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Generalized Length and an Inequality of Cesari for Surfaces Defined over Two-Manifolds. (**)

1. - Introduction.

In his book **Surface Area** CESARI introduced the notion of a contour of a surface and used techniques associated with this concept to dovelop some of the most basic properties of FRÉCHET surfaces. CESARI and the present author, both jointly and separately, have published other papers [**1** b, c; **2** *b*; **5** a, b, c] which investigate properties of contours in more detail. The use of methods involving contours yields intrinsic methods of proof of theorems in the theory of surfaces and avoids the necessity of using coordinate systems and other methods which are essentially extraneous to the true notion of a surface. Thus it would appear that the possibility of generalizing the theory along certain lines is much enhanced by using such techniques.

One of this directions in which advances of this type can be made is that of generalizing the domain space of a mapping which defines a surface. The work on contours which has been done to date by CESARI and the author has largely been confined to contours in a simply connected JORDAN region in the plane. More recently, results of this nature have been obtained for mappings on a multiply connected JORDAN region. CESARI [**1** d, e] has introduced and studied the concepts of retraction and fine-cyclic elements for such mappings.

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These notions have been studied from a topological point of view by NEUGEBAUER [7 a, b] and results on representations of surfaces over such sets and over manifolds have been obtained by FLEMING [4 a, b]. CESARI and NEUGEBAUER [3] have also obtained results on the coincidence of the LEBESGUE and GEÖCZE areas of such surfaces. Since the structure of open subsets on compact two-manifolds is more complicated than in the plane, the discussion given in [1 a, sect. 20] will not be valid for mappings from such spaces. The author in [5 c] has developed a prime end theory for open subsets of two-manifolds which helps to surmount the difficulties involved and these properties are used here in defining and studying generalized lengths for contours and, in particular, in proving the CAVALIERI inequality of CESARI [1 a, p. 327] for mappings on two-manifolds. This inequality has already proved extremely useful in the study of mappings defined on planar sets and its validity for mappings on two-manifolds should prove equally useful in developing similar theorems for that case.

2. - Notations and basic definitions.

Let M be a compact two dimensional metric manifold. By this we mean that M is a metric space such that if $p \in M$ then there exists a neighborhood N_p of p which is homeomorphic either to an open disk in E_2 with the counter image of p as the center or to an open half disk plus its bounding diameter with the counter image of p as the center of the diameter. Points of the latter type are called boundary points. of M. Neighborhoods of this type will be called *coordinate neighborhoods* of p in M. We assume that M may or may not have boundary and that M may be orientable or non-orientable. The classification of manifolds of this type is well known [6] and, in particular, it is known that they are triangulable, a fact which we shall use later on.

Let Q be a connected open subset of M and denote its boundary components by $\{\gamma\}_Q$. Let $\gamma \in \{\gamma\}_Q$. Then the set $A(Q, \gamma)$ is defined to be the component of $M - \gamma$ which contains Q. Ends are defined for $A(Q, \gamma)$ as follows. If $p \in \gamma$ is accessible from A, an arc from a point of A to p intersecting γ only at p defines an *end* η of A ending at p. Two arcs b_1, b_2 of this type ending at p are equivalent if either for every neighborhood of p, $(b_1 - (p)) \cap (b_2 - (p)) \neq 0$, or in every neighborhood of p there exists a third arc c joining a point of b_1 to a point of b_2 such that $b_1 \cup b_2 \cup c$ bounds a simply connected JORDAN region in A. An end η of A is *admissible* if some arc defining η intersects Q in a set of points which have p as a limit point. The set of ends of $A(Q, \gamma)$ ending on γ can be divided into *segments* as follows [5 c]. Let γ be covered by a finite family $\mathfrak{M}$ of coordinate neighborhoods $\{N^i\}$ $(i = 1, 2, \ldots, n)$. This is possible because of the compactness of M. Let K be a component of $(N^i - \gamma) \cap A$ for some i.

Then K has a portion of γ on its boundary and ends of K ending on γ can be defined and assigned an ordering as explained in [**1** a]. Since the N^i overlap, this ordering can be extended along γ both ways from K and an order can be built up for a portion of γ. A maximal ordered set of ends $\sigma_{\mathfrak{M}}$ formed by such a construction we call a segment of ends corresponding to the covering $\mathfrak{M}$. As in [**5** c] for different $\mathfrak{M}$ these form a lattice under inclusion and the maximal element of this lattice is called a *segment* of ends ending on γ. Since the ordering is linear, prime ends may also be defined on the segment and the ordered set of ends an prime ends will be called simply a *segment.* Evidently every end belongs to a unique segment. These concepts have been introduced in [**5** c] and their properties have been studied. In particular, it was shown that every segment of ends consists entirely of admissible ends or entirely of inadmissible ends. It is also shown that if $\eta_1 < \eta_2$ are in the segment σ and are defined by arcs b_1, b_2 and if $\varepsilon > 0$ there exists an arc $c_\sigma(\eta_1, \eta_2)$ which begins at b_1, ends at b_2 and is such that every end in σ can be defined by an arc wich intersects $c_\sigma(\eta_1, \eta_2)$ in a point within ε distance of γ. By appropriately joining these arcs at their end points, the union c_σ of all such arcs for a segment σ is either a simple arc or an indefinite arc, i.e. a homeomorphic map of the open or half open unit interval. If the arc is indefinite in one of the above cases it defines one or two *prime ends of second kind.*

Let $T : M \to E_N$ be a continuous mapping from the manifold M into N dimensional euclidean space E_N. Then T will define a surface S over M and the usual definition of Fréchet equivalence will give a class of mappings defining S. The Lebesgue area of S can be defined as follows. Let P_n a triangulation of M and let $T_n : P_n \to E_n$ $(n = 1, 2, \dots)$ be a sequence of quasi linear mappings, each from P_n to E_N such that the mapping T_n approaches T uniformly in the sense that given any $\varepsilon > 0$ there exists a k_ε such that $| T_n(p) - T(p) | < \varepsilon$ for $n > k_\varepsilon$ (the absolute value denotes distance in E_N). For each (T_n, P_n) let $a(T_n, P_n)$ be the sum of the areas of the triangles $T_n(\Delta_i)$, where Δ_i are elements of the triangulation P_n. Define $L(T) = L(T, Q) = \inf (\underline{\lim}_{n\to\infty} a(T_n, P_n)$, where the infimum is taken over all such sequences of mappings T_n and triangulations P_n (see [**3**]). Admissible subsets of M and areas of surfaces defined over such subsets can also be defined as in [**1** a].

3. - Contours and generalized length.

We define contours and generalized length in a manner similar that of Cesari [**1** a]. The development follows in much the same manner and the result are similar.

Let T be a mapping from M into E_N as defined in section **2**. Let $[S]$ denote the set of points in E_N occupied by the surface S and let $f : [S] \to$ Reals be a real valued continuous function defined on $[S]$. Since M is compact, f has a maximum and a minimum value t_1 and t_2 respectively. If $t_1 \leqslant t \leqslant t_2$, let $D^-(t)$, $D^+(t)$, $C(t)$ be defined as the set of points of M for which $f(T(p)) < t$, $f(T(p)) > t$, $f(T(p)) = t$ respectively. Since f is continuous, $C(t)$ is closed and $D^-(t)$, $D^+(t)$ are open. $C(t)$ is said to be the *contour* determined by t, f, T. Evidently the boundaries of $D^-(t)$, $D^+(t)$ in M both lie in $C(t)$.

To define the generalized length of the image of a contour we proceed as follows. Let Q be a component of $D^-(t)$ for some fixed t. Let $\{\gamma\}_Q$ be the family of boundary components of Q in M. Let $A(Q, \gamma)$ be the set defined in section **2**. Let σ be a segment of admissible ends of A ending on γ. Let $\{\eta_1, \eta_2, \eta_3, \ldots, \eta_k\}$ be any finite subfamily of ends of σ, $\eta_1 < \eta_2 < \eta_3 < \ldots < \eta_k$. Let $p_1, p_2, \ldots, p_k$ be the end points in M of arcs defining $\eta_1, \eta_2, \eta_3, \ldots, \eta_k$ respectively. Let $s = \sum_{i=1}^{k-1} | T(p_i) - T(p_{i-1}) |$ (it is possible that $p_1 = p_k$ in which case the obvious minor modifications in this definition must be made).

Let $\lambda_{\sigma,\gamma} = \sup s$, where the supremum is taken over all ordered subfamilies $\{\eta_i\}$ of σ of the above type. Let $\lambda_\gamma = \sum_\sigma \lambda_{\sigma,\gamma}$, where the sum is taken over all segments σ. Let $\lambda_Q = \sum_\gamma \lambda_\gamma$, where γ runs over all elements $\gamma \in \{\gamma\}_Q$. Let finally $l(t;\, T,\, M,\, f) = \sum_Q \lambda_Q$, where the sum is taken over all components Q of $D^-(t)$. The quantity $l(t)$ is called the generalized length of the image under T of the boundary of $D^-(t)$ in M or, more briefly, the generalized length of the image under T of the contour $C(t)$. (A similar definition could be made for $D^+(t)$ and the length in general would be different but we shall use $D^-(t)$ throughout). As in [**1** a, p. 317] it can be seen that if $l(t)$ is finite then only countable families of the $\lambda_{\sigma,\gamma}$, λ_γ, λ_Q can be different from 0 and $\lambda_\gamma = 0$ if and only if γ is a continuum of constancy for T in M. Also from the definition above, it is easily seen that for any $\varepsilon > 0$ if $l(t) < \infty$ there exist finitely many elements $\{\eta_{ijk}\}$, $\{\sigma_{jk}\}$, $\{\gamma_k\}$ with $\eta_{ijk} \in \sigma_{jk}$, all i, σ_{jk} a segment of γ_k for all j and such that

$$l(t) - \varepsilon \leqslant \sum_{k=1}^{n} \sum_{j=1}^{m_k} \sum_{i=1}^{l_{jk}-1} | T(p_{ijk}) - T(p_{i+1,jk}) | \leqslant l(t) .$$

If $l(t) = \infty$, then for appropriately chosen $\{\eta_{ijk}\}, \{\sigma_{jk}\}$ $\{\gamma_k\}$ the sum can be made greater than $1/\varepsilon$.

If $l(t) < \infty$ and ω is any prime end of first or second kind with limiting continuum E_ω, then T is constant on E_ω. This follows again in exactly the same manner as in [**1** a, p. 319] and the fact proved in [**1** b] that the limiting set of any prime end is connected.

Theorem 3.1: *Let σ be a segment of ends and prime ends of $A(Q, \gamma)$ and let γ_σ be the set of points of γ which are end points of the ends and prime ends in σ. Then if $\lambda_\sigma < \infty$ the set $T(\gamma_\sigma) \subset E_N$ is a continuous curve of length λ_σ.*

Proof: Let ω_1 be the first prime end of σ, ω_2 the last. By the definition of prime ends of first and second kind it can be seen that such prime ends exist, although they may coincide. Then σ consists of all prime ends ω for which $\omega_1 \leqslant \omega \leqslant \omega_2$ and where all ω can be defined locally as described in [5 c]. Let η', η'' $(\eta' < \eta'')$ be two ends in σ and consider the interval $\eta' \leqslant \omega \leqslant \eta''$. The proof given in [1 a, p. 319, iv], shows that the mapping T operating on the sets E_ω for this interval defines a continuous curve in E_N and the generalized length of the image of these sets is the same as the ordinary length of the continuous curve. Now assume that the interval $\eta' \leqslant \omega \leqslant \eta''$ is considered to correspond to the real interval $1/3 \leqslant t \leqslant 2/3$ and that the mapping defining the images of the corresponding E_ω as a continuous curve be considered as mapping this interval into E_N. Let $\{\eta^i\}$ $(i = \pm 1, \pm 2, \pm 3, \ldots)$ be ends in σ such that $\ldots < \eta^{-2} < \eta^{-1} < \eta^1 < \eta^2 < \ldots$ define a sequence of intervals I^i in σ which cover σ. Then each such interval defines a continuous curve. Let η^i correspond to 3^i for $i < 0$ and to $1 - 3^{-i}$ for $i > 0$. Then the intervals I_i will correspond to a sequence of intervals filling the open interval $0 < t < 1$ and each one can be mapped continuously into E_N by a mapping τ such that $\tau(t) = T(p_t)$, where p_t is in the set E_ω of the prime end ω corresponding to t, $0 < t < 1$. Since $\lambda_\sigma < \infty$, T is constant in E_{ω_2} and on E_{ω_2} and if $p_i \to p \in \omega_1$, $T(p_i) \to T(p) = T(E_{\omega_1})$ by continuity of T. Since the intervals I_i have terminal elements approaching E_{ω_1} as $I \to -\infty$, and E_{ω_2} as $i \to \infty$, if we let $\tau(0) = T(\omega_1)$ and $\tau(1) = T(\omega_2)$, the entire map τ from the closed interval $0 \leqslant t \leqslant 1$ into E_N is continuous and hence defines a continuous map of $0 \leqslant t \leqslant 1$ into E_N. However, $\tau[0, 1] = T(\bigcup_{\omega \in \sigma} E)$ and hence the image of E_ω, $\omega_1 \leqslant \omega \leqslant \omega_2$, is a continuous curve.

4. - Semi continuity of the generalized length.

From the definition of $l(t) = l(t; T, M, f)$ it can be seen that $l(t)$ is a function of t, T, M, f. We shall assume that M is a fixed manifold and we shall investigate here the dependence of l upon the other three entities. In particular, it will be shown that in a certain sense the function l is lower semi continuous in t, T, f. These properties are needed in the proof of CESARI's inequality. Theorem 4.1, which follows, is the key theorem in this investigation and the other semi continuity theorems will follow from similar considerations.

Theorem 4.1: *Let M, T be given as before. Let $\{f_s(x)\}$ be a one parameter set of continuous real valued functions defined on a compact subset V in E_N which contains the points of the surface defined by T, where the parameter s ranges over a set S of real numbers with upper bound s_0 (possibly infinite). Let the family $\{f_s\}$ have the properties* (a) $f_{s_1}(x) > f_{s_2}(x)$ *if* $s_1 < s_2$, (b) $f_s(x) > f_{s_0}(x)$ *for* *all* $s \in S$, $s \neq s_0$, (c) $\lim_{s \to s_0} f_s(x) = f_{s_0}(x)$ *uniformly on* V*: For each* $f_s(x)$ *let* $l_s(t)$ *be the generalized length defined in section* **3**. *Then* $l_{s_0}(t) \leqslant \liminf_{s \to s_0} l_s(t)$ *for all* t, $-\infty < t < \infty$.

Proof: If $l_{s_0}(t) = 0$ the theorem is trivial. Thus assume $l_{s_0}(t) > 0$ and let $D = D^-(t)$ for the function f_{s_0}, $D_s = D_s^-(t)$ for the function $f_s(x)$. Then D and D_s are all open subsets of M and $D_{s_1} \subset D_{s_2}$ for $s_1 < s_2$; $s_1, s_2 \in S$, $\overline{D}_s \subset D$ for all $s \in S$, $s \neq s_0$. This follows since we have assumed that $f_{s_1}(x) > f_{s_2}(x)$ for $s_1 < s_2$ and hence that the set of all x for which $f_{s_1}(x) < t$ is included in those for which $f_{s_2}(x) < t$. Hence all boundary components of D_{s_1} are included in D and in D_s for $s > s_1$.

We proceed to the proof of the theorem by first proving two lemmas.

Lemma 1: For any $\varepsilon > 0$ there exists an $s_\varepsilon \in S$ such that for $s > s_\varepsilon$, every point of D_s^* in M is within ε distance of some point of D^* in M and every point of D^* in M is within ε distance of some point of D_s^* in M, i.e. $\lim_{s \to s_0} D_s^* = D^*$ in M.

Proof: The uniformity of the convergence of f_s to f_{s_0} implies that for every $\delta > 0$ there exists an s_δ such that $f_s(x) - f_{s_0}(x) < \delta$ for $s > s_\delta$ and all $x \in V$. Consider the set P_ε of all points $p \in \overline{D}$ such that the distance $d(p, D^*) \geqslant \varepsilon$. Let $t - \delta = \sup_{p \in P_\varepsilon} f_{s_0}(T(p))$. Since P_ε is compact, $f_{s_0}(x)$ attains its maximum value on $T(P_\varepsilon)$ and hence $\delta > 0$ since if $\delta = 0$ there would be a point $x_0 \in T(P_\varepsilon)$ with $f_{s_0}(x_0) = t$. However, this would imply that $P_\varepsilon \cap D^* \neq 0$ contrary to the construction of P. Now let s_δ be chosen so that $f_s(x) - f_{s_0}(x) < \delta$ for $s > s_\delta$, $x \in V$. Thus $f_s(x) > f_{s_0}(x) + \delta$ for $x \in V$, $s > s_\delta$ and $f_s(x) < t - \delta + \delta = t$ for $x \in T(P_\varepsilon)$. Thus all boundary points of D_s are in the complement of P_ε and are hence within ε of some point of D^*.

Suppose that there exists a point $p \in D^*$ in M such that if N_ε is an ε sphere about p, then $N_\varepsilon \cap D \cap D_s = 0$ for all D_s. Let $p' \in N \cap D$ and let $f_{s_0}(T(p')) = t - \varrho$. However, by the uniform convergence of the family f_s to f_{s_0}, there exists an s such that for $s > s_\varrho$, $f_s(x) - f_{s_0}(x) < \varrho/2$ for all $x \in V$. Thus at p' we have $f_s(T(p')) < t - \varrho + (\varrho/2) = t - (\varrho/2) < t$ and $p' \in D_s$. Thus every point of D^* is within ε of a point of D_s^* for all sufficiently large s.

Lemma 2: Let Q be a component of D, γ a component of Q^* in M, σ a segment of ends and prime ends of $A(Q, \gamma)$ ending on γ which are admissible and let $\eta_1 < \eta_2 < \eta_3 < \ldots < \eta_m$ be a finite set of ends of σ (η_1 and η_m may coincide). For $\delta > 0$, let C_σ be an arc (possibly indefinite) corresponding to σ as in [**5** c] such that each point of C_σ lies within δ distance of the portion of γ which consists of end points for ends and prime ends in σ, $C_\sigma \cap \gamma = 0$. Let $b_1, b_2, \ldots, b_m$ be arcs which determine $\eta_1, \eta_2, \ldots, \eta_m$ respectively, and which do not intersect each other except possibly in points on γ and let $b_j \cap C_\sigma = p_j$ ($j = 1, 2, \ldots, m$), where $p_j \in Q$ for $j = 1, 2, \ldots, m$. Let k be fixed and let $b_k \in \{ b_j \}$. Then there exists an index $s_\delta \in S$ such that for $s > s_\delta$, D_s has a boundary component γ_s which intersects either all arcs $b_1, b_2, \ldots, b_k$ or $b_k, b_{k+1}, \ldots, b_m$ and if $H \subset A$ is the set bounded by $b_1 \cup b_m \cup C_\sigma \cup \gamma$, then a single component of $H \cap \gamma_s$ intersects these arcs.

Proof: By definition of the sets D_k and by lemma 1, if $F \subset D$ is any closed subset of D there exists an s' such that $F \subset D_s$ for $s > s'$. Since η_k is admissible, there exists an infinite sequence of points $q_1, q_2, \ldots \subset b_k \cap Q$ converging to a point of γ. Since $q_i \in Q$ for each i and since Q contains points not in H by the choice of the p_i, if $p_0 \in Q - H$ there exists arcs l_i connecting q_i to p_0 for $i = 1, 2, \ldots$, all lying in Q and disjoint except for p_0. Since $l_i \cap \gamma = 0$ for all i, each l_i must intersect H^* in b_1, C_σ, or b_m. If for some i, the first intersection point of l_i with H^* is on b_1 or b_m, then $l_i \cap H$ intersects all arcs $b_1, b_2, \ldots, b_k$ or $b_k, b_{k+1}, \ldots, b_m$. However, for some $s \in S$, there exists a component of D_s which contains $l_i \cap H$. Since $\gamma \cap D_s = 0$, there must be a component of D_s^* which separates l_i from γ and this component must intersect all the b_i between b_k and b_i or between b_k and b_m and furthermore, the portion of this component lying in the set H_k bounded by $b_1, \gamma, b_k, C_\sigma$ or $b_m, b_k, \gamma, C_\sigma$ which separates l_i from γ has the ends determined by the b_i as admissible ends.

If the first intersection of each l_i with H^* is on C_σ, let q_i' be the first point of $C_\sigma \cap l_i$ for each i, then for each set l_i there exists an index $s_i \in S$ such that $l_i \subset D_s$ for $s > s_i$ and some component γ_{s_i} of D_{s_i} separates $l_i \cap H$ from γ. If γ_{s_i} does not intersect all the arcs b_i for $i = 1, 2, \ldots, k$ or $i = k, k+1, \ldots, m$, then there exists a first q_i'' point on C (where points are considered as ordered both ways from p_k) in which γ_{s_i} intersects C_σ since γ_{s_i} must then intersect C_σ. For each i, choose an s_i and a γ_{s_i} with this property, where the s_i are chosen in such a way that $\lim_{i\to\infty} s_i = s_0$. Then the points q_i approach a point on γ. If for some i, $\gamma_{s_i} \cap b_1 \neq 0$ or $\gamma_{s_i} \cap b_m \neq 0$, then γ_{s_i} is a component of $D_{s_i}^*$ with the desired properties. If this is not the case, the points $\{ q_i'' \} \subset C_\sigma$ have a limit point q_0'' on C_σ. If $q_{ij}'' \to q_0''$ as $j \to \infty$, then $\{ q_{ij} \}$ approach a point on γ and the continua γ_{s_i} also have a limiting continuum β. However, by lemma 1, $\beta \subset D^*$ and since $\beta \cap \gamma \neq 0$, $\beta \subset \gamma$. However, $q_0' \in \beta \cap C_\sigma$ and $\gamma \cap C_\sigma = 0$ which yields a contradiction. This proves that for some s_i, γ_{s_i} must intersect all b_i for $i = 1, 2, \ldots, k$ or for $i = k, k+1, \ldots, m$.

To prove the theorem, let $l(t)$ be defined and choose $\varepsilon > 0$. We can assume $l(t) > 0$ since the trivial case has already been discussed. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be a finite set of segments of admissible ends of D, each corresponding to a portion γ_i of D^*, each having length λ_i ($i = 1, 2, \ldots, n$) and such that $\sum_{i=1}^{n} \lambda_i >$

$> l(t) - \varepsilon/3$ this is possible by the definition of $l(t)$. Let $\delta > 0$ be chosen in such a way that if $p_1,\ p_2 \in M,\ |p_1 - p_2| < \delta$, then $|T(p_1) - T(p_2)| < \varepsilon/(12\,n)$. In each segment σ_i, choose a finite set of ends $\eta_j(i)$ $(j = 1, 2, \ldots, k_i;\ i = 1, 2, \ldots, n)$, $\eta_j(i) > \eta_{j+1}(i)$ for each i, j such that if $p_{ij} \in \gamma_i$ is the end point of $\eta_j(i)$ then for each i, $|p_{ij} - p_{i,j+1}| < \delta$ and such that for each i, $\sum_{j=1}^{k_i-1} |T(p_{ij}) - T(p_{i,j+1})| > \lambda_i - \varepsilon/(3n)$. Then

$$\sum_{i=1}^{n} \sum_{j=1}^{k_i-1} |T(p_{ij}) - T(p_{i,j+1})| \geqslant \sum_{i=1}^{n} \lambda_i - \varepsilon/3 \geqslant l(t) - 2\varepsilon/3 .$$

Let $k = \sum_{i=1}^{n} k_i$ and let $\delta' > 0$ be chosen in such a way that if $q_1,\ q_2 \in M$, $|q_1 - q_2| < \delta'$, then $|T(q_1) - T(q_2)| < \varepsilon/(6\,kn)$. For each $i = 1, 2, \ldots, n$ choose an index $s_i \in S$ such that for $s > s_i$ the set D_s^* has components of the type found in lemma 2 which are within δ' of λ_i. Let $p'_{ij}(s)$ be the first intersection points of these components with the defining arcs $b_j(i)$ of $\eta_j(i)$. Let $s_\varepsilon = \max_i s_i$. Then for each $i = 1, 2, \ldots, n$, let l_i be a number k in lemma 2 such that there exists a component $\gamma' \subset D_s^*$ which intersects all arcs $b_1, b_2, \ldots, b_{k_i}$ or two components γ', γ'' with γ' intersecting $b_1, b_2, \ldots, b_{l_i-1}$ and γ'' intersecting $b_{l_i}, b_{l_i+1}, \ldots, b_{k_i}$. This is possible since by the lemma, if a component γ' intersects $b_1, b_2, \ldots, b_{l-1}$ but no component intersects $b_1, b_2, \ldots, b_l$, then there must be a component γ'' which intersects $b_l, b_{l+1}, \ldots, b_{k_i}$. In this case the following relation holds:

$$\sum_{j=1}^{l_i-1} |T(p'_{ij}(s)) - T(p'(s))| + \sum_{j=l_i}^{k_i-1} |T(p'_{ij}(s)) - T(p'_{i,j+1}(s))| \geqslant$$
$$\geqslant \sum_{j=1}^{k_i-1} |T(p_{ij}) - T(p_{i,j+1})| - \varepsilon/(6n) - \varepsilon/(6n) .$$

Let $l_s(t)$ be the generalized length of the image of the contour determined by D_s. Then adding with respect to the index i:

$$l_s(t) \geqslant \sum_{i=1}^{n} \Big[\sum_{j=1}^{l_i-1} |T(p'_{ij}(s)) - T(p'_{i,j+1}(s))| + \sum_{j=l_i}^{k_i-1} |T(p'_{ij}(s)) - T(p'_{i,j+1}(s))|\Big] \geqslant$$
$$\geqslant \sum_{i=1}^{n} \sum_{j=1}^{k_i-1} |T(p_{ij}) - T(p_{i,j+1})| - \varepsilon/3 \geqslant \sum_{i=1}^{n} \lambda_i - 2\varepsilon/3 \geqslant l(t) - \varepsilon ,$$

for all $s > s_\varepsilon$. This implies then that $\liminf_{s\to s_0} l_s(t) \geqslant l(t)$.

If $l(t) = \infty$, the same type of argument shows also that $\liminf_{s\to s_0} l_s(t) = \infty$.

Theorem 4.2: *Let T be a continuous mapping of M into E_N and let f be continuous over V. Let $l(t)$ be defined as above, where $l(t) = 0$ if t is outside the range of f. Then $l(t) \leqslant \liminf l(\tau)$ as $\tau \to t - 0$ for all t, $-\infty < t < \infty$.*

Proof: If $\tau < t$, let $\mu = t - \tau$ and $f_\mu(x) = f(x) + \mu$ for all $x \in V$. Then $f_{\mu_1}(x) > f_{\mu_2}(x)$ for $\mu_1 > \mu_2$ and $\lim_{\mu \to 0^+} f_\mu(x) = f(x)$ uniformly for $x \in V$. Then as in the preceding theorem if we define $l(\tau) = l_\mu(t)$ we have $\liminf_{\mu \to 0^+} l_\mu(t) = \liminf_{\tau \to t-0} l(\tau) \geqslant l(t)$.

Theorem 4.3: *Let $\{ T_n \}$ be a sequence of mappings from M into E_N with $\lim_{n\to\infty} T_n(p) = T(p)$ uniformly for $p \in M$. Let f be a real valued continuous function from some compact set $V \subset E_N$ which contains each of the sets $\{ T_n(M) \}$, $T(M)$. Let $l(t)$ be as defined above and let $l_n(t) = l(t; T_n, M, f)$. Then for all t, $-\infty < t < \infty$,*

$$l(\tau) \leqslant \liminf_{\tau \to t - \infty} [\liminf_{n\to\infty} l_n(\tau)] .$$

Proof: The proof is analogous to the proof in theorem 4.1 and references for details will be made to that proof. We first prove a lemma similar to lemma 2 of that theorem and from this lemma the theorem follows as in 4.1.

Lemma: Let σ be a segment of admissible ends and prime ends in $A(D^-(t), \gamma)$, where γ is some component of the boundary of $D^-(t)$. Let η_1, η_2, ..., η_m be a finite set of admissible ends of σ with defining arcs b_1, b_2, ..., b_m and with the arc C_σ as defined in lemma 2 of theorem 4.1. Let b_k be any such arc. Then there exists a number $\tau < t$ and an integer n_0 such that for all $n > n_0$, the set $D_n^-(\tau) = \{ p \in M \mid f(T_n(p)) < \tau \}$ has a boundary component which intersects b_k and all arcs b_1, b_2, ..., b_{k-1}, or b_{k+1}, b_{k+2}, ..., b_m or all arcs b_1, b_2, ..., b_m in points lying between C_σ and γ on the respective arcs.

Proof: By theorem 4.1, lemma 2 and by theorem 4.2, there exists a $\tau' < t$ such that $D^-(\tau')$ has boundary components having the above property. Let τ be a number with $\tau' < \tau < t$. Choose $\varrho < \min[(\tau - \tau'), (t - \tau)]$. Let n_0 be chosen so that

$$| f(T_n(p)) - f(T(p)) | < \varrho \qquad \text{for} \qquad n > n_0, \ p \in M .$$

Evidently $D^-(\tau') \subset D_n^-(\tau) \subset D^-(t)$ and the inclusion in each case is proper for $n > n_0$. Thus if γ' is a component of $[D^-(\tau)]^*$ intersecting all the $\{ b_j \}$ or if γ', γ'' are two components which together intersect all the $\{ b_j \}$, then $D_n^-(\tau')$ has boundary components which must do the same since these components must separate γ' and γ'' from γ and hence must cross the $\{b_j\}$. The proof that these arcs define admissible ends proceeds in the same manner as in 4.1. Thus the lemma holds in this case.

By the same proof as that used in theorem 4.1, for any $\varepsilon > 0$ there exists $\tau < t$ and an n_0 such that for all $n > n_0$, $l_n(\tau) \geqslant l(t) - \varepsilon$. Thus $\liminf_{\tau \to t-0} [\liminf_{n\to\infty} l_n(\tau)] \geqslant l(t)$.

5. - The inequality of Cesari.

The establishment of the semi-continuity theorems for the function $l(t; T, M, f)$ in the last section enables us to prove the CAVALIERI inequality of CESARI in essentially the same manner as in [1 a, p. 327]. We first state several lemmas given in [1 a] and then state and prove the inequality. The proof differs from that of CESARI on only minor details but we include it for the sake of completeness.

Lemma 1: Every real single valued function $F(t)$, $a \leqslant t \leqslant b$, $-\infty < F(t) < \infty$, satisfying the relation $F(t) \leqslant \liminf_{\tau \to t-0} F(\tau)$ for $a \leqslant t \leqslant b$ is measureable in $[a, b]$.

Lemma 2: If Δ is a triangle in the w plane, $w = (u, v)$, if $\varphi(w) = au + bv + c$ is a linear non constant function, if t_1, t_2 are respectivelr the minimum and maximum of $\varphi(w)$ in Δ, if $\lambda(t)$, $t_1 \leqslant t \leqslant t_2$ is the length of the segment of Δ on which $\varphi(w) = t$, then $\sqrt{a^2 + b^2}\,[\alpha(\Delta)] = \int_{t_1}^{t_2} \lambda(t)\,dt$, where $\alpha(\Delta)$ denotes the area of Δ.

Lemma 3: Given $\varepsilon > 0$, a compact set $K \subset E_N$ and a real function $f(x)$, $x \in E_N$ with $|f(x) - f(x')| < G \cdot |x - x'|$ for every x, $x' \in E_N$ and some constant $G > 0$, then there exists a real piecewise linear function $\varphi(x)$ on E_N with $|\varphi(x) - f(x)| < \varepsilon$ and $|\operatorname{grad} \varphi| < G + \varepsilon$ for all $x \in K$.

Lemma 1, 2, 3 are proved in [1 a; pp. 325, 326] and their proofs will not be repeated here.

Theorem 5.1: *If T is any continuous mapping from the manifold M into the space E_N, if $f(x)$, $x \in E_N$ is any real single valued function such that $|f(x) - f(x')| \leqslant K \cdot |x - x'|$ for all x, $x' \in E_N$ with $K > 0$ a constant, if $l(t) = l(t; T, M, f)$ is the generalized length of the image of the contour $C(t)$, $-\infty < t < \infty$ as defined in section* **3** *and $L(M, T)$ the Lebesgue area of T. Then*

$$K \cdot L(M, T) \geqslant \int_{-\infty}^{\infty} l(t)\,dt .$$

Proof: The proof will be divided into parts (a), (b), (c) as in [1 a, p. 328].

(a) Suppose that M has been triangulated, that T is piecewise linear from M into E_N and that f is a piecewise linear function defined over a triangulation of E_N. Then by considering the intersection of the triangulation of E^N over which f is piecewise linear with the triangles defined in $T(M)$, it is possible to refine the triangulation of M in such a way that T is piecewise linear over the new triangulation and f is linear over each image under T of a triangle in the triangulation of M.

Let $\{\Delta\}$ be the family of all non-degenerate triangles in $T(M)$ over which f is non constant and let $\{t\}$ be the family of triangles in M from which $\{\Delta\}$ arises. For a fixed $\Delta < T(M)$ choose a planar cartesian coordinate system (ξ, η). Then $f(x) = a\xi + b\eta + c$ for $(\xi, \eta) \in \Delta$ and for some real constants a, b, c. Let g, g' be respectively the gradient of $f(x)$ in Δ and in the cell of linearity for f in E_N which contains Δ. Then evidently $0 < (a^2 + b^2)^{1/2} = g' \leqslant g \leqslant K$ by lemma 3. Let t_1, t_2 be the minimum and maximum of $f(x)$ in Δ and let $l'(t)$, $t_1 \leqslant t \leqslant t_2$ be the length of the segment of Δ on which $a\xi + b\eta + c = t$, $l'(t_1) = l'(t_2) = 0$ and we define $l'(t) = 0$, $t < t_1$ or $t > t_2$. Then on M, the boundary of $D^-(t)$ is the union of segments $\lambda'(t)$ contained in triangles τ and whose images are the corresponding segments in E_N plus sides of triangles of constancy for T plus segments $\lambda'(t)$ whose images are points. Each segment $\lambda'(t)$ of the first kind separates $D^-(t)$ and $D^+(t)$ in τ. $f(T(P))$ is constant on each of the other segments. Thus, since the number of the τ is finite, there is at most a finite collection of values of t, $t_1 \leqslant t \leqslant t_2$ for which $\lambda'(t)$ does not separate $D^-(t)$ and $D^+(t)$ in triangles τ. For all other t, the number $l(t) = l(t; T, M, f)$ is equal to the sum $\sum l'(t)$ of all $l'(t)$ for the given value of t. Thus by lemma 2

$$K\alpha(t) \geqslant K \sum_{\{\tau\}} \alpha(\Delta) \geqslant \sum_{\{\tau\}} \int_{-\infty}^{\infty} l'(t)\,dt \geqslant \int_{-\infty}^{\infty} \sum_{\{\tau\}} l'(t)\,dt = \int_{-\infty}^{\infty} l(t)\,dt\,.$$

(b) Now assume $T: M \to E_N$ is still piecewise linear over a triangulation of M but that f is any Lipschitzian function in E_N with constant $K > 0$. By lemma 3 there exists a sequence $f_n'(x)$, $x \in E_N$ of continuous functions piecewise linear in E_N such that $|f_n'(x) - f(x)| < 1/n$, $|\operatorname{grad} f_n'| \leqslant K + 1/n$ for all $x \in E_N$. If $f_n(x) = f_n'(x) + 1/n$ for all $x \in E_N$ then $f(x) < f_n(x) < f(x) + 2/n$ and $|\operatorname{grad} f_n| = |\operatorname{grad} f_n'| \leqslant K + 1/n$. By (a)

$$\left(K + \frac{1}{n}\right)\alpha(T) \geqslant \int_{-\infty}^{\infty} l_n(t)\,dt\,,$$

where $l_n(t) = l(t;\ T,\ M,\ f_n)$. By theorem 4.1 and by FATOU's lemma

$$\lim_{n\to\infty}\left\{ \left(K + \frac{1}{n}\right) \alpha(T) \right\} = K\alpha\,(T) \geqslant \lim_{n\to\infty} \int_{-\infty}^{\infty} l_n(t)\ \mathrm{d}t \geqslant \int_{-\infty}^{\infty} l(t)\ \mathrm{d}t\ .$$

Thus if T is piecewise linear and f any Lipschitzian function, the inequality holds.

(c) Finally let T be a continuous mapping from M into E_N and let f be any Lipschitzian function on E_N. If $L(M,\ T) = \infty$ or if $K = 0$ the inequality is trivially satisfied. Thus assume $K > 0$, $L(M,\ T) < \infty$. Let $T_n : M \to E_N$ be a sequence of piecewise linear mappings from M into E_N with $\lim\limits_{n\to\infty} T_n(p) = T(p)$ uniformly on M and such that $L(M,\ T) = \lim\limits_{n\to\infty} \alpha(T_n,\ M)$, where $\alpha(T_n,\ M)$ is the area of the piecewise linear surface determined by T_n as computed in the ordinary manner. The definition of $L(M,\ T)$ insures that such a sequence exists. Let $l_n(t) = l(t;\ T_n,\ M,\ f)$. Let $\varphi(t) \leqslant \liminf\limits_{n\to\infty} l_n(t)$ for all t, $-\infty < t < \infty$. By part (b), $K\,\alpha(M,\ T_n) \geqslant \int_{-\infty}^{\infty} l_n(t)\ \mathrm{d}t$. If $n \to \infty$, by FATOU's lemma we have $K\,L(M,\ T) \geqslant \int_{-\infty}^{\infty} \varphi(t)\ \mathrm{d}t$. The substitution $t = t' - h$ leaves the integral unchanged and hence

$$K\,L(M,\ T) \geqslant \int_{-\infty}^{\infty} \varphi(t-h)\ \mathrm{d}t\ .$$

However by theorem 4.3, $l(t) \leqslant \liminf\limits_{h\to 0^+} \varphi(t-h)$ and again, by using FATOU's lemma,

$$K\,L(M,\ T) \geqslant \int_{-\infty}^{\infty} l(t)\ \mathrm{d}t\ .$$

Thus the inequality is established for mappings T on the two manifold M.

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