if $(P/E)\,\alpha\,(P;\ t)$ is a solution of eqn (1.3), so is also $(P/E)\,\alpha\,(P;\ \gamma t)$ where γ is an arbitrary constant. Also, in the experiment of tension of a rod under stepwise varying load described in Section 3, the predicted strain for $t > t_1$ would be now

$$\varepsilon^{(p)}(t) = (P_2/E)\,\alpha\,(P_2;\ at),$$

where the constant a must satisfy the equation

$$P_2\,\alpha\,(P_2;\ at_1) = P_1\,\alpha\,(P_1;\ t_1)\,.$$

When the dependence of T on t is more complex than that envisaged in formula (4.6) the material must possess at least one time modulus, say ν. It will be convenient later to put the function T in the form

$$T = tH\,[(P/E);\ \varepsilon^{(p)};\ (t/\nu)]\,. \tag{4.7}$$

5. — Creep under complex stress.

The behaviour of a cylindrical rod under tension and of a few other systems under stress can be described on the basis of eqn (2.2); most stress analyses, however, involve complex stress distributions. To deal with these a general form of the stress/strain-rate relation must be first agreed upon, as the extension of formula (2.2) is not unique; a possible choice is indicated in this Section. We start by recalling some classical definitions and properties; the hypotheses on which the generalization is based can then be clearly stated.

At any point in a body $\mathcal{C}$ a general state of stress can be characterized by the components τ_{hk} $(h, k = 1, 2, 3)$ of the stress tensor, given with reference to a cartesian frame of co-ordinates (O, x_1, x_2, x_3); the tensor τ_{hk} can be split into the sum of its spherical part, the mean normal stress τ:

$$\tau = \frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33}) \tag{5.1}$$

and a deviatoric part, as follows

$$\tau_{hk} = \tau\delta_{hk} + (\tau_{hk} - \tau\delta_{hk}) \tag{5.2}$$

[δ_{hk} is the KRONECKER delta]. This partition is so convenient for our purposes to call for a specific notation for the components of the second tensor on the right-hand side of formula (5.2):

$$s_{hk} = \tau_{hk} - \tau\delta_{hk}; \tag{5.3}$$

the tensor s_{hk} is termed stress deviation.

The conditions of equilibrium for $\mathcal{C}$ assure that τ_{hk} (and hence s_{hk}) is symmetric. Furthermore if we call F_k $(k = 1, 2, 3)$ the components of the body force acting on $\mathcal{C}$, and f_k $(k = 1, 2, 3)$ the components of the surface forces acting on the boundary $\sum$ of $\mathcal{C}$, the following equations must be satisfied

$$\sum_1^3{}_k \frac{\partial \tau_{hk}}{\partial x_k} = F_h \ \dots \text{ in } \mathcal{C}, \tag{5.4}$$

$$\sum_1^3{}_k \tau_{hk}\, \nu_k = f_h \ \dots \text{ on } \sum . \tag{5.5}$$

In (5.5) ν_k are the direction cosines of the exterior normal to $\sum$. Reference is made here to the equations of equilibrium rather than to the dynamic equations for $\mathcal{C}$ because the motions associated with creep are usually so slow that inertia forces can be safely disregarded.

We will measure the strain in accordance with the classical linear definition, thus assuming that the displacements of the points of $\mathcal{C}$ are small. This is in line with the definition of strain given in Sect. 1 where the change of length of a rod was referred to its initial length, not to its current length. If u_h ($h = 1, 2, 3$) are the components of the displacement, the components of the strain tensor are

$$e_{hk} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_h} + \frac{\partial u_h}{\partial x_k}\right), \quad (h, k = 1, 2, 3) . \tag{5.6}$$

The tensor e_{hk} can be split into the sum of the mean normal strain e and the strain deviation γ_{hk}

$$e_{hk} = \gamma_{hk} + e\delta_{hk} . \tag{5.7}$$

e, defined by

$$e = \frac{1}{3}(e_{11} + e_{22} + e_{33}),$$

measures the increment of volume per unit initial volume.

As long as the displacements are small the components of the strain-rate tensor can be simply expressed in terms of the components v_h of the velocity vector

$$\dot{e}_{hk} = \frac{1}{2}\left(\frac{\partial v_h}{\partial x_k} + \frac{\partial v_k}{\partial x_h}\right); \tag{5.8}$$

here, of course, $v_h = \frac{du_h}{dt}$, $(h = 1, 2, 3)$.

All definitions and relations given so far in this Section apply to any continuum and do not suffice in general to determine the mechanical state of $\mathcal{C}$. We still lack the analytical expression of the material properties of $\mathcal{C}$: the relation between stress, strain and strain rate. For sake of simplicity we seek here that relation under the hypothesis that $\mathcal{C}$ is isotropic.

Then, if the deformation were perfectly elastic, the two tensor s_{hk} and γ_{hk} and the two scalars τ and e would be simply proportional

$$\tau = 3Ke, \tag{5.9}$$
$$s_{hk} = 2G\gamma_{hk}, \qquad (h,\ k = 1,\ 2,\ 3)\,.$$

K and G are respectively the bulk modulus and the shear modulus; both moduli are proportional to the elastic constant E already introduced, but their definition involves also Poisson's ratio μ

$$K = \frac{E}{3(1-2\mu)}\,, \qquad G = \frac{E}{2(1+\mu)}\,. \tag{5.10}$$

This shows that the recording of the elastic lengthening of a rod under tension is not sufficient as an experimental basis of a three-dimensional theory of elastic deformation; the accompanying lateral contraction must be also examined. Similarly the effect of creep on volume changes must be studied, before eqn (2.2) can be generalized. The point seems to be clearly decided on experimental basis; but to proceed to an analytical statement we must introduce a preliminary hypothesis to this effect:

In a body subject to elastic deformation and creep the total strain can be split into the sum of elastic and creep strains, separately obeying different laws:

$$e_{hk} = e_{hk}^{(e)} + e_{hk}^{(p)}; \tag{5.11}$$
$$e = e^{(e)} + e^{(p)}, \qquad \gamma_{hk} = \gamma_{hk}^{(e)} + \gamma_{hk}^{(p)};$$

in particular the elastic portion of the strain in still related to the stress through eqns (5.9).

We may note incidentally that the statement does not imply that the displacement itself can be split in a manner similar to the strain: in general only the total strain can be associated with a displacement in accordance with the eqn (5.6), not the individual $e_{hk}^{(e)}$, $e_{hk}^{(p)}$. From a geometrical point of view one could in fact choose either of these two tensors arbitrarily; on the contrary the sum e_{hk} must satisfy the conditions of compatibility of de Saint Venant, conditions which follow from the fact that there are three components of the displacement available to fit the six independent components of the strain tensor. The fact that sometimes the removal of external loads from a body subject to creep is not followed by the vanishing of stress must be attributed to the geometrical circumstance just mentioned.

Let us go back now to the question of the effect of stress on change of volume: experimental evidence seems to show that creep deformation is not associated with any appreciable change in volume and that a hydrostatic pressure does not cause creep ([**15**], Appendix 11). Hence our second hypothesis:

During creep the mean normal strain e is purely elastic

$$e^{(p)} = 0, \qquad e = e^{(e)};$$

e is related to the mean normal stress by the formula

$$\tau = 3Ke\,. \tag{5.12}$$

The formula which corresponds in our case to the second elastic formula (5.9) remains to be stated. In this respect we have remarked in Sect. **1** that a direct relation seems to exist between stress and creep strain rate rather than between stress and creep strain; eqn (1.3) already reflects this circumstance. However, the possible generalizations of the unidirectional formula (1.3) are still manifold, even when the choice is restricted by the assumption of isotropy. The similar question in the field of plastic deformation has been dealt with conclusively in [**16**] and [**17**], where the most general relation has been obtained; this involves linearly the two tensors $\dot{\gamma}^{(p)}_{hk}$ and s_{hk}, and also the deviation of the square of s_{hk} with coefficients depending on the invariants of the tensor s_{hk}.

Here we must on the one hand allow also for the effect of strain-hardening; but otherwise we follow a criterion of simplicity, and we assume that *the relation between stress and strain rate involves only the tensors $\dot{\gamma}^{(p)}_{hk}$ and s_{hk} with coefficients which depend on time and on the second invariants of $\gamma^{(p)}_{hk}$ and s_{hk} only*

$$(5.13)\qquad \begin{aligned} J &= \frac{1}{2}\sum_{hk} s^2_{hk} = \frac{1}{2}(s^2_{11}+s^2_{22}+s^2_{33}) + s^2_{12}+s^2_{23}+s^2_{31}\,. \\ I &= 2\sum_{hk}[\gamma^{(p)}_{hk}]^2 = 2\{[\gamma^{(p)}_{11}]^2+[\gamma^{(p)}_{22}]^2+[\gamma^{(p)}_{33}]^2\} + 4\{[\gamma^{(p)}_{12}]^2+[\gamma^{(p)}_{23}]^2+[\gamma^{(p)}_{31}]^2\}\,; \end{aligned}$$

explicity the relation is written

$$\dot{\gamma}^{(p)}_{hk} = \frac{3s_{hk}}{2ET[\sqrt{3J}, \sqrt{I/3}\,; t]}\,, \tag{5.14}$$

where the numerical coefficients are arranged so that uniformity with formula (1.3) is obtained; in fact for the case of uniaxial tension P in the direction of the x_3-axis one has

$$s_{11} = s_{22} = -P/3, \quad s_{33} = 2P/3; \quad s_{hk} = 0 \quad \text{for} \quad h \neq k;$$

$$J = P^2/3; \quad I = 3\,[\varepsilon^{(p)}]^2; \quad \dot{\gamma}_{33}^{(p)} = \dot{\varepsilon}^{(p)}.$$

6. — The special case of constant stress or constant strain.

It is of interest to examine in some detail the consequences of the hypotheses made in Sect. **5** in the particular case of constant stress or constant strain. Some of the implication appear then more clearly; also cases of technical interest can be studied in all generality.

When the tensor s_{hk} is constant its components appear in eqns (5.14) as mere parameters; with the substitution

$$\gamma_{hk}^{(p)} = \frac{3}{2}\,\frac{s_{hk}}{\sqrt{3J}}\,F(t)$$

all eqns (5.14) reduce to the single one

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\sqrt{3J}}{ET\,[\sqrt{3J};\,F;\,t]}\;;$$

the solution of this equation, which corresponds to the initial condition $F = 0$ for $t = 0$, can be given if the function α, introduced in Section **1**, is known

$$F(t) = (\sqrt{3J}/E)\;\alpha\,(\sqrt{3J};\;t);$$

in fact $(P/E)\,\alpha\,(P,\;t)$ satisfies eqn (1.3), and vanishes for $t = 0$. We have then for the components of the creep tensor

$$\gamma_{hk}^{(p)}\,(t) = (3\,s_{hk}/2E)\;\alpha\,(\sqrt{3J},\;t),$$

and for the total strain deviation

$$\gamma_{hk}\,(t) = \left\{\frac{1}{2G} + \frac{3}{2}\,\frac{\alpha(\sqrt{3J},\,t)}{E}\right\} s_{hk}\,. \tag{6.1}$$

This means in particular that *for a body subject to constant and uniform stress the strain and the displacement is expressed by the same formulae which are valid in the pure elastic case if only the shear modulus G is substituted by a pseudo shear modulus Γ given by the formula*

$$\Gamma = G\left\{1 + \frac{3\alpha(\sqrt{3J}, t)}{2(1+\mu)}\right\}^{-1}. \tag{6.2}$$

The theorem has an immediate application in the prediction of the behaviour of thin-walled tubes under internal pressure, axial load and torsion.

When the total strain is constant eqns (5.9) and (5.14) can be combined to give

$$\dot{s}_{hk} = -\frac{3s_{hk}}{2(1+\mu)T[\sqrt{3J}, \sqrt{I/3}; t]} \tag{6.3}$$

and the problem of finding the dependence of s_{hk} on time can be reduced to the simpler problem of relaxation treated in Sect. **5** *provided that the body is completely incompressible*. In that case $\mu = 1/2$ so that eqn (6.3) can be simplified

$$\dot{s}_{hk} = -\frac{s_{hk}}{T[\sqrt{3J}; \sqrt{I/3}; t]}. \tag{6.4}$$

Note that the hypothesis of incompressibility is not too restrictive: at high temperatures Poisson's ratio for some metals and alloys has been found to approach the limiting value 1/2 [**18**].

If $\bar{\gamma}_{hk}$, say, are the constant values of the components of strain deviation, with the substitution

$$s_{hk} = \frac{2}{3}\sqrt{\frac{3}{\bar{I}}}\sqrt{3J}\;\bar{\gamma}_{hk},$$

(where $\bar{I}$ is the second invariant of the tensor $\bar{\gamma}_{hk}$) all eqns (6.4) can be reduced to a single one

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sqrt{3J}) = \frac{\sqrt{3J}}{T\{\sqrt{3J}; \sqrt{\bar{I}/3} - \sqrt{3J}/E; t\}}; \tag{6.5}$$

in fact the creep-strain tensor can be written

$$\gamma_{hk}^{(p)} = \bar{\gamma}_{hk} - 3s_{hk}/2E = \bar{\gamma}_{hk}\left(1 - \sqrt{\frac{3}{\bar{I}}}\frac{\sqrt{3J}}{E}\right),$$

and its second invariant

$$I = \bar{I}\left(1 - \sqrt{\frac{3}{\bar{I}}}\,\frac{\sqrt{3J}}{E}\right)^2.$$

On the other hand eqn (6.5) can be solved explicitly if the function β defined in Sect. **2** is known; by comparing eqn (6.5) with eqn (2.4), and using the solution (2.5) one obtains

$$\sqrt{3J} = E\sqrt{\bar{I}/3}\left\{1 - \beta\left(\sqrt{\bar{I}/3},\ t\right)\right\}$$

and hence

$$s_{hk} = \frac{2}{3}\,E\,\bar{\gamma}_{hk}\left\{1 - \beta\left(\sqrt{\bar{I}/3},\ t\right)\right\}.$$

At this point one may wonder if it were possible to extend also the results of Sect. **3** regarding the behavior of a rod under stepwise varying load. In fact the extension is possible under the time-hardening hypothesis but not in general under the strain-hardening hypothesis. In the first instance the system of eqns (5.14) can be written more simply

$$\dot{\gamma}_{hk}^{(p)} = \frac{3s_{hk}}{2ET(\sqrt{3J};\,t)}\,, \tag{6.6}$$

and it can be immediately realized that, if $E_{hk}(t)$ is a solution of this system, so is also $E_{hk}(t) + G_{hk}$, where G_{hk} is an arbitrary constant tensor. This property can be used directly to find the creep strain in a body where the level of stresses is represented by the constant tensor $s_{hk}^{(1)}$ in the interval of time $(0,\ t_1)$ and by another constant tensor $s_{hk}^{(2)}$ in $(t_1,\ t_2)$. If the second invariants of $s_{hk}^{(1)}$, $s_{hk}^{(2)}$ are written respectively $J^{(1)}$, $J^{(2)}$ the creep strain in $(0,\ t_1)$ is of course

$$\gamma_{hk}^{(p)}(t) = (3s_{hk}^{(1)}/2E)\,\alpha\left(\sqrt{3J^{(1)}},\ t\right) \tag{6.7}$$

and in $(t_1.\ t_2)$

$$\gamma_{hk}^{(p)}(t) = (3s_{hk}^{(2)}/2E)\left\{\alpha\left(\sqrt{3J^{(2)}},\ t\right) - \alpha\left(\sqrt{3J^{(2)}},\ t_1\right)\right\} + \tag{6.8}$$

$$+ (3s_{hk}^{(1)}/2E)\,\alpha\left(\sqrt{3J^{(1)}},\ t_1\right).$$

In fact the function (6.8) not only satisfies eqn (6.6), but also assures continuity with the function (6.7) for $t = t_1$.

Under the strain-hardening hypothesis it remains true that if $E_{hk}(t)$ is a solution of

$$\dot{\gamma}^{(p)}_{hk} = \frac{3s_{hk}}{2ET(\sqrt{3J};\ \sqrt{I/3})}, \tag{6.9}$$

so is also $E_{hk}(t+\tau)$, where τ is an arbitrary constant; but this property is not sufficient to build a continuous solution of (6.9) under conditions of stepwise varying load when the components of $s^{(2)}_{hk}$ are not all in the same ratio to the corresponding components of $s^{(1)}_{hk}$.

7. — A restricted time-hardening hypothesis. Flexure and torsion of a beam.

To treat, to some degree explicitly, particular stress analyses, we adopt here a restricted time-hardening hypothesis: we assume the time function to be of the type [see formula (4.7)]

$$T(\sqrt{3J};\ t) = t\, T_1(\sqrt{3J}/E)\, T_2(t/\nu)\,. \tag{7.1}$$

The restriction does not make the original hypothesis much more onerous in practice: most time functions used so far are of the type (7.1): on the other hand the form (7.1) for the function T is attractive from an analytical point of view because it leads to developments substantially identical with those which follow from the much more restrictive hypothesis of constant creep rate.

In fact eqn (5.14) can be reduced to the form

$$\frac{\mathrm{d}\gamma^{(p)}_{hk}}{\mathrm{d}t_1} = \frac{3s_{hk}}{2ET_1(\sqrt{3J}/E)}, \tag{7.2}$$

by introducing the « modified » non-dimensional time

$$t_1 = \int_0^t \frac{\mathrm{d}t}{tT_2(t/\nu)}\,. \tag{7.3}$$

We intend also to disregard in this Section the elastic strain, hence restricting further our considerations to cases where the creep strain is of a larger order of magnitude.

We consider first the problem of the flexure of a beam under the action of constant couples of intensity M acting on the bases. For convenience we take the z-axis as the line of the centroids of the cross-sections and the plane (x, z) as the plane of the couples; the y-axis has then the direction of the vectors which represent the moments of the couples. We will show that it is possible to solve the problem by taking all components of the stress tensor to be zero except τ_{33}, and τ_{33} to depend on x and y only

$$\tau_{33} = E\varphi(x, y) .$$

The conditions of equilibrium (5.4), (5.5) are identically satisfied inside the beam and on the lateral surface. The exact stress distribution on the bases is not specified, only the stress resultant and the resultant couples are given; hence the equations of equilibrium impose upon φ only the global conditions

$$\int_A \varphi(x, y)\, \mathrm{d}A = 0,$$

$$\text{(7.4)} \qquad E\int_A x\,\varphi(x, y)\, \mathrm{d}A = M, \qquad \int_A y\,\varphi(x, y)\, \mathrm{d}A = 0,$$

where A is one of the bases.

The stress/strain rate relations (7.2) assure that all components of strain with $h = k$ are zero and that γ_{11} and γ_{22} both coincide with $-(\gamma_{33}/2)$; putting $\varepsilon = \gamma_{33}$, to simplify the notation, the only relevant stress/strain relation can be written

$$\text{(7.5)} \qquad \varepsilon = \varphi\cdot t_1\,[T_1(|\varphi|)]^{-1}.$$

Eqn (7.5) shows that ε depends on x, y and t_1 and we will take simply

$$\text{(7.6)} \qquad \varepsilon = (a\,x + by + c)\,t_1$$

where a, b and c are three constants, which we will choose properly later.

We have not yet stated a property of the function $\varphi/T_1(|\varphi|)$ (which is obvious on mechanical grounds): that of being a monotonically increasing function of φ; the property is needed here to make the inversion of the relation (7.5) possible, giving say

$$\varphi = \Phi(\varepsilon/t_1),$$

or, because of (7.6),

$$\varphi = \Phi\,(ax + by + c), \tag{7.7}$$

where Φ is an odd monotonic function of its argument.

The expression (7.7) of φ must now be introduced in eqns (7.4). For the solution of these equations in terms a, b, c, approximate numerical methods will be usually required; but in principle the problem is solved.

We consider next the problem of torsion of a beam having circular cross-section, of radius R, under the action of twisting couples of constant magnitude M. For convenience we take a cylindrical system of reference (ϱ, θ, z) with polar axis along the axis of the cylinder. We will show that the problem can be solved by taking all stress components to be zero except $\tau_{\theta z}$, and $\tau_{\theta z}$ to depend on ϱ only

$$\tau_{\theta z} = E\,\psi\,(\varrho/R)\,.$$

The conditions of equilibrium are thus identically satisfied inside the beam and on the lateral surface. On the bases one must have

$$\int_A \psi\,(\varrho/R)\,\varrho\,\mathrm{d}A = M/E\,. \tag{7.8}$$

The stress/strain rate relations assure that all components of the strain tensor are zero except $\gamma_{\theta z}$ which must be a function of ϱ and t_1 only

$$\gamma_{\theta z} = \eta\,(\varrho/R,\ t_1) = \frac{3\psi(\varrho/R)}{2ET_1(\sqrt{3}\,|\,\psi(\varrho/R)\,|)}\,t_1\,. \tag{7.9}$$

We will take for η simply

$$\eta = B\varrho t_1,$$

where B is a constant; hence by inversion of eqn (7.9) we have, say,

$$\psi = \Psi\,(\eta/t_1) = \Psi\,(B\varrho)\,.$$

Finally by introduction of this function in (7.8) an equation is obtained whence B can be determined; the problem is thus solved in principle.

8. – Flexure of a beam of rectangular cross-section.

We give here further attention to the problem of flexure of a beam already studied in Sect. 7. Here, however, we will not disregard the elastic strain; on the other hand we will restrict our analysis to the case of a beam of rectangular cross-section bent in one of its planes of symmetry. That plane can then be chosen as the plane (x, z) for instance; with the consequence that the function φ does not depend on y and the first and third eqns (7.4) are automatically satisfied. The second becomes

$$\int_{-a/2}^{a/2} x\,\varphi\,(x,\ t_1)\,\mathrm{d}x = M/bE, \tag{8.1}$$

if a and b are now the lengths of the sides of the cross-section. When the elastic strain is not neglected the following relation between strain rate and stress takes the place of eqn (7.5)

$$\frac{\partial e_{33}}{\partial t_1} = \frac{\partial \varphi}{\partial t_1} + \frac{\varphi}{T_1(\,|\varphi|)}\,, \tag{8.2}$$

as can be seen by combining and specializing eqns (5.9), (5.10), (5.11), (5.14). The problem is made definite by taking again for e_{33} an expression which is linear in x; the type of the dependence of e_{33} on t_1 cannot be immediately specified, however,

$$e_{33} = \frac{2x}{a}\,A(\,t_1)\,. \tag{8.3}$$

Then our problem is reduced to that of finding two functions, $\varphi\,(x, t_1)$ and $A(t_1)$, which are defined for $-a/2 \leqslant x \leqslant a/2$, $t_1 \geqslant 0$, satisfy condition (8.1) and the equation

$$(2x/a)\,\frac{\mathrm{d}A}{\mathrm{d}t_1} = \frac{\partial \varphi}{\partial t_1} + \frac{\varphi}{T_1(\,|\varphi|)}\,, \tag{8.4}$$

and for $t_1 = 0$ reduce to

$$\varphi(x,\ 0) = \frac{Ma}{2EI}\,\frac{2x}{a}\,, \qquad A(0) = \frac{Ma}{2EI}\,. \tag{8.5}$$

The reason for the last condition is that for $t = 0$ the stress distribution is that given by the theory of elasticity.

An approximate solution of eqns (8.4), (8.5), (8.1) valid for small values of t_1 can be given explicitly; when t_1 is small one can take

$$\varphi(x, t_1) = \frac{Ma}{2EI}\frac{2x}{a}[1 + t_1\gamma(x)],$$

$$A(t_1) = \frac{Ma}{2EI}(1 + Ct_1),$$

and determine the function $\gamma(x)$ and the constant C so that eqns (8.4), (8.1) are satisfied within quantities of the first order in t_1: from eqn (8.4)

$$\gamma(x) = C - \left[T_1\left(\frac{M|x|}{EI}\right)\right]^{-1},$$

and from eqn (8.1)

$$C = \frac{12}{a^3}\int_{-a/2}^{a/2}\frac{x^2\mathrm{d}x}{T_1(M|x|/EI)}.$$

Actually the process could be repeated to an approsimation of arbitrary degree, leading to power-series developments in t_1 for φ and A.

In practice it is usually found more convenient to transform the differential problem in a finite-difference problem. One first introduces in the strip $0 \leqslant x \leqslant a/2$, $t_1 \geqslant 0$ a rectangular mesh defined by mesh points (x_n, t_m) $(n = 0, 1, \dots N)$, $(m = 0, 1, 2, \dots)$; one replaces the equations (8.1), (8.4), (8.5) with equivalent finite-difference equations, for instance

$$\sum_{1}^{N}{}_n\, x_n(x_n - x_{n-1})\varphi_{n,m+1} = M/2bE,$$

$$(8.6)\qquad (2x_n/a)(A_{m+1} - A_m) = \varphi_{n,m+1} - \varphi_{n,m} + \varphi_{n,m}(t_{m+1} - t_m)[T_1(|\varphi_{n,m}|)]^{-1},$$

$$\varphi_{n,0} = (Ma/2EI)(n/N), \qquad A_0 = (Ma/2EI),$$

and computes the unknown functions at the mesh points by successive steps.

Finite difference methods are of course powerful tools for the approximate solution of differential problems; more will be said in the next Section on their application to problems of creep. Sometimes, however, they may fail because of « instability »; in simpler cases concerning linear systems, when « explicit » formulae of the type (8.6) are used, the phenomenon of instability occurs if the step in time is greater than twice the minimum relaxation time in the system. Although the extension to problems of our type is rather far fetched, it has been found in practical calculations that the rule still holds approximately.

9. — A general method of stress analysis for bodies subject to creep.

We put forward here an approsimate method of calculation of the strain developed in a body subject to creep over an interval of time $(0, t)$. Precisely we show how the calculation can be carried out in a number of steps in each of which the solution of a « pseudo-elastic » problem is required, i.e. of a problem of linear type as in the classical theory of elasticity. The method obtains under the time-hardening hypothesis and is similar to one proposed by Ilyushin for the solution of elasto-plastic problems [**19**]; it is based on a simple generalization of the remarks of Sect. **6** regarding the development of creep under stepwise varying load.

The generalization in question is most simply introduced if we write first the stress-strain relations (5.9), (5.11), (5.12), (5.14) in integral form:

$$\tau(t) = Ke(t)\,,$$

(9.1)

$$\gamma_{hk}(t) = [s_{hk}(t)/2G] + \frac{3}{2}\int_0^t \frac{s_{hk}(\xi)}{ET[\sqrt{3J(\xi)},\,\xi]}\,d\xi\,.$$

These relations show that, if the stress distribution is suddenly changed at a certain instant t_m from $s_{hk}^{(m)}$ to $s_{hk}^{(m+1)}$ $[\tau^{(m)}$ to $\tau^{(m+1)}]$ and then kept constant over an interval of time (t_m, t_{m+1}), the strain at t_{m+1} is given by

$$Ke(t_{m+1}) = \tau^{(m+1)}$$

(9.2)

$$\gamma_{hk}(t_{m+1}) = \gamma_{hk}(t_m) - [s_{hk}^{(m)} - s_{hk}^{(m+1)}]/2G + \frac{3}{2}\int_{t_m}^{t_{m+1}} \frac{s_{hk}^{(m+1)}}{ET(\sqrt{3J^{(m+1)}},\,\xi)}\,d\xi;$$

the second formula can be written in an explicit form, if the function α is known

$$(9.3)\qquad \gamma_{hk}(t_{m+1}) = \gamma_{hk}(t_m) - [s_{hk}^{(m)} - s_{hk}^{(m+1)}]/2G +$$

$$+ (3s_{hk}^{(m+1)}/2E)\,[\alpha(\sqrt{3J^{(m+1)}},\, t_{m+1}) - \alpha(\sqrt{3J^{(m+1)}},\, t_m)]\,.$$

Now, if we want to determine approsimately the amount of creep developed over an interval of time $(0, \bar{t})$ in a stressed body, we proceed as follows.

We divide the interval of time $(0, \bar{t})$ in a number of subintervals (t_0, t_1), (t_1, t_2), ... (t_{n-1}, t_n) [with $t_0 = 0$, $t_n = \bar{t}$] so that we can presume the change of stress to be negligibile in each of them. We then accept the approximate formulae

$$(9.4_1) \qquad \begin{aligned} s_{hk}(t) &= s_{hk}^{(m)}, \quad (h, k = 1, 2, 3), \\ \tau(t) &= \tau^{(m)} \qquad \text{for} \qquad t_{m-1} < t \leqslant t_m; \end{aligned}$$

and

$$(9.4_2) \qquad J(t) = J^{(m-1)} \qquad \text{for} \qquad t_{m-1} \leqslant t < t_m;$$

where $s_{hk}^{(m)}$, $\tau^{(m)}$, $J^{(m)}$ form a succession of quantities independent of time; we also indicate with $s_{hk}^{(0)}$, $\tau^{(0)}$, $J^{(0)}$ the measures of stress deviation, mean normal stress and Mises' invariant which are related to the initial elastic condition for $t = 0$.

By introducing the approximations (9.4) in (9.2) we obtain in particular

$$(9.5) \qquad \begin{aligned} \gamma_{hk}(t_1) &= s_{hk}^{(1)} \left\{ \frac{1}{2G} + \frac{3}{2E} \alpha (\sqrt{3J^{(0)}}, t_1) \right\}, \\ Ke(t_1) &= \tau^{(1)}, \end{aligned}$$

also

$$(9.6) \qquad \begin{aligned} \gamma_{hk}(t_2) - \{ \gamma_{hk}(t_1) - [s_{hk}^{(1)}/2G] \} &= \\ = s_{hk}^{(2)} \{ (2G)^{-1} + 3(2E)^{-1} [\alpha (\sqrt{3J^{(1)}}, t_2) &- \alpha (\sqrt{3J^{(1)}}, t_1)] \}, \\ Ke(t_2) &= \tau^{(2)}, \end{aligned}$$

and so on.

Formulae (9.5), which determine the strain at t_1 can be interpreted as stress-strain relations for a pseudo-elastic non-homogeneous body, whose bulk modulus is still K, but whose shear modulus is reduced from G to

$$\Gamma^{(1)} = EG \{ E + 3G\alpha (\sqrt{3J^{(0)}}, t_1) \}^{-1}.$$

Similarly formulae (9.6) can be interpreted as stress-strain relations for a pseudo-elastic non-homogeneous body, whose bulk modulus is still K, but whose shear modulus is reduced from G to

$$\Gamma^{(2)} = EG\left\{ E + 3G\left[\alpha\left(\sqrt{3J^{(1)}},\ t_2\right) - \alpha\left(\sqrt{3J^{(1)}},\ t_1\right)\right]\right\}^{-1},$$

and which is subject to a non-uniform and anisotropic pseudo-thermal expansion

$$\gamma_{hk}(t_1) - \left[s_{hk}^{(1)}/2G\right].$$

Therefore, in the cases where a method of calculation of elastic strain is available, which can care for lack of homogeneity and isotropy, it is also possible to carry out an approximate calculation of creep strain if the function α is known.

For a more complete discussion of this numerical metod see Ref. **20**.

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Riassunto.

Si fa un esame delle ipotesi, a volte contrastanti, che stanno alla base delle teorie fenomenologiche delle deformazioni per scorrimento (creep) dei metalli e delle leghe ad alta temperatura e si mettono a confronto alcune semplici conseguenze di tali ipotesi. Si precisa anche come si possa giungere ad una decisione circa l'effettivo valore fisico di alcune delle ipotesi attraverso esperienze relativamente semplici. Infine si discutono alcuni problemi di interesse tecnico.

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