Riv. Mat. Univ. Parma 10 (1959), 183-200

L. CARLITZ (*)

A Class of Symmetric q-Polynomials. (**)

1. - Introduction.

Put

$$\prod_0^\infty (1 - q^n t)^{-1} (1 - q^n xt)^{-1} = \sum_0^\infty H_n(x) \frac{t^n}{(q)_n}, \tag{1.1}$$

$$\prod_0^\infty (1 - q^n t) (1 - q^n xt)^{-1} = \sum_0^\infty (-1)^n q^{n(n-1)/2} G_n(x) \frac{t^n}{(q)_n}, \tag{1.2}$$

where

$$(q)_0 = 1, \quad (q)_n = (1-q) \dots (1-q^n),$$

and $|q| < 1$. Properties of the polynomials $H_n(x)$, $G_n(x)$ have been discussed in a number of papers [2], [3], [4], [5], [7], [9]. In particular, SZEGÖ [7] and WIGERT [9] have obtained orthogonality relations satisfies by these polynomials; see also [3].

A possibile generalization of (1.1) and (1.2) is furnished by [1], [8]:

$$\prod_{n=0}^\infty \prod_{r=1}^k (1 - q^n x_r t)^{-1} = \sum_{n=0}^\infty H_n(x_1, \dots, x_k) \frac{t^n}{(q)_n}, \tag{1.3}$$

$$\prod_{n=0}^\infty \prod_{r=1}^k (1 - q^n x_r t) = \sum_{n=0}^\infty (-1)^n q^{n(n-1)/2} G_n(x_1, \dots, x_k) \frac{t^n}{(q)_n}, \tag{1.4}$$

(*) Address: Department of Mathematics, Duke University, Durham, North Carolina, U.S.A..

(**) Received February 17, 1959.

where $k \geqslant 2$. Clearly $H_n = H_n(x_1, \ldots, x_k)$ and $G_n = G_n(x_1, \ldots, x_k)$ are polynomials symmetric in the x_j; the coefficients are themselves polynomials in q.

Let E_j denote the operator that replaces x_j by $q\,x_j$ but leaves the remaining x_i unchanged. If we apply E_j to both sides of (1.3), we get

$$\sum_{n=0}^{\infty} E_j\, H_n(x_1, \ldots, x_k)\, \frac{t^n}{(q)_n} = (1 - x_j\, t) \sum_{n=0}^{\infty} H_n(x_1, \ldots, x_k)\, \frac{t^n}{(q)_n},$$

which yields

$$E_j\, H_n = H_n - (1 - q^n)\, x_j\, H_{n-1}. \tag{1.5}$$

If we now define the operators

$$\Omega_r = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_k \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{k-2} & x_2^{k-2} & \ldots & x_k^{k-2} \\ x_1^r\, E_1 & x_2^r\, E_2 & \ldots & x_k^r\, E_k \end{vmatrix} \qquad (r = 0, 1, 2, \ldots),$$

then it follows from (1.5) that

$$\Omega_r\, H_n = 0 \qquad (0 \leqslant r \leqslant k-3), \tag{1.6}$$

while

$$\Omega_{k-2}\, H_n = (1 - q^n)\, T\, H_{n-1}, \tag{1.7}$$

where T is the VANDERMONDE determinant:

$$T = | x_s^{r-1} | \qquad (r, s = 1, 2, \ldots, k). \tag{1.8}$$

Also it is easily verified that

$$\Omega_{k-1}\, H_n = T\left(H_n - (1 - q^n)\, H_1\, H_{n-1}\right). \tag{1.9}$$

If S_n is any homogeneous symmetric polynomial in $x_1, \ldots, x_k$, then $T^{-1}\,\Omega_r\,S_n$ is either 0 or some symmetric polynomial of degree $n + r - k + 1$. In particular for $r = k - 1$ it is of interest to consider the equation

$$T^{-1}\,\Omega_{k-1}\,S_n = \lambda\,S_n, \tag{1.10}$$

where λ is independent of the x's. In the present paper we consider a number of equations of this kind. However we limit the discussion to the case $k = 3$. We construct several bases for the set of symmetric polynomials of degree n and also show in particular that (1.10) is solvable if and only if $\lambda = q^r$ $(0 \leqslant r \leqslant n)$.

2. - Notation.

We shall use the following notation. Put

$$X = z - y, \qquad Y = x - z, \qquad Z = y - x, \tag{2.1}$$

$$T = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \sum x^2\,X, \tag{2.2}$$

$$\Omega_r = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^r\,E_x & y^r\,E_y & z^r\,E_z \end{vmatrix} \qquad (r = 0,\ 1,\ 2,\ \ldots), \tag{2.3}$$

where E_x replaces x by qx, and similarly for E_y and E_z;

$$\Omega = \Omega_0 = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ E_x & E_y & E_z \end{vmatrix}. \tag{2.4}$$

Then clearly

$$\Omega_r = \sum x^r\,X\,E_x = x^r\,X\,E_x + y^r\,Y E_y + z^r\,Z\,E_z. \tag{2.5}$$

We next put

$$e(t) = \prod_{0}^{\infty} (1 - q^n t)^{-1} \tag{2.6}$$

and define H_n, G_n by means of

$$\prod_{x,y,z} e(xt) = \sum_{0}^{\infty} H_n(x, y, z) t^n, \tag{2.7}$$

$$\prod_{x,y,z} (e(qxt))^{-1} = \sum_{0}^{\infty} G_n(x, y, z) t^n. \tag{2.8}$$

Notice that the new definitions differ slightly from (1. 3), (1.4); the new definition leads to somewhat more compact formulas.

We shall also require polynomials K_n, L_n, defined by

$$\prod_{x,y,z} e(xyt) = \sum_{0}^{\infty} K_n(x, y, z) t^n, \tag{2.9}$$

$$\prod_{x,y,z} (e(qxyt))^{-1} = \sum_{0}^{\infty} L_n(x, y, z) t^n. \tag{2.10}$$

It is clear from (2.7), (2.8), (2.10) that

$$\begin{cases} K_n(x, y, z) = H_n(yz, zx, xy), \\ L_n(x, y, z) = G_n(yz, zx, xy); \end{cases} \tag{2.11}$$

moreover

$$\begin{cases} K_n(yz, zx, xy) = (xyz)^n H_n(x, y, z), \\ L_n(yz, zx, xy) = (xyz)^n G_n(x, y, z). \end{cases} \tag{2.12}$$

Since

$$e(t) = \sum_{0}^{\infty} \frac{t^n}{(q)_n}, \qquad (e(qt))^{-1} = \sum_{0}^{\infty} (-1)^n \frac{(q)^{n(n+1)/2}}{(q)_n} t^n,$$

it follows that

$$H_n = \sum_{i+j+k=n} \frac{x^i y^j z^k}{(q)_i (q)_j (q)_k}, \tag{2.13}$$

$$G_n = (-1)^n \sum_{i+j+k=n} q^{\{i(i+1)+j(j+1)+k(k+1)\}/2} \frac{x^i y^j z^k}{(q)_i (q)_j (q)_k}. \tag{2.14}$$

If $f(q)$ is a rational function of q we define

$$f^*(q) = f(1/q). \tag{2.15}$$

Comparison of (2.13) with (2.14) shows that

$$G_n = H_n^*, \qquad G_n^* = H_n, \tag{2.16}$$

and in view of (2.11) we have also

$$L_n = K_n^*, \qquad L_n^* = K_n. \tag{2.17}$$

We also define

$$P_{m,n} = \sum_{r=0}^{\min(m,n)} \frac{q^r (xyz)^r}{(q)_r} G_{m-r} L_{n-r}, \tag{2.18}$$

$$Q_{m,n} = \sum_{r=0}^{\min(m,n)} (-1)^r \frac{q^{r(r-1)/2} (xyz)^r}{(q)_r} H_{m-r} K_{n-r}. \tag{2.19}$$

Then if follows that

$$P_{m,n}^* = Q_{m,n}, \qquad Q_{m,n}^* = P_{m,n}, \tag{2.20}$$

and, by (2.11), (2.12),

$$\begin{cases} P_{m,n}(yz, zx, xy) = (xyz)^n P_{m,n}(x, y, z), \\ Q_{m,n}(yz, zx, xy) = (xyz)^n Q_{m,n}(x, y, z). \end{cases} \tag{2.21}$$

We remarck that (2.18) and (2.19) can be inverted. The formulas

$$G_m L_n = \sum_{r=0}^{\min(m,n)} (-1)^r \frac{q^{r(r-1)/2}(xyz)^r}{(q)_r} P_{m-r,n-r}, \tag{2.22}$$

$$H_m K_n = \sum_{r=0}^{\min(m,n)} \frac{(xyz)^r}{(q)_r} Q_{m-r,n-r} \tag{2.23}$$

are easily verified. We also note that

$$\sum_{m,n=0}^{\infty} Q_{m,n} u^m v^n = \frac{\prod_{x,y,z} e(xu) \quad e(yzv)}{e(xyzuv)}, \tag{2.24}$$

$$\sum_{m,n=0}^{\infty} P_{m,n} u^m v^n = \frac{e(qxyzuv)}{\prod_{x,y,z} e(qxu) \quad e(qyzv)}. \tag{2.25}$$

3. – An immediate consequence of (2.7) is

$$(1 - E_x) H_n = x H_{n-1}, \tag{3.1}$$

with like formulas for E_y and E_z. Similarly from (2.8) we get

$$(1 - E_x) G_n = - qx E_x G_{n-1}. \tag{3.2}$$

Similarly (2.9) and (2.10) yield

$$E_x K_n = K_n - x(y + z) K_{n+1} + x^2 yz K_{n-2}, \tag{9.3}$$

$$L_n = E_x L_n - qx(y + z) E_x L_{n+1} + q^2 x^2 yz E_x L_{n-2}. \tag{3.4}$$

In each case there are like formulas involving E_y and E_z. Note that in view of (2.16) and (2.17), (3.1) and (3.2) are equivalent, also (3.3) and (3.4) are equivalent.

Returning to (2.7), if we replace t by q^t we get

$$(1 - xt)(1 - yt)(1 - zt) \sum_0^\infty H_n t^n = \sum_0^\infty H_n q^n t^n,$$

which implies

$$(1 - q^n) H_n - (\textstyle\sum x) H_{n-1} + (\textstyle\sum xy) H_{n-2} - xyz H_{n-3} = 0. \tag{3.5}$$

Again, replacing t by qt in (2.8), we get

$$(1 - qxt)(1 - qyt)(1 - qzt) \sum_0^\infty G_n q^n t^n = \sum_0^\infty G_n t^n,$$

so that

$$q^n \{ G_n - (\sum x) G_{n-1} + (\sum xy) G_{n-2} - xyz\, G_{n-3} \} = G_n . \tag{3.6}$$

The corresponding formulas for K_m and L_m are

$$(1 - q^n) K_n - (\sum xy) K_{n-1} + xyz (\sum x) K_{n-2} - x^2 y^2 z^2 K_{n-3} = 0 , \tag{3.7}$$

$$q^n \{L_n - (\sum xy) L_{n-1} + xyz\, (\sum x) L_{n-2} - x^2 y^2 z^2 L_{n-3}\} = L_n . \tag{3.8}$$

Various combinations of the above formulas are of some interest. For example, from (3.1) we get

$$(E_x - E_y) H_n + (y - x) H_{n-1} = 0, \qquad (y E_x - xE_y) H_n = (y - x) H_{n-1},$$

while (3.3) yelds

$$(\sum E_x) K_n = 3 K_n - 2 (\sum xy) K_{n-1} + xyz (\sum x) K_{n-2},$$

$$(E_x - E_y) K_n = (y - x) z K_{n-1} - xyz (y - x) K_{n-2} .$$

Many other formulas of this sort can be obtained without difficulty.

4. – For brevity put

$$\Phi(u) = \prod_{x,y,z} (e(qxu))^{-1}, \qquad \Psi(v) = \prod_{x,y,z} (e(qxyv))^{-1}. \tag{4.1}$$

Then we have

$$T^{-1} \Omega_2(\Phi(u) \Psi(v)) = \Phi(qu) \Phi(qv) \cdot T^{-1} \sum x^2 X (1 - qyu)(1 - qzu)(1 - qyzv) .$$

Now it is easily verified that

$$\sum x^2 X = T, \qquad \sum x^2 X (y+z) = 0, \qquad \sum x^2 X yz = 0,$$

$$\sum x^2 X (y+z)yz = xyz \sum xX (y+z) = -xyz\, T, \quad \sum x^2 X y^2 z^2 = (xyz)^2 \sum X = 0.$$

Consequently

$$T^{-1} \Omega_2(\Phi(u)\Psi(v)) = \Phi(qu)\Psi(qv)(1 - q^2 xyzuv).$$

If we now make use of (2.8) and (2.10), we at once obtain

$$T^{-1} \Omega_2(G_m L_n) = q^{m+n}(G_m L_n - xyz\, G_{m-1} L_{n-1}), \tag{4.2}$$

where it is understood that if $mn = 0$, then the right member of (4.2) is $q^{m+n} G_m L_n$.

Next, referring to the definition of $P_{m,n}$ in (2.18), we have

$$T^{-1} \Omega_2 P_{m,n} = \sum_{r=0}^{\min(m,n)} \frac{q^{2r}(xyz)^r}{(q)_r} T^{-1} \Omega_2(G_{m-r} L_{n-r}) =$$

$$= \sum_r \frac{q^{2r}(xyz)^r}{(q)_r} = q^{m+n-2r}(G_{m-r} L_{n-r} - xyz\, G_{m-r-1} L_{n-r-1}) =$$

$$= q^{m+n} \left\{ \sum_r \frac{(xyz)^r}{(q)_r} G_{m-r} L_{n-r} - \sum_r \frac{(xyz)^r}{(q)_{r-1}} G_{m-r} L_{n-r} \right\} =$$

$$= q^{m+n} \sum_r \frac{(xyz)^r}{(q)_r} (1 - (1 - q^r)) G_{m-r} L_{n-r} = q^{m+n} \sum_r \frac{q^r (xyz)^r}{(q)_r} G_{m-r} L_{n-r},$$

and therefore

$$\boxed{T^{-1} \Omega_2 P_{m,n} = q^{m+n} P_{m,n}}. \tag{4.3}$$

In the next place, we have, with the notation (4.1),

$$T^{-1} \Omega_1(\Phi(u)\Psi(v)) = \Phi(qu)\Psi(qv) \cdot T^{-1} \sum x X (1 - qyu)(1 - qzu)(1 - qyzv).$$

Since

$$\sum x X = 0, \qquad \sum x X (y + z) = T,$$

$$\sum x X (y + z) yz = 0, \qquad \sum x X y^2 z^2 = xyzT,$$

it is clear that

$$T^{-1} \Omega_1 \left(\Phi(u) \Psi(v)\right) = \Phi(qu) \Psi(qv) (qu - q^3 xyzT u^2 v) .$$

It accordingly follows that

$$T^{-1} \Omega_1(G_m L_n) = q^{m+n} (G_{m-1} L_n - xyz G_{m-2} L_{n-1}) . \tag{4.4}$$

Consequently

$$T^{-1} \Omega_1 P_{m,n} = \sum_r \frac{q^{2r} (xyz)^r}{(q)_r} q^{m+n-2r} (G_{m-r-1} L_{n-r} - xyz G_{m-r-2} L_{n-r-1}) =$$

$$= q^{m+n} \sum_r \frac{(xyz)^r}{(q)_r} \left(1 - (1 - q^r)\right) G_{m-r-1} L_{n-r} = q^{m+n} \sum_r \frac{q^r (xyz)^r}{(q)_r} G_{m-r-1} L_{n-r} ,$$

so that

$$\boxed{T^{-1} \Omega_1 P_{m,n} = q^{m+n} P_{m-1,n}} . \tag{4.5}$$

We next consider

$$T^{-1} \Omega(\Phi(u) \Psi(v)) = \Phi(qu) \Psi(qv) T^{-1} \sum X (1 - qyu) (1 - qzu) (1 - qyzv).$$

Since

$$\sum X = 0, \qquad \sum X (y + z) = 0, \qquad \sum Xyz = T,$$

$$\sum X (y + z) yz = (\sum x) (\sum Xyz) - \sum Xxyz = T \sum x, \qquad \sum X y^2 z^2 = T \sum xy,$$

we find that

$$T^{-1} \Omega(\Phi(u) \Psi(v)) = \Phi(qu) \Psi(qv) \left\{ q^2 u^2 - qv - q^2 uv \sum x - q^3 u^2 v \sum xy \right\},$$

which yields

$$T^{-1}\,\Omega(G_m L_n) = q^{m+n}\,(G_{m-2} L_n - G_m L_{n-1} + \sum x\, G_{m-1} L_{n-1} - \sum xy\, G_{n-2} L_{n-1}) =$$

$$= q^{m+n}\, G_{m-2} L_n - q^{m+n}\,(G_m - \sum x\, G_{n-1} + \sum xy\, G_{m-2})\, L_{n-1}\,.$$

Using (3.6), this becomes

$$q^{m+n}\, G_{m-2} L_n - q^n\,(G_m + q^m\, xyz\, G_{m-3})\, L_{n-1}\,,$$

so that

$$T^{-1}\,\Omega(G_m L_n) = q^{m+n}\,(G_{m-2} L_n - xyz\, G_{m-3} L_{n-1}) - q^n\, G_m L_{n-1}\,. \tag{4.6}$$

It follows from (4.6) that

$$T^{-1}\,\Omega\, P_{m,n} =$$

$$= \sum_r \frac{q^{2r}\,(xyz)^r}{(q)_r} \left\{ q^{m+n-2r}\,(G_{m-r-2} L_{n-r} - xyz\, G_{m-r-3} L_{n-r-1}) - q^{n-r}\, G_{m-r} L_{n-r-1} \right\},$$

and a little manipulation leads to

$$\boxed{T^{-1}\,\Omega\, P_{m,n} = q^{m+n}\, P_{m-2,n} - q^n\, P_{m,n-1}}\,. \tag{4.7}$$

The reduction of $\Omega_3 P_{m,n}$ is slightly more involved. Since

$$T^{-1}\,\Omega_3(\Phi(u)\,\Psi(v)) = \Phi(qu)\,\Psi(qv)\cdot T^{-1} \sum x^3\, X\,(1-qyu)\,(1-qzu)\,(1-qyzv)$$

and

$$\sum x^3 X = T \sum x, \qquad \sum x^3 X\,(y+z) = T\left\{ (\sum x)^2 - \sum x^2 - \sum xy \right\} = T \sum xy,$$

$$\sum x^3 X\, yz = T\, xyz, \qquad \sum x^3 X\,(y+z)\, yz = xyz \sum x^2 X\,(y+z) = 0, \qquad \sum x^3 X y^2 z^2 = 0,$$

we get

$$T^{-1}\,\Omega_3(G_m L_n) = q^{m+n}\,(\sum x\, G_m L_n - \sum xy\, G_{m-1} L_n + xyz\, G_{m-2} L_n - xyz\, G_m L_{n-1})\,.$$

But by (3.6)

$$q^{m+1}\left(\sum x\, G_m - \sum xy\, G_{m-1} + xyz\, G_{m-2}\right) = (q^{m+1}-1)\, G_{m+1},$$

which gives

$$T^{-1}\,\Omega_3(G_m\, L_n) = q^{n-1}\,(q^{m+1}-1)\,(G_{m+1}\, L_n - q^{m+n}\, xyz\, G_m\, L_{n-1}\,. \tag{4.8}$$

Using (4.8) it is easy to obtain

$$\boxed{T^{-1}\,\Omega_3\, P_{m,n} = (q^{\,m+n} - q^{n-1})\, P_{m+1,n}}\,. \tag{4.9}$$

The formulas (4.3), (4.5), (4.7), (4.9) are concerned with $P_{m,n}$. It is also possible to obtain a number of like results involving $Q_{m,n}$. Thus, since

$$T^{-1}\,\Omega\Big\{\prod_{x,y,z} e(xu)\, e(yzv)\Big\} = \prod_{x,y,z} e(xu)\, e(yzv)\cdot T^{-1} \sum X\,(1-xu)\,(1-xyv)\,(1-xzu),$$

we find that

$$T^{-1}\,\Omega(H_m\, K_n) = H_m\, K_{n-1} - xyz\, H_{m-1}\, K_{n-2}\,. \tag{4.10}$$

Now using (2.19) we get

$$\boxed{T^{-1}\,\Omega\, Q_{m,n} = Q_{m,n-1}}\,. \tag{4.11}$$

In a similar manner we obtain

$$\boxed{T^{-1}\,\Omega_1\, Q_{m,n} = -\,q^n\, Q_{m-1,n} + xyz\, Q_{m,n-2}}\,. \tag{4.12}$$

A formula of a somewhat different kind that may be mentioned is

$$\boxed{T^{-1}\,\Omega_2(L_m\, L_n) = q^{m+n}\, L_m\, L_n}\,. \tag{4.13}$$

Returning to (4.9) it would be of interest to evaluate

$$T^{-1}\,\Omega_r\, P_{m,n} \tag{4.14}$$

for $r > 3$. However even for $r = 4$ the computation becomes involved. Thus, if h_r denotes the complete symmetric function of weight r, we find that

$$T^{-1}\,\Omega_r(G_m\,L_n) = q^{m+n}\Big\{\big[\,h_{r-2}\,G_m - (h_1\,h_{r-2} - h_{r-1})\,G_{m-1} + xyz\,h_{r-3}\,G_{m-2}\big]\,L_n - $$

$$- xyz\big[h_{r-3}\,G_m - (h_1\,h_{r-3} - h_{r-2})\,G_{m-1} + xyz\,h_{r-4}\,G_{m-2}\big]\,L_{n-1}\Big\}.$$

This does not seem to imply a usable formula for (4.14).

5. – It is familiar that the number of linearly independent homogenous symmetric polynomials of weight n is the number of solutions in non-negative integers of the equation

$$r + 2s + 3t = n. \tag{5.1}$$

See for example [**6**, Chapters 5, 6].

We shall now prove the following

Theorem 1. *The set*

$$P_{r,s,t} = (xyz)^t\,P_{r,s}, \tag{5.2}$$

where r, s, t *are non-negative integers that satisfy* (5.1) *constitute a basis for symmetric polynomials of weight* n.

Assume a relation

$$\sum_{r+2s+3t=n} a_{rst}\,P_{r,s,t} = 0, \tag{5.3}$$

where the a_{rst} are independent of x, y, z. Since, by (4.3) and (5.2),

$$T^{-1}\,\Omega_2\,P_{r,s,t} = q^{r+s+t}\,P_{r,s,t}, \tag{5.4}$$

(5.3) implies

$$\sum_{r+2s+3t=n} q^{(r+s+t)j}\,a_{rst}\,P_{r,s,t} = 0 \qquad (j = 0, 1, 2, \ldots). \tag{5.5}$$

If we put

$$A_k = \sum_{\substack{r+2s+3t=n\\ r+s+t=k}} a_{rst}\,P_{r,s,t} \qquad (0 \leqslant k \leqslant n),$$

then it follows from (5.5) that

$$\sum_{k=0}^{n} q^{kj} A_k = 0 \qquad (j = 0, 1, \dots, n).$$

Hence $A_k = 0$ $(k = 0, 1, \dots, n)$. Next applying (4.5) we get

$$\sum_{s+2t=n-k} a_{rst} P_{0,s,t} = 0 \qquad (r = 0, 1, 2, \dots). \tag{5.6}$$

Again applying the operator $T^{-1}\Omega_2$ and repeating the above argument we get

$$a_{rst} P_{0,s,t} = 0, \tag{5.7}$$

where $s + 2t = n - k$, $s + t = j$. This implies $t = n - k - j$; since k, j are arbitrary it follows that s, t are also arbitrary. Consequently (5.7) implies $a_{rst} = 0$ for all r, s, t.

Thus the $P_{r,s,t}$ are linearly independent. Since the number of $P_{r,s,t}$ satisfying (5.1) is equal to the number of linearly independent symmetric polynomials of weight n, the theorem follows.

Applying (2.20) we get the following corollary.

Theorem 2. *The set*

$$Q_{r,s,t} = (xyz)^t Q_{r,s,t}, \tag{5.8}$$

where r, s, t *are non-negative integers that satisfy* (5.1), *constitute a basis for symmetric polynomials of weight* n.

We now consider the equation

$$T^{-1}\Omega_2 S_n = \lambda S_n, \tag{5.9}$$

where S_n denotes a (homogeneous) symmetric polynomial of weight n. In view of Theorem 1, we may put

$$S_n = \sum_{r+2s+3t=n} a_{rst} P_{r,s,t}. \tag{5.10}$$

By (5.4) and (5.9)

$$\lambda \sum_{r+2s+3t=n} a_{rst} P_{r,s,t} = \sum_{r+2s+3t=n} q^{r+s+t} a_{rst} P_{r,s,t}.$$

In view of the linear independence of the $P_{r,s,t}$ we infer that $\lambda = q^{r+s+t}$ for some set of non-negative integers r, s, t satisfying $r + 2s + 3t = n$. Thus $\lambda = q^k$ with $0 \leqslant k \leqslant n$. This proves

Theorem 3. *The equation* (5.9) *is solvable in symmetric polynomials of weight* n *only when*

$$\lambda = q^k \qquad (0 \leqslant k \leqslant n). \tag{5.11}$$

When (5.11) *holds the general solution of* (5.9) *is given by*

$$S_n = \sum_{\substack{r+2s+3t=n \\ r+s+t=k}} a_{rst} P_{r,s,t}, \tag{5.12}$$

where the a_{rst} *are arbitrary*.

It is clear from Theorem 1 and (5.4) that the operator $T^{-1}\Omega_2$ induces a non-singular linear transformation on the space R_n of symmetric polynomials of weight n. The rank of R_n is the number of solutions of (5.1); the characteristic values of the linear transformation are given by (5.11). Moreover it follows from (5.4) that the matrix of the linear transformation is in diagonal form relative to the basis $P_{r,s,t}$.

In the next place we may consider the equation

$$T^{-1}\Omega_1 S_n = S_{n-1}, \tag{5.13}$$

where S_{n-1} is assigned. In view of Theorem 1 it will suffice to discuss the case

$$S_{n-1} = P_{r,s,t},$$

where r, s, t are fixed integers such that $r + 2s + 3t = n - 1$. Then (4.5) yields the particular solution

$$S_n = q^{-r-s-t-1} P_{r+1,s,t}.$$

Also it is clear that the general solution of

$$T^{-1}\Omega_1 S_n = 0$$

is furnished by

$$Z_n = \sum_{2s+3t=n} a_{0st} P_{0,s,t}, \tag{5.14}$$

where the a_{0st} are arbitrary. We have therefore

Theorem 4. *The general solution of the equation*

$$T^{-1}\,\Omega_1\,S_n = P_{r,s,t}\,, \tag{5.15}$$

where r, s, t are fixed integers such that $r+2s+3t=n-1$, is given by

$$S_n = q^{-r-s-t-1}\,P_{r+1,s,t} + Z_n, \tag{5.16}$$

where Z_n is defined by (5.14).

As for the equation

$$T^{-1}\,\Omega\,S_n = S_{n-2},$$

it is advantageous to use $Q_{r,s,t}$ rather than $P_{r,s,t}$; this is clear when (4.11) is compared with (4.7). We obtain the following result

Theorem 5. *The general solution of the equation*

$$T^{-1}\,\Omega\,S_n = Q_{r,s,t}\,, \tag{5.17}$$

where r, s, t are fixed integers such that $r+2s+3t=n-2$, is given by

$$S_n = q^{-t}\,Q_{r,s+1,t} + Z_n', \tag{5.18}$$

where

$$Z_n' = \sum_{r+3t=n} a_{r0t}\,Q_{r,0,t}$$

and the a_{r0t} are arbitrary.

The final equation we discuss is

$$T^{-1}\,\Omega_3\,S_n = S_{n+1}\,. \tag{5.19}$$

Since (4.9) implies

$$T^{-1}\,\Omega_3\,P_{r,s,t} = (q^{r+s} - q^{s-1})\,q^t\,P_{r+1,s,t},$$

it follows thzt (5.19) is solvable only when

$$S_{n+1} = \sum_{r+2s+3t=n} a_{rst}\,P_{r+1,s,t}\,. \tag{5.20}$$

We may state

Theorem 6. *The equation* (5.19) *is solvable for* S_n *if and only if* S_{n+1} *is of the form* (5.20), *in which case*

$$S_n = \sum_{r+2s+3t=n} \frac{a_{rst}}{(q^{r+s} - q^{s-1})\, q^t} P_{r,s,t}\,.$$

In particular, when (5.19) *is solvable, the solution is unique.*

6. – As an application of (4.13) we consider the representation

$$L_m L_n = \sum_{r+2s+3t=2m+2n} a_{rst} P_{r,s,t}; \tag{6.1}$$

the possibility of such a representation is clear from Theorem 1. Now, using (4.13) and (5.4), we get

$$q^{m+n} L_m L_n = \sum_{r+2s+3t=2m+2n} q^{r+s+t} a_{rst} P_{r,s,t}\,.$$

Comparison with (6.1) indicates that $a_{rst} = 0$ unless $r + s + t = m + n$. Since also $r + 2s + 3t = 2m + 2n$, it follows that $r = t$, $s = m + n - 2r$, so that (6.1) reduces to

$$L_m L_n = \sum_{2r \leq m+n} a_r P_{r,m+n-2r,r}\,, \tag{6.2}$$

where $a_r = a_{r,m+n-2r,r}$. Next, making use of (2.21), we find that (6.2) becomes

$$G_m G_n = \sum_{2r \leq m+n} a_r P_{m+n-2r,r}; \tag{6.3}$$

in view of (2.16) and (2.20) we have also

$$H_m H_n = \sum_{2r \leq m+n} a_r^* Q_{m+n-2r,r}\,, \tag{6.4}$$

where the asterisk has the same meaning as in (2.15). It remains to determine a_r.

If we put $z = 0$ in (6.4) and note that

$$Q_{m,n}(x, y, 0) = H_m(x, y, 0)\, K_n(x, y, 0)\,.$$

But

$$K_n(x, y, 0) = (xy)^r/(q)_r,$$

so that (6.4) becomes

$$H_m(x, y)\, H_n(x, y) = \sum_r a_r^* \frac{(xy)^r}{(q)_r} H_{m+n-2r}(x, y),$$

where $H_m(x, y) = H_m(x, y, 0)$. But by [**4**, formula (1.7)], we have

$$H_m(x, y)\, H_n(x, y) = \sum_{r=0}^{\min(m,n)} \begin{bmatrix} m+n-2r \\ m-r \end{bmatrix} \frac{(xy)^r}{(q)_r} H_{m+n-2r}(x, y),$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{(q)_m}{(q)_r\,(q)_{m-r}}.$$

Thus

$$a_r^* = \begin{bmatrix} m+n-2r \\ m-r \end{bmatrix}, \qquad a_r = q^{-(m-r)(n-r)} \begin{bmatrix} m+n-2r \\ m-r \end{bmatrix}.$$

Therefore (6.3) and (6.4) become

$$G_m\, G_n = \sum_{r=0}^{\min(m,n)} q^{-(m-r)(n-r)} \begin{bmatrix} m+n-2r \\ m-r \end{bmatrix} P_{m+n-2r,r}, \tag{6.5}$$

$$H_m\, H_n = \sum_{r=0}^{\min(m,n)} \begin{bmatrix} m+n-2r \\ n-r \end{bmatrix} Q_{m+n-2r,r}; \tag{6.6}$$

we have also

$$L_m\, L_n = \sum_{r=0}^{\min(m,n)} q^{-(m-r)(n-r)} \begin{bmatrix} m+n-2r \\ n-r \end{bmatrix} P_{r,m+n-2r,r}, \tag{6.7}$$

$$K_m\, K_n = \sum_{r=0}^{\min(m,n)} \begin{bmatrix} m+n-2r \\ m-r \end{bmatrix} Q_{r,m+n-2r,r}. \tag{6.8}$$

These formulas may be compared with (2.22) and (2.23).

References.

[1] W. A. SALAM, *A generalization of some polynomials related to the theta function*, Riv. Mat. Univ. Parma **8** (1957), 381-395.

[2] H. BATEMAN, *Functions orthogonal in the Hermitian sense*, Proc. Nat. Acad. Sci. U.S.A. **20** (1934), 63-66.

[3] L. CARLITZ, *Note on orthogonal polynomials related to theta functions*, Publ. Math. Debrecen **5** (1958), 222-228.

[4] L. CARLITZ, *Some polynomials related to the theta functions*, Ann. Mat. Pura Appl. (4) **41** (1955), 359-373.

[5] W. HAHN, *Über orthogonal Polynome, die q-Differenzengleichungen genugen*, Math. Nachr. **2** (1949), 4-34.

[6] D. E. LITTLEWOOD, *The Theory of Group Characters*, Clarendon, Oxford 1940.

[7] G. SZEGÖ, *Ein Beitrag zur Theorie der Thetafunktionen*, Sitzungsber. Preuss. Akad. Wiss. Phys-Math. Kl. (1926), 242-253.

[8] M. WARD, *A calculus of sequences*, Amer. J. Math. **58** (1936), 255-266.

[9] S. WIGERT, *Sur les polynomes orthogonaux et l'approximation des fonctions continues*, Arkiv Math. Astr. Fys. **17** (1923), 1-15.