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On some dynamical problems arising in the theory of lubrication. (**)

1. – Introduction.

Great technical interest lies in the study of the movements of a cylindrical body within a fixed hollow cylinder of larger internal radius when the clearance space is filled with a viscous fluid (lubricant) and the speed of rotation of the first cylinder around its axis is kept constant by an external drive. This mechanical system can be taken as a simplified model of more complex devices associated with rotating machinery (journal bearings). The study can be split in three successive stages. In the first an analysis of the movements of the lubricant for any kinematic condition of the inner cylinder (which we will call the journal) is required. An exact solution of the general equations of hydrodynamics is of course out of the question; but an approximate treatment is possible. In the second stage the forces and couples which act on the journal and are due to the pressure and friction in the lubricant must be calculated. Finally the dynamic behaviour of the journal must be explored; from a mathematical point of view this implies a qualitative analysis of a system of non-linear equations.

2. – Generalities.

Before we proceed to the first stage of our work we must describe in greater detail the system which we are considering. The necessary notation can then be introduced, the boundary conditions stated and the grounds on which simplifying assumptions are accepted can be made clear.

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The external cylinder (the bearing) is taken as immovable throughout our analysis; there is consequently an advantage in choosing for reference a system $\mathfrak{S}_f$ of cylindrical co-ordinates $(\Omega, \varrho, \vartheta, \zeta)$, with the origin at midway along the axis of the bearing, so that the inner surface is represented by the equations

$$\varrho = R \quad (R \text{ radius of the bearing}),$$

$$-b/2 \leqslant z \leqslant b/2 \quad (b \text{ width of the bearing}).$$

The journal is a homogeneous cylinder of radius $R-c$ (c radial clearance) and height b, having a mass m and hence principal moments of inertia

$$A = B = m\,\frac{3(R-c)^2 + b^2}{12}, \qquad C = \frac{m}{2}\,(R-c)^2.$$

Its freedom of movement is thought to be restricted only by the condition that its centre of gravity O be on the plane $\zeta = 0$; how this restriction can be in practice enforced need not worry us here.

To define the instantaneous position of the journal we introduce a moving frame of reference $\mathfrak{S}_m(O, x, y, z)$ fixed to the journal; it is natural to choose the z-axis as the axis of symmetry. The position of $\mathfrak{S}_m$ is then defined by the modulus e and anomaly β of the vector ΩO in the plane $\zeta = 0$, and by the Euler angles θ, ψ, φ of the axes of $\mathfrak{S}_m$ with reference to $\mathfrak{S}_f$.

The annular gap between journal and bearing can be defined as the region, $\mathfrak{R}$, where $R-h \leqslant \varrho \leqslant R$, $0 \leqslant \vartheta < 2\pi$, $-b/2 \leqslant z \leqslant b/2$, if we call h the radial measure of the gap. This annular space is filled with lubricant, which can also flow freely at either end of the bearing to and from conveniently placed reservoirs where the pressure is taken to be constant. Hence, the radial, transverse and axial components of the speed of the lubricant, v_1, v_2, v_3 and the pressure p satisfy in $\mathfrak{R}$ the general equations of viscous flow:

$$(2.1.1) \qquad \delta\left(\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial \varrho} + \frac{v_2}{\varrho}\frac{\partial v_1}{\partial \vartheta} - \frac{v_2^2}{\varrho} + v_3\frac{\partial v_1}{\partial \zeta}\right) = -\frac{\partial p}{\partial \varrho} + $$

$$+\eta\left(\frac{\partial^2 v_1}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial v_1}{\partial \varrho} + \frac{1}{\varrho^2}\frac{\partial^2 v_1}{\partial \vartheta^2} + \frac{\partial^2 v_1}{\partial \zeta^2} - \frac{v_1}{\varrho^2} - \frac{2}{\varrho^2}\frac{\partial v_2}{\partial \vartheta}\right),$$

$$(2.1.2) \qquad \delta\left(\frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial \varrho} + \frac{v_2}{\varrho}\frac{\partial v_2}{\partial \vartheta} + \frac{v_1 v_2}{\varrho} + v_3\frac{\partial v_2}{\partial \zeta}\right) = -\frac{\partial p}{\varrho\,\partial \vartheta} + $$

$$+\eta\left(\frac{\partial^2 v_2}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial v_2}{\partial \varrho} + \frac{1}{\varrho^2}\frac{\partial^2 v_2}{\partial \vartheta^2} + \frac{\partial^2 v_2}{\partial \zeta^2} + \frac{2}{\varrho^2}\frac{\partial v_1}{\partial \vartheta} - \frac{v_2}{\varrho^2}\right),$$

(2.1.3) $$\delta\left(\frac{\partial v_3}{\partial t} + v_1\frac{\partial v_3}{\partial \varrho} + \frac{v_2}{\varrho}\frac{\partial v_3}{\partial \vartheta} + v_3\frac{\partial v_3}{\partial \zeta}\right) = -\frac{\partial p}{\partial \zeta} + \eta\left(\frac{\partial^2 v_3}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial v_3}{\partial \varrho} + \frac{1}{\varrho^2}\frac{\partial v_3}{\partial \vartheta^2} + \frac{\partial^2 v_3}{\partial \zeta^2}\right),$$

(2.1.4) $$\frac{\partial v_1}{\partial \varrho} + \frac{v_1}{\varrho} + \frac{1}{\varrho}\frac{\partial v_2}{\partial \vartheta} + \frac{\partial v_3}{\partial \zeta} = 0,$$

here δ and η are respectively the constant density and viscosity of the lubricant. Further, p is a constant and may be taken to vanish for $z = \pm b/2$; v_1, v_2, v_3 vanish for $\varrho = R$ and coincide with the respective components of the surface speed of the journal, say B_1, B_2, B_3, for $\varrho = R - h$. As we have already remarked, an exact analysis of eqns (2.1) cannot be contemplated. The usual approach, when dealing with the flow in lubricating films, is in the nature of an approximation, where account is taken of an important circumstance not yet mentioned but a first reflection of which can be found in the simple boundary condition specified for p. We are interested in cases where the clearance c of the bearing is so much smaller than its radius R and width b that in all our formulae it is possible to disregard inessential terms of higher order in c/R or c/b. This circumstance leads to overriding simplifications; for instance, apart from terms of higher order in c/R, the function h can be written as

(2.2) $$h(\vartheta,\ \zeta) = c - e\cos(\vartheta - \beta) + \theta\zeta\sin(\vartheta - \psi).$$

Within the same approximation B_1, B_2 and B_3 can be specified as follows

(2.3)
$$B_1 = \dot e\cos(\vartheta-\beta) + e\dot\beta\sin(\vartheta-\beta) - \omega e\sin(\vartheta-\beta) + (\dot\psi-\omega)\zeta\theta\cos(\vartheta-\psi) - \zeta\dot\theta\sin(\vartheta-\psi),$$

$$B_2 = -\dot e\sin(\vartheta-\beta) + e\dot\beta\cos(\vartheta-\beta) - \theta\zeta\dot\psi\sin(\vartheta-\psi) + \omega(R-c) - \zeta\dot\theta\cos(\vartheta-\psi),$$

$$B_3 = R[\dot\theta\sin(\vartheta-\psi) + (\omega-\dot\psi)\theta\cos(\vartheta-\psi)].$$

It is perhaps in order to explain briefly how one arrives at these formulae. The first two addenda in the expressions of B_1 and B_2 are simply a reflection of the movement of O; all the remaining terms must be put in relation with the rotation of the journal around O. Now, the components of the angular speed on the axes of $\mathfrak{T}_m$ are given by the well-known formulae

$$\dot\theta \cos\varphi + \theta\dot\psi \sin\varphi, \quad -\dot\theta \sin\varphi + \dot\psi\theta \cos\varphi, \quad \dot\psi + \dot\varphi,$$

where, in accordance with our assumption, θ has been written for $\sin\theta$ and 1 for $\cos\theta$. It follows that the components on the same axes of the speed of a point P, which is on the surface of the journal and belongs to the plane (x, z), are

$$(2.4) \qquad z(-\dot\theta \sin\varphi + \dot\psi\theta\cos\varphi), \quad (R-c)\omega - z(\dot\theta\cos\varphi + \theta\dot\psi\sin\varphi),$$
$$(R-c)(\dot\theta\sin\varphi - \dot\psi\theta\cos\varphi).$$

The triad $\mathfrak{T}_p$ of axes in the radial, tangential and axial directions at P does not coincide with the triad $\mathfrak{T}_m$ of axes x, y and z respectively. To bring about the coincidence within the usual approximation, $\mathfrak{T}_m$ must be rotated through an angle $-R^{-1}[e\sin(\psi+\varphi-\beta) + z\theta\cos\varphi]$ around the z-axis and through an angle $-\theta$ around the line of nodes. Hence to obtain B_1, B_2, B_3 at P one must transform the vector of components (2.4) with the matrix

$$\begin{pmatrix} 1 & -R^{-1}[e\sin(\psi+\varphi-\beta) + z\theta\cos\varphi] & \theta\sin\varphi \\ R^{-1}[e\sin(\psi+\varphi-\beta) + z\theta\cos\varphi] & 1 & -\theta\cos\varphi \\ -\theta\sin\varphi & \theta\cos\varphi & 1 \end{pmatrix}.$$

The condition for P to fall on the plane (x, z) is in no way restrictive as the choice of the x-axis was left open so far; but it is better to avoid in the final formulae any reference to an arbitrary element and use $\vartheta - \psi$ instead of φ; this is permitted within the usual approximation.

An this point only the significance of the symbol ω remains to be explained in formulae (2.3); the sum $\dot\psi + \dot\varphi$ is replaced by ω to mark the fact that the angular speed around the z-axis (or, apart from a negligible error, around the ζ-axis) is a constant in our problem.

3. – Reynolds' equation.

In Section 2 emphasis has been laid on the simplifications which ensue in the expressions of h, B_1, B_2, B_3 from the hypothesis of smallness of the ratios c/R, c/b. Even more drastic simplifications can be introduced in the hydrodynamic equations if we assume on similar grounds, but with wider implications, that: (i) if the variable ϱ is substituted by a new variable $R(1-\xi)$

in v_1, v_2, v_3, p these functions are developable in power series of ξ in the interval $(0,\ h/R)$

$$v_i = \sum_{0}^{\infty}{}_{\nu}\, v_i^{(\nu)}(\vartheta,\ \zeta)\xi^{\nu} \quad (i=1,\,2,\,3), \qquad p = \sum_{0}^{\infty}{}_{\nu}\, p^{(\nu)}(\vartheta,\ \zeta)\xi^{\nu}; \tag{3.1}$$

(ii) in these developments only the first two terms, which do not vanish as immediate consequence of eqns (2.1) or of the boundary conditions, are significant; (iii) finally it is possible to disregard c/R and c/b when compared with unity.

The full use of these assumptions leads to a partial differential equation involving only the first term $p^{(0)}$ in the development (3.1) of p; this is the well-known « Reynolds' equation » of the theory of lubrication

$$\frac{\partial}{R\partial\vartheta}\left[\frac{h^3}{\eta}\,\frac{\partial p^{(0)}}{R\partial\vartheta}\right] + \frac{\partial}{\partial\zeta}\left[\frac{h^3}{\eta}\,\frac{\partial p^{(0)}}{\partial\zeta}\right] = -12B_1 + 6h^2\frac{\partial}{R\partial\vartheta}\left(\frac{B_2}{h}\right) + 6h^2\frac{\partial}{\partial\zeta}\left(\frac{B_3}{h}\right). \tag{3.2}$$

In fact, under assumption (i), because of the boundary conditions we have

$$v_1^{(0)} = v_2^{(0)} = v_3^{(0)} = 0; \tag{3.3}$$

further, as a consequence of the equation of continuity [eqn (2.1.4)], we get

$$\begin{aligned} &v_1^{(1)} = 0,\\ &2v_1^{(2)} = \frac{\partial v_2^{(1)}}{\partial\vartheta} + R\,\frac{\partial v_3^{(1)}}{\partial\zeta},\\ &3v_1^{(3)} = v_1^{(2)} + \frac{\partial v_2^{(1)}}{\partial\vartheta} + \frac{\partial v_2^{(2)}}{\partial\vartheta} + R\,\frac{\partial v_3^{(2)}}{\partial\zeta}; \end{aligned} \tag{3.4}$$

and from the Navier-Stokes equations [the first three eqns (2.1)]

$$\begin{aligned} &Rp^{(1)} = -2\eta v_1^{(2)},\\ &\frac{R}{\eta}\,\frac{\partial p^{(0)}}{\partial\vartheta} = 2v_2^{(2)} - v_2^{(1)},\\ &\frac{R^2}{\eta}\,\frac{\partial p^{(0)}}{\partial\zeta} = 2v_3^{(2)} - v_3^{(1)}. \end{aligned} \tag{3.5}$$

Note that, because of eqns (3.3), no trace is left in eqns (3.5) of the inertia terms.

Taking now into account the remaining boundary conditions for the speed at the surface of the journal and assumption (ii) above we get

$$(3.6)\qquad v_1^{(2)}\left(\frac{h}{R}\right)^2 + v_1^{(3)}\left(\frac{h}{R}\right)^3 = B_1,$$

$$v_2^{(1)}\frac{h}{R} + v_2^{(2)}\left(\frac{h}{R}\right)^2 = B_2,$$

$$(3.7)\qquad v_3^{(1)}\frac{h}{R} + v_3^{(2)}\left(\frac{h}{R}\right)^2 = B_3\,.$$

At this stage it is possible to obtain from eqns (3.4), (3.5), (3.7) expressions of $v_1^{(2)}, v_1^{(3)}, v_2^{(1)}, v_2^{(2)}, v_3^{(1)}, v_3^{(2)}, p^{(1)}$ in terms of $p^{(0)}$ and B_2, B_3:

$$(3.8)\qquad \begin{aligned} 2v_1^{(2)} &= \frac{\partial}{\partial\vartheta}\left[\left(1+\frac{h}{2R}\right)^{-1}\left(B_2\frac{R}{h}+\frac{R}{\eta}\frac{\partial p^{(0)}}{\partial\vartheta}\right)-\frac{R}{\eta}\frac{\partial p^{(0)}}{\partial\vartheta}\right]+\\ &\quad + R\frac{\partial}{\partial\zeta}\left[\left(1+\frac{h}{2R}\right)^{-1}\left(B_3\frac{R}{h}+\frac{R^2}{\eta}\frac{\partial p^{(0)}}{\partial\zeta}\right)-\frac{R^2}{\eta}\frac{\partial p^{(0)}}{\partial\zeta}\right],\\ 2v_1^{(3)} &= 2\frac{\partial}{\partial\vartheta}\left[\left(1+\frac{h}{2R}\right)^{-1}\left(B_2\frac{R}{h}+\frac{R}{\eta}\frac{\partial p^{(0)}}{\partial\vartheta}\right)\right]-\frac{3}{2}R\frac{\partial}{\partial\vartheta}\left(\frac{1}{\eta}\frac{\partial p^{(0)}}{\partial\vartheta}\right)+\\ &\quad + R\frac{\partial}{\partial\zeta}\left[\left(1+\frac{h}{2R}\right)^{-1}\left(B_3\frac{R}{h}+\frac{R^2}{\eta}\frac{\partial p^{(0)}}{\partial\zeta}\right)\right]-\frac{R^3}{2}\frac{\partial}{\partial\zeta}\left(\frac{1}{\eta}\frac{\partial p^{(0)}}{\partial\zeta}\right);\\ v_2^{(1)} &= \left(1+\frac{h}{2R}\right)^{-1}\left(B_2\frac{R}{h}+\frac{R}{\eta}\frac{\partial p^{(0)}}{\partial\vartheta}\right)-\frac{R}{\eta}\frac{\partial p^{(0)}}{\partial\vartheta},\\ v_2^{(2)} &= \left(2+\frac{h}{R}\right)^{-1}\left(B_2\frac{R}{h}+\frac{R}{\eta}\frac{\partial p^{(0)}}{\partial\vartheta}\right); \end{aligned}$$

$$(3.9)\qquad \begin{aligned} v_3^{(1)} &= \left(1+\frac{h}{2R}\right)^{-1}\left(B_3\frac{R}{h}+\frac{R^2}{\eta}\frac{\partial p^{(0)}}{\partial\zeta}\right)-\frac{R^2}{\eta}\frac{\partial p^{(0)}}{\partial\zeta},\\ v_3^{(2)} &= \left(2+\frac{h}{R}\right)^{-1}\left(B_3\frac{R}{h}+\frac{R^2}{\eta}\frac{\partial p^{(0)}}{\partial\zeta}\right);\\ p^{(1)} &= -\frac{2\eta}{R}v_1^{(2)}\,. \end{aligned}$$

If the expressions (3.8) of $v_1^{(2)}$, $v_1^{(3)}$ are now introduced in eqn (3.6) and account is taken of assumption (iii) the Reynolds' equation (3.2) results.

In the right-hand side of this equation B_1, B_2, B_3 must be specified in accordance with formulae (2.3); after substitution simplifications may be again achieved by disregarding terms of the second order in h/R or θ and the final result is

$$\frac{\partial}{R\partial\vartheta}\left[\frac{h^3}{6\eta}\frac{\partial p^{(0)}}{R\partial\vartheta}\right]+\frac{\partial}{\partial\zeta}\left[\frac{h^3}{6\eta}\frac{\partial p^{(0)}}{\partial\zeta}\right]=-2\dot{e}\cos(\vartheta-\beta)+ \tag{3.10}$$

$$+(\omega-2\dot{\beta})e\sin(\vartheta-\beta)+2\zeta\dot{\theta}\sin(\vartheta-\psi)+(\omega-2\dot{\psi})\zeta\theta\cos(\vartheta-\psi).$$

Formulae (3.8), (3.9) and the first eqn (3.5) can be also combined to give, within the usual approximation,

$$\begin{aligned} v_1 &= \xi^2\left(\frac{R}{h}\right)^3\left[\xi B_1+\left(\frac{h}{R}\right)^2 v_1^{(2)}\left(\frac{h}{R}-\xi\right)\right],\\ v_2 &= \xi\,\frac{R}{h}\left[B_2+\left(\xi-\frac{h}{R}\right)\frac{h}{2\eta}\frac{\partial p^{(0)}}{\partial\vartheta}\right],\\ v_3 &= \xi\,\frac{R}{h}\left[B_3+\left(\xi-\frac{h}{R}\right)\frac{h}{2\eta}R\frac{\partial p^{(0)}}{\partial\zeta}\right],\\ p &= p^{(0)}-\frac{2\eta}{R}\xi v_1^{(2)}. \end{aligned} \tag{3.11}$$

Here $v_1^{(2)}$ has not been written down explicitly; it is given by the formula

$$2v_1^{(2)}=\frac{1}{\eta}\frac{\partial h}{\partial\vartheta}\frac{\partial p^{(0)}}{\partial\vartheta}+\frac{R^2}{\eta}\frac{\partial h}{\partial\zeta}\frac{\partial p^{(0)}}{\partial\zeta}+\frac{6R^2B_1}{h^2}-2R\frac{\partial}{\partial\vartheta}\left(\frac{B_2}{h}\right)-2R^2\frac{\partial}{\partial\zeta}\left(\frac{B_3}{h}\right). \tag{3.12}$$

At this point the problem might appear overdetermined because the boundary condition that remains to be satisfied (i.e. $p=0$ for $\zeta=\pm b/2$) seems to imply that both $p^{(0)}$ and $v_1^{(2)}$ must vanish for $z=\pm b/2$ [compare the last eqn (3.11)]. Whereas the first implication satisfactorily completes the specification of the differential problem (3.10), it might be impossible to reconcile the second with formula (3.12). However, if we want to be consistent, the boundary condition for p must be stated as follows: for $\zeta=\pm b/2$, p vanishes apart for terms of higher order in c/R or c/b; this imposes upon $v_1^{(2)}$, only a qualitative condition, which is always satisfied. In fact from eqn (3.10) it appears that $p^{(0)}$ is of the order of $(R/c)^2$, whereas $v_1^{(2)}/R$ is of the order of R/c, so that the correction introduced by the second term in the right-hand side of the last eqn (3.11) is of the second order. On these grounds we will not worry henceforth about the term $\xi p^{(1)}$ and we will write simply p for $p^{(0)}$.

4. – Solution of Reynolds' equation for small values of eccentricity ratio and angle of tilt.

The integration of eqn (3.10) presents grave difficulties in general, but there are several cases where the integration succeeds by elementary means. The simplest of these cases obtains when eccentricity ratio $a = e/c$ and angle of tilt θ are negligible if compared with 1, i.e. when the conditions prevail which, as we shall see later, are of interest in studies of stability. The case is discussed in some detail in this Section.

Solution of eqn (3.10) follows from the remark that, apart from a coefficient $c^3/6\eta$, the right-hand side is reduced to the laplacian of p, when a is infinitesimal, because h^3 can be then replaced by c^3. p is simply

$$p = \frac{6R^2\eta}{c^3}\left[1 - \frac{\cosh(\zeta/R)}{\cosh(b/2R)}\right]\left[(2\dot\beta - \omega)e\sin(\vartheta - \beta) + 2\dot e\cos(\vartheta - \beta) + \right. \tag{4.1}$$

$$\left. + \zeta\theta(2\dot\psi - \omega)\cos(\vartheta - \psi) - 2\zeta\dot\theta\sin(\vartheta - \psi)\right].$$

The force and the moment of the couple acting on the journal because of the presence of lubricant in the bearing can now be calculated. In general the component F_e of the force in the direction of the vector ΩO is given by the integral

$$F_e = -\int_{-b/2}^{b/2} d\zeta \int_0^{2\pi} p\cos(\vartheta - \beta)R\, d\vartheta, \tag{4.2}$$

and the component F_n in the normal direction by the integral

$$F_n = -\int_{-b/2}^{b/2} d\zeta \int_0^{2\pi} p\sin(\vartheta - \beta)R\, d\vartheta; \tag{4.3}$$

the moment M_1 of the couple acting around the line of nodes is

$$M_1 = \int_{-b/2}^{b/2} \zeta\, d\zeta \int_0^{2\pi} p\sin(\vartheta - \psi)R\, d\vartheta, \tag{4.4}$$

and the moment M_2 of the couple acting around a line normal to the line of nodes in the plane $\zeta = 0$ is

$$M_2 = -\int_{-b/2}^{b/2} \zeta \, d\zeta \int_0^{2\pi} p \cos(\vartheta - \psi) R \, d\vartheta. \tag{4.5}$$

All these formulae are approximate, with an error of the order of c/R.

Substitution of the expression (4.1) for p in formulae (4.2), (4.3), (4.4), (4.5) gives

$$F_e = -2\mathfrak{F} c\dot{a}, \qquad F_n = \mathfrak{F}(\omega - 2\dot{\beta})ca, \tag{4.6}$$

with

$$\mathfrak{F} = \frac{6\pi R^3 b\eta}{c^3}\left[1 - \frac{\tanh(b/2R)}{(b/2R)}\right], \tag{4.7}$$

and

$$M_1 = -2M\dot{\theta}, \qquad M_2 = M(\omega - 2\dot{\psi})\theta, \tag{4.8}$$

with

$$M = \frac{6\pi R^6 \eta}{c^3}\left\{\frac{b^3}{12R^3} - \frac{b}{R}\left[\frac{b}{2R}\coth\frac{b}{2R} - 1\right]\right\}; \tag{4.9}$$

both $\mathfrak{F}$ and M are positive quantities.

5. – Stability of a perfectly aligned journal.

We devote this Section to a study of the behaviour of a rotating journal for which perfect alignement with the bearing is assumed. We will not discuss here the devices that may in practice assure the fulfilment of this condition; we will deal only with the analytical consequences of the hypothesis $\theta = 0$. Hence, our concern will be the motion of the centre O of the journal in the plane $\zeta = 0$ under the force $\boldsymbol{F}$ of components F_e, F_n. We will disregard effects of weight on the journal, but we will assume that an elastic restoring force acts on O, besides $\boldsymbol{F}$; this assumption does not prevent us from obtaining in particular the results for the case when such force is absent and, on the other hand, it leads to results which have some bearing in the study of mechanical systems which are of greater complexity and practical interest than the system under consideration.

In polar co-ordinates the equations of motion of O become then

$$(5.1)\qquad \begin{aligned} mc\ddot{a} &= mca\dot{\beta}^2 + F_e - kca, \\ mca\ddot{\beta} &= -2mc\dot{a}\dot{\beta} + F_n, \end{aligned}$$

where k is the stiffness of the spring which provides the restoring force.

In eqns (5.1) we were unable to specify in general F_e and F_n as functions of a, $\dot{a}$ and $\dot{\beta}$. However, we can say at least this: F_e and F_n vanish if a and $\dot{a}$ vanish (and, as specified in this Section, θ is identically zero); in fact, if $\theta \equiv 0$ and $a = \dot{a} = 0$, the right-hand side of eqn (3.10) and hence p itself is zero. The property is almost trivial, but had to be stated explicitly: now we can infer from it that the system (5.1), has a « steady-state » solution $a \equiv 0$.

This solution corresponds, of course, to a uniform rotation during which journal and bearing are coaxial.

Actually our main concern here is a discussion of the stability of this uniform rotation; for the purpose we need only to know the behaviour of the solutions of the system (5.1) which correspond to small values of a, $\dot{a}$. These solutions can be studied using for F_e and F_n the expressions (4.6):

$$(5.2)\qquad \begin{aligned} m\ddot{a} &= ma\dot{\beta}^2 - 2\mathfrak{F}\dot{a} - ka, \\ ma\ddot{\beta} &= -2m\dot{a}\dot{\beta} + \mathfrak{F}(\omega - 2\dot{\beta})a\,. \end{aligned}$$

This system admits of some particular solutions of the simple type

$$(5.3)\qquad a = \mathfrak{A}e^{nt}, \qquad \dot{\beta} = \gamma,$$

where $\mathfrak{A}$, n and γ are suitable constants. As we will show, these solutions alone give already a fairly complete picture of the behaviour of the journal. They represent whirls of constant angular speed and increasing or decreasing amplitude according to whether n is positive or negative; circular whirls correspond to $n = 0$.

For (5.3) to represent a solution of (5.2) n and γ must satisfy the equations

$$(5.4)\qquad \begin{aligned} \gamma^2 &= n^2 + \frac{2\mathfrak{F}}{m}n + \frac{k}{m}, \\ \gamma &= \frac{\mathfrak{F}\omega}{2(mn + \mathfrak{F})}\,. \end{aligned}$$

These represent in the (γ, n) -plane two hyperbolae, which we call here respectively K_1 and K_2. K_1 has the straight lines

$$n = -\frac{\mathfrak{F}}{m} + \gamma, \qquad n = -\frac{\mathfrak{F}}{m} - \gamma$$

as asymptotes and actually degenerates into the asymptotes themselves for $k = \mathfrak{F}^2/m$; it has only one point in common with the positive γ -axis: the point of abscissa $\sqrt{k/m}$. K_2 has the straight lines

$$n = -\frac{\mathfrak{F}}{m} \quad \text{and} \quad \gamma = 0$$

as asymptotes and cuts the γ -axis in the point of abscissa $\omega/2$. From these properties it follows that K_1 and K_2 have only two real points in common; one, Q_1, in the half plane $\gamma > 0$ and the other, Q_2, in the half plane $\gamma < 0$. In other words, there are only two real solutions of the system (5.4): (γ_1, n_1) and (γ_2, n_2), where the first pair represents the co-ordinates of Q_1 and the second pair the co-ordinates of Q_2.

The solutions of (5.2) of the type (5.3), which correspond to the root of (5.4) represented by Q_2, have little interest. They represent highly damped whirls (as n_2 is less than $-\mathfrak{F}/m$), whose sense of rotation is opposite to the sense of rotation of the journal; but there is little chance that these movements can be ever realized by experiment, because, as we will show in the next Section, their character can be completely altered by a small disturbance.

The main features of the behaviour of the system under consideration are shown in evidence by the properties of the solutions of (5.2) which have the form (5.3) and correspond to the root of (5.4) represented by Q_1. For the usual values of the parameters, γ_1 is not much different from $\omega/2$ (i.e. the frequency of the whirl is approximately half the running frequency). At the same time n_1 is positive or negative according to whether ω is greater or smaller than $2(k/m)^{1/2}$; n_1 is zero if $\omega = 2(k/m)^{1/2}$. The bare existence of solutions of the system (5.2) of the form

$$(5.5) \qquad \alpha = \mathfrak{A}e^{n_1 t}, \qquad \dot{\beta} = \gamma_1$$

indicates that whirls of increasing amplitude are possible when the frequency of rotation of the journal is more than twice the frequency of the vibrations, which the journal would perform under the action of the restoring force alone. But the importance of the solutions (5.5) is much increased by the following property (for which evidence will be provided in the next Section): all solutions of the system (5.2), (with the only exception of those related to Q_2 which we discussed above) behave asymptotically for $t \to +\infty$ as do the particular solutions (5.5).

6. – Complements to the discussion of stability.

Some general properties of the solutions of the system (5.2), which are relevant to the discussion of stability, are proved here. They can be derived most easily if account is taken of the following remarks.

In correspondence to any non-trivial solution of (5.2)

$$a = a(t), \qquad \beta = \beta(t)$$

consider, in the plane (γ, n), the curve $\mathcal{C}$ whose parametric equations are

$$\gamma = \gamma(t) = \frac{d\beta}{dt}, \qquad n = n(t) = \frac{1}{a}\frac{da}{dt}. \tag{6.1}$$

The set of all curves $\mathcal{C}$ coincides with the set of characteristics of the autonomous system of two differential equations of the first order

$$\frac{dn}{dt} = \gamma^2 - \frac{2\mathfrak{F}}{m}n - n^2 - \frac{k}{m}, \tag{6.2}$$

$$\frac{d\gamma}{dt} = -2\gamma n + \frac{\mathfrak{F}}{m}(\omega - 2\gamma).$$

This system has properties which reflect those of the system (5.2) but is much simpler and can be studied with known methods of the theory of non-linear differential equations.

For the sake of brevity the discussion is carried out below under the assumption $\mathfrak{F}/m > \sqrt{k/m}$; it could be shown, however, that the results do not change substantially in the alternative case.

We have already mentioned in Section 5 the hyperbolae K_1 and K_2 which are defined by eqns (5.4). K_1 and K_2 divide the (γ, n) -plane in seven regions. The sign of one of the direction cosines of the isoclines of the system (6.2) changes in passing from one region to a neighbouring one. On K_1 the isoclines are parallel to the γ -axis, on K_2 they are parallel to the n -axis.

The two real points of intersection between K_1 and K_1, Q_1 and Q_2, are the singular points of the system (6.2); it is easy to realize that they are both focal points. But whereas Q_1 is a stable focal point, Q_2 is an unstable one. So a point $Q(t)$ of co-ordinates $\gamma(t)$, $n(t)$ which at $t = 0$ was very near to Q_2 tends to move away from it along a characteristic $\mathcal{C}$. At the contrary, if $Q(t)$ had been very near to Q_1, it would have moved towards Q_1.

The property that $Q(t)$ tends to Q_1 for $t \to +\infty$, if it is sufficiently close to Q_1 at any instant, can be generalized. For instance we can show that along any curve $\mathcal{C}$ which crosses the positive n-axis we have

$$\lim_{t \to +\infty} n(t) = n_1, \qquad \lim_{t \to +\infty} \gamma(t) = \gamma_1 . \tag{6.3}$$

The statement can be proved as follows. Let $\mathfrak{L}$ be a closed line having the form of the perimeter of a rectangle with the sides parallel to the axes of n and γ and with the corners smoothed by quarter-circles each of which belongs to one of the four regions which have Q_1 as vertex and the hyperbolae K_1 and K_2 as boundary. One of the sides of $\mathfrak{L}$ can be chosen to lie on the n-axis and, of course, $\mathfrak{L}$ must belong to the half-plane $n > -\mathfrak{F}/m$.

We call G the set of points which contains Q_1 and has $\mathfrak{L}$ as a boundary; G and $\mathfrak{L}$ have the following properties:

(i) G can be made to contain points whose ordinate is arbitrarily big.

(ii) G does not contain limit-cycles of the system (6.2). In fact, if we call $P_1(\gamma, n)$ and $P_2(\gamma, n)$ respectively the right-hand sides of the eqns (6.2), we have

$$\frac{\partial P_1}{\partial n} + \frac{\partial P_2}{\partial \gamma} = -4\left(n + \frac{\mathfrak{F}}{m}\right).$$

As this quantity is always negative in G no limit-cycles can exist in G by a theorem of Bendixon.

(iii) $\mathfrak{L}$ is a cycle without contact for the characteristics of the system (6.2). Therefore any point $Q(t)$ which describes a characteristic $\mathcal{C}$ and enters G can never leave it for $t \to +\infty$ but necessarily approaches the singular point Q_1.

So formulae (6.3) are proved. We can interpret them as follows: Suppose we have caused a small disturbance on the journal (which was rotating uniformly in steady-state conditions) so that its centre O is slightly displaced ($a > 0$) and has a small positive radial speed ($\mathrm{d}a/\mathrm{d}t > 0$). The movement that follows is regulated by eqns (5.2) and, because of the properties (i), (ii), (iii), must develop in such a way that

$$\lim_{t \to +\infty} \frac{1}{a(t)} \frac{\mathrm{d}a}{\mathrm{d}t} = n_1, \qquad \lim_{t \to +\infty} \gamma(t) = \gamma_1 .$$

As a consequence of the first of these equations the asymptotic behaviour of $a(t)$ is given by

$$a(t) \sim \mathcal{A} e^{n_1 t}, \tag{6.4}$$

$\mathcal{A}$ being a suitable constant. In (6.4) the sign of n_1 is the same as that of the difference $\omega - 2\sqrt{k/m}$; hence if ω is less than $2\sqrt{k/m}$, O will move again towards the steady-state position Ω, the eccentricity decreasing exponentially after a transient. If instead ω is greater than $2\sqrt{k/m}$ the eccentricity will tend to increase exponentially; although it is not possible to follow the successive stages of the phenomenon in this case (beacause the validity of our equations is limited to small values of a and $\dot{a}$), we can conclude that for $\omega > 2\sqrt{k/m}$ the journal is in an unstable condition.

For $\omega = 2\sqrt{k/m}$ circular whirls are possible, with a frequency equal to half the running frequency; during these whirls the oil forces vanish and the journal moves as if it were under the action of the elastic restoring force alone.

The reader may remark that the results just stated are completely independent of the dimensions of the bearing and of the value of the viscosity of the lubricant. It may be also observed that, although the considerations of this and the previous Section are carried out on the hypothesis that an elastic force acts on the journal besides the force due to the lubricant, there is no difficulty in stating the corresponding results for the case when such elastic force is absent.

In most formulae it is sufficient to put $k = 0$; and one obtains in particular that the uniform rotation of the journal is unstable whatever is its speed ω.

7. – Whirls of large amplitude.

In Section 6 we found that any disturbance in the position of the centre of a rotating journal subject to lubricant forces only is magnified with time. However, we were not able there to describe the successive phases of the phenomenon because the formulae involved have no meaning when the eccentricity ratio a is not small. If we want now to push our analysis further we must restrict our considerations to cases for which a solution of Reynolds equation is available without restrictions as to the size of a. One such case obtains when the bearing is very long: F_e and F_n are given then by

$$\begin{aligned} F_e &= -12\pi R^3 b c^{-2} \eta \dot{a} (1 - a^2)^{-3/2}, \\ F_n &= 12\pi R^3 b c^{-2} \eta a (\omega - 2\dot{\beta})(2 + a^2)^{-1}(1 - a^2)^{-1/2}, \end{aligned} \tag{7.1}$$

and the system (5.1) becomes, in absence of the elastic force,

$$(7.2)\qquad \begin{aligned} m\ddot{a} &= ma\dot{\beta}^2 - 2\mathfrak{F}_1\,\dot{a}(1-a^2)^{-3/2},\\ ma\ddot{\beta} &= -2m\dot{a}\dot{\beta} + \mathfrak{F}_1\,a(\omega - 2\dot{\beta})\left(1+\frac{a^2}{2}\right)^{-1}(1-a^2)^{-1/2}, \end{aligned}$$

with

$$(7.3)\qquad \mathfrak{F}_1 = 6\pi R^3 b c^{-3}\eta .$$

A complete analysis of the behaviour of the solutions of the system (7.2) is difficult; we endeavour here only to prove some properties, whose statement allows us to decide upon an important question: does the eccentricity ratio a increase in unstable conditions towards its maximum permissible value 1 or does it tend to a value less than 1?

To answer this we start by assuming that a and $\dot{\beta}$ have approximately constant values, A and γ say:

$$(7.4)\qquad a(t) = A - \delta(t), \qquad \dot{\beta}(t) = \gamma + \varepsilon(t);$$

here δ and ε are taken to be small so that appropriate simplifications can be carried out when the expressions (7.4) are introduced in eqns (7.2). The most relevant of these simplifications causes the disappearance of ε from the first eqn (7.2), so that an equation in δ only is obtained. To write the equation (which is particularly relevant to us, because our interest centres around the behaviour of δ) we must distinguish two cases: (i) A is less than 1, (ii) A is equal to 1. In the first case we obtain

$$A = 0, \quad \ddot{\delta} + (2\mathfrak{F}_1/m)\,\dot{\delta} - \gamma^2\,\delta = 0,$$

and hence for δ

$$(7.5)\qquad \delta = C\exp\left\{-(\mathfrak{F}_1/m) + \sqrt{(\mathfrak{F}_1/m)^2+\gamma^2}\right\} + D\exp\left\{-(\mathfrak{F}_1/m) - \sqrt{(\mathfrak{F}_1/m)^2+\gamma^2}\right\}.$$

In the second case we have instead

$$\dot{\delta} = \frac{m\gamma^2\sqrt{2}}{\mathfrak{F}_1}\,\delta^{3/2},$$

and hence for δ

$$\delta = \frac{2\mathfrak{F}_1^2}{m^2\gamma^4(t+C)^2}\,. \tag{7.6}$$

In (7.5), (7.6) C and D are arbitrary constants.

The interpretation of formulae (7.5) and (7.6) is clear; the assumption that $a(t)$ has a nearly constant value less than 1 is contradictory: the absolute value of the first addendum in the right-hand side of formula (7.5) increases without limit when $t \to +\infty$. Formula (7.6) suggests that a tends instead rapidly to 1, the difference $1 - a$ being infinitesimal of the second order in $1/t$.

8. - Stability of a journal suspended by its centre of gravity.

In this Section a second problem of stability of the steady rotation of a journal is studied: the assumption is made at the outset that the centre of gravity O of the journal is kept immovable in the centre of the bearing by a universal joint without friction, while arbitrary rotations are allowed.

In absence of lubricant the movements of the journal would then be regular precessions with a speed of precession which is $[1-(C/A)]$ times the angular speed ω around the ζ-axis. This fact has a decisive influence on the rule of stability which we shall obtain; it will become apparent that the steady rotation is stable if, and only if, ω is less than twice the « natural » speed of precession of the journal. Stated in this way the rule is attractice, because it sounds similar to that reached under different circumstances at the end of Section 5; but, in fact, it might be misleading. No condition on the value of ω is actually involved, because the « natural » speed of precession is itself proportional to ω. The restriction is purely geometrical: stability is ensured at any speed if, and only if, the moment of inertia C is greater than $A/2$.

To study the movement of the journal, we use the nodal equations, i.e. the equations which involve: (i) the components of the moment of momentum on a nodal system of reference and (ii) the moments M_1, M_2 of the couples caused by the action of the lubricant, as defined by formulae (4.4), (4.5). If λ and μ are the components of the angular speed along the line of nodes and in a normal direction in the plane $\zeta = 0$ respectively, the first two nodal equations can be written as follows

$$\begin{aligned} A\dot{\lambda} - (A-C)\omega\mu + A\mu\dot{\varphi} &= M_1, \\ A\dot{\mu} + (A-C)\omega\lambda - A\lambda\dot{\varphi} &= M_2\,. \end{aligned} \tag{8.1}$$

The third equation has no interest here; it would only specify the moment of the couple required to maintain the angular speed around the ζ-axis constant, in accordance with our hypothesis. λ, μ and ω are bound to the values of θ, ψ, φ and their derivatives by the approximate relations

$$\lambda = \dot{\theta}, \qquad \mu = \theta\dot{\psi}, \qquad \omega = \dot{\varphi} + \dot{\psi},$$

so that, in terms of θ and ψ, eqns (8.1) become

$$\begin{aligned} A\ddot{\theta} + C\omega\theta\dot{\psi} - A\theta\dot{\psi}^2 &= M_1, \\ A\theta\ddot{\psi} + 2A\dot{\theta}\dot{\psi} - C\omega\dot{\theta} &= M_2 . \end{aligned} \tag{8.2}$$

In these equations we cannot specify in general M_1 and M_2 as functions of θ, $\dot{\theta}$, $\dot{\psi}$; but what we know from the results of Sects 3 and 4 is enough to assure us that the system (8.2) has a steady-state solution $\theta \equiv 0$. The solution corresponds to the steady rotation, the stability of which we intend to study here; for this restricted purpose we need to discuss the behaviour of the solutions of (8.2) only for small values of θ, hence we may use for M_1 and M_2 the expressions obtained at the end of Section 4:

$$\begin{aligned} A\ddot{\theta} + C\omega\theta\dot{\psi} - A\theta\dot{\psi}^2 + 2M\dot{\theta} &= 0, \\ A\theta\ddot{\psi} + 2A\dot{\theta}\dot{\psi} - C\omega\dot{\theta} - M\theta(\omega - 2\dot{\psi}) &= 0 . \end{aligned} \tag{8.3}$$

A procedure can now be followed, which is very similar to that described in Sections 5 and 6. First of all some particular solutions are found; from the properties of these solutions it is possible already to decide for the instability of the steady rotation of the journal, if C is less than $A/2$. The proof of the rule can be then completed, again on the basis of general theorems on non-linear differential equations.

The particular solutions we were referring to above are of the type

$$\theta = \bar{\theta}e^{nt}, \qquad \dot{\psi} = \gamma, \tag{8.4}$$

with $\bar{\theta}$, n and γ suitable real constants. Obviously the movements represented by formulae (8.4) are conical whirls of increasing or decreasing amplitude according to whether n is positive or negative; conical whirls of steady amplitude correspond to $n = 0$.

For (8.4) to be a solution of (8.3) n and γ must satisfy the equations

$$(8.5)\qquad An^2 + C\omega\gamma - A\gamma^2 + 2Mn = 0,$$

$$\gamma = \frac{\omega}{2}\frac{Cn + M}{An + M}.$$

These represent two hyperbolae in the $(\gamma,\ n)$ plane, which have only two real points in common: one, Q_1, in the half-plane $n > -M/A$ the other, Q_2, in the half-plane $n < -M/A$. In other words, there are only two real solutions of the system (8.5): $(\gamma_1,\ n_1)$ and $(\gamma_2,\ n_2)$ where the first pair represents the co-ordinates of Q_1 and the second pair the co-ordinates of Q_2. n_2 in always negative, whereas n_1 is positive or negative according to whether $A/2$ is greater or smaller than C and it is zero for $C = A/2$.

The results obtained so far already prove that whirls of increasing amplitude may occur if C is less than $A/2$. That all solutions of (8.3) (with the only exception of those « related » to Q_2) behave asymptotically for $t \to +\infty$ as do

$$\theta = \bar{\theta}e^{n_1 t}, \quad \dot{\psi} = \gamma_1,$$

can be proved with developments which are so similar to those involved in the discussion of the system (5.2) that it is hardly worth entering into details here.

The conclusion is that if C is greater than $A/2$, all solutions of (6.3) tend exponentially to zero after a transient, whatever is the value of the angular speed ω.

9. - Bibliographical note.

Theoretical and experimental papers on lubrication of bearings are so numerous that a compilation of a list of references would be a major undertaking. Here we quote only a few recent papers which had direct influence on our work. The early analyses of REYNOLDS (1886), SOMMERFELD (1904), HARRISON (1913, 1919), STODOLA (1925), ROBERTSON (1933) have been referred to so often that they need not be cited once more; for this reason we have used in Section 7 the expressions of the oil forces for long bearings without further explanation.

In our derivation of Reynolds equation from the general equations of hydrodynamics we have generalized a procedure of G. H. WANNIER [Quart. Appl. Math. **8** (1950), 1-32]. The developments of Section 4 extend results obtained by M. MUSKAT and F. MORGAN [J. Appl. Phys., **9** (1938), 393] for conditions of steady state and perfect alignment. The problem of stability of Sects **5**, **6** and **7** has been partially dealt with before: see H. Poritsky, Trans. A.S.M.E., **75** (1953), 1153-1161. There are many recorded cases of « bearing instability »; the experimental investigation in which the set of idealized conditions assumed here was satisfied more closely is that carried out by G. F. Boeker and B. Sternlicht [Trans. A.S.M.E., **78** (1956), 13-19]. Some of the results obtained here have been applied in a study « *On the Vibrations of Shafts Rotating on Lubricated Bearings* », which will appear in 1960 in the Annali di Matematica Pura e Applicata. Note added in proof: For a more complete treatment of the topics of Section 3 see a paper by H. G. Elrod in the Quart. Appl. Math. (January 1960).

10. - Acknowledgement.

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Sunto.

Un completo studio del comportamento cinematico della pellicola di lubrificante in un supporto cilindrico di un corpo ruotante ha notevole interesse tecnico, perchè in tale comportamento va spesso ricercata la causa di indesiderabili vibrazioni autoeccitate. Nel presente lavoro si è cercato di porre su basi razionali lo studio in questione (mettendo, ad esempio, in diretta relazione la equazione di Reynolds della teoria della lubrificazione con le equazioni generali della idrodinamica) e si sono discussi in dettaglio i problemi di stabilità per il perno a perfetto allineamento e per il perno a cui sono permesse solo rotazioni attorno al baricentro: due sistemi meccanici piuttosto idealizzati, le cui proprietà riflettono però abbastanza fedelmente quelle di dispositivi più complessi, tanto che sono stati oggetto di varie ricerche sperimentali.