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R. F. WILLIAMS (*)

Lebesgue Area Zero, Dimension and Fine-cyclic Elements. (**)

« If a surface has area zero, it is really a curve », is the proposition to be discussed in this paper. With a suitably loose interpretation of « is really a curve », the proposition is valid in a very general setting (Therorem 1). A more satisfying interpretation leads to an affirmative solution in a special case (Theorem 2). This is a generalization of a theorem of RADÓ [10]. LEBESGUE m-dimensional « area » and extensions of it are used.

An m-dimensional area is a function whose domain is a certain class of maps $f: X \rightarrow E_n$, $m \leqslant n$. The largest such class considered here consists of arbitrary maps (continuous transformations) from compact, m-dimensional HAUSDORFF spaces, X. This generality seems justified by our corollary that LEBESGUE area zero is a topological property.

1. - Definitions.

If L is a complex of dimension $\leqslant m$ and $g: L \rightarrow E_n$ is simplicial, relative to a subdivision of E_n (for these and other terms concerning simplicial complexes, see [4], [5]), the elementary area of g is defined by

$$e_m(g) = \sum \mu_m(g(\sigma)),$$

where μ_m is LEBESGUE m-dimensional measure and the summation is taken over all m-simplexes σ of L.

(*) Address: Mathematics Department, University of Chicago, Chicago 37, Illinois, U.S.A. .

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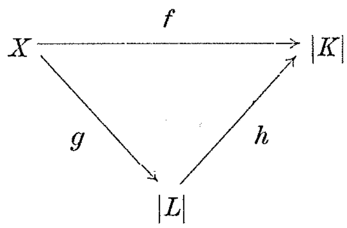
Lebesgue area of a map $f: X \to E_n$ is defined by means of *ε-triangles*:



where K and L are complexes of dimension $\leqslant m$, $|K| \subset E_n$, $h: L \to K$ is simplicial, and $\varrho(f, hg) < \varepsilon$. Specifically,

$$L_m(f) = \text{least number } k$$

such that for every $\varepsilon > 0$, there is an ε-triangle, where

1) g is a homeomorphism; and

2) $e_m(h) \leqslant k + \varepsilon$ [1].

Condition 2) requires triangulability of the space X. If this restriction is relaxed, one has the alternative functionals,

$$L_m^p(f) = \text{least number } k$$

such that for every $\varepsilon > 0$ and every open cover $\mathfrak{U}$ of X, there is an ε-triangle, where

1 p) g is a $\mathfrak{U}$-map; and

2 p) $e_m(h) \leqslant k + \varepsilon$.

And

$$L_m^*(f) = \text{least number } k$$

such that for every $\varepsilon > 0$ and every open cover $\mathfrak{U}$ of X, there is an ε-triangle, where

1*) g is a $\mathfrak{U}$-map; and

2*) $e_m^*(g, L, h) \leqslant k + \varepsilon$;

where e_m^* is defined in terms of g and L as well as the map h.

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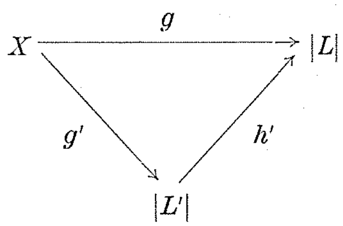
(1) More customarily, $L_m(f) = \liminf_{g \to f} e_m(g)$.

In [12], L_m^p and a type of L_m^* are discussed [2], where e_m^* is defined in terms of the map h and arbitrarily fine « δ-triangles » of the form



Though the inequality $L_m^* \leqslant L_m$ follows easily from the definitions and holds whenever X is triangulable, the contrary inequality is in general false [11]; that $L_m^p \leqslant L_m^*$ is easily proved though it is not known whether this inequality can be reversed. However, if any one of these functionals is zero, then they all are. This is contained in Theorem 1, below.

2. - Statement of results.

Theorem 1. *Suppose X is a compact Hausdorff space of dimension $\leqslant m$ and $f : X \to E_n$. Then the following are equivalent:*

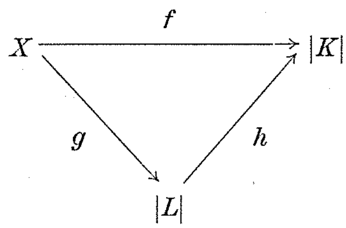
a. $L_m^p(f) = 0$;

b. $L_m^*(f) = 0$;

c. *for every $\varepsilon > 0$, there exists a map $f' : X \to E_n$ such that $\varrho(f, f') < \varepsilon$ and $f'(X)$ is a polyhedron of dimension $\leqslant m-1$;*

d. *for every $\varepsilon > 0$, there exists a map $f' : X \to E_n$ with $\varrho(f, f') < \varepsilon$ such that f' can be factored through a space of dimension $\leqslant m-1$;*

e. *for every $\varepsilon > 0$ and every open cover $\mathfrak{U}$ of X there is an ε-triangle*



such that (1) *g is a $\mathfrak{U}$-map, and* (2) $e_m^*(g, L, h) = e_m(h) = 0$.

(2) The definitions given here differ slightly from those in [11]. See the appendix.

If in addition X is triangulable, we may add:

f. $L_m(f) = 0$; *and*

g. *for every $\varepsilon > 0$ there is an ε-triangle*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & |K| \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & |L| & \end{array}$$

such that (1) *g is a homeomorphism, and* (2) $e_m(h) = 0$.

Corollary 1.1. *The ε in part* (2) *of the definition of L_m, L_m^p, and L_m^* may be deleted.*

Corollary 1.2. *Suppose X and Y are compact Hausdorff spaces of dimension $\leqslant m$, that the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\quad g \quad} & E_n \\ {\scriptstyle h_1}\downarrow & & \uparrow{\scriptstyle h_2} \\ X & \xrightarrow{\quad f \quad} & f(X) \subset E_n \end{array}$$

is commutative, and that $L_m^*(f) = 0$. Then $L_m^*(g) = 0$ and in particular, Lebesgue area zero is a topological property.

For reference we state a special hypothesis:

(#) X is compact, metric, locally connected, and of finite degree of multicoherence.

Inasmuch as every 1-dimensional unicoherent space satisfying (#) is a dendrite and as (#) is preserved by monotone maps, the following is a generalization of theorem 2.13 of Radó [10].

Theorem 2. *If X is of dimension $\leqslant 2$, satisfies* (#), *and*

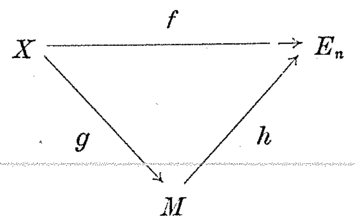
$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & E_n \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & M & \end{array}$$

where g and h are the monotone, light, factors of f, then the following are equivalent:

1. $L_2^*(f) = 0$;
2. *dim* $M \leqslant 1$;
3. *M has no fine-cyclic element* (see § **4**, below).

In justification of the special hypothesis (#), we give an

Example. *There exists a 2-dimensional, locally connected continuum X in the plane and a map $f: X \to E_3$ with $L_2^*(f) = 0$, such that if*



is any factorization whatsoever of f, then dim $M \geqslant 2$.

3. - A key lemma and proof of Theorem 1.

A version of the « pigeon hole principal » states: *if k is a positive integer and $H_0, \ldots, H_k$ are measurable subsets of H_0 such that $\mu_m(H_i) \geqslant \dfrac{k - 1/2}{k} \mu_m(H_0)$ $(i = 1, \ldots, k)$ then $\mu_m(\bigcap_{i=1}^k H_i) \geqslant (1/2)\,\mu_m(H_0)$.* We will use this in the proof of the

Lemma. *Suppose C_0 is the unit n-cell in E_n, $k = \binom{n}{m}$, M is a compact subset of C_0, and that $\mu_m(\pi_i M) < \varepsilon$, where $\pi_1, \ldots, \pi_k$ are the orthogonal projections of E_n onto its m-dimensional coordinates planes. Then there exists a polyhedron $K \subset E_n$ of dimension $< m$ and a retraction $r: M \to K$ such that no point of M is moved more than $d = 2n^{1/2}(2k\varepsilon)^{1/m}$ by r.*

Proof.
If $\varepsilon \geqslant 1/(2k)$, then $d = 2n^{1/2}$ and r can be chosen to map all of M into any one point of C_0. Now suppose $\varepsilon < 1/(2k)$ and let u be the positive integer such that $1/\{2k\,(u+1)^m\} \leqslant \varepsilon < 1/(2ku^m)$. It follows that $(2k\varepsilon)^{1/m} < 1/u \leqslant 2\,(2k\varepsilon)^{1/m}$.

Let $\mathcal{C}$ be the collection of u^n equilateral n-cells of edge length $1/u$ which fills C_0. For such a collection $\mathcal{C}$, the notation $\mathrm{Sk}_i\,\mathcal{C}$ will be used to denote the union of all i-dimensional faces of cells of $\mathcal{C}$. The technique (³) is to show that there is a retraction $r: M \to \mathrm{Sk}_{m-1}\,\mathcal{C}$ which moves no point out of any n-cell of $\mathcal{C}$ which contains it. As the diameter of each n-cell of $\mathcal{C}$ is $n^{1/2}/u$, r will have the required properties.

Note that $\varepsilon < \{1 - (k - 1/2)/k\}/u^m$. Let $W = (0, \ldots, 0)$ be the origin, $F_1, \ldots, F_k$ be the m-dimensional faces of C_0 which contain W, and let $\mathcal{C}_i = \{C_{ij}\}_{j=1}^{u^m}$ be the collection of all m-cells in which the n-cells of $\mathcal{C}$ intersect F_i $(i = 1, \ldots, k)$. Choose the notation so that C_{i1} is the m-cell of $\mathcal{C}_i$ which contains W, and let $t_{ij}: C_{ij} \to C_{i1}$ be the translation $(i = 1, \ldots, k;\ j = 1, \ldots, u^m)$. Let C_1 be the cell of $\mathcal{C}$ which contains W and $\alpha = (k - 1/2)/k$.

Computing,

$$\mu_m\,[\textstyle\bigcup_{j=1}^{u^m} t_{ij}\,(C_{ij} \cap \pi_i M)] \leqslant$$

$$\leqslant \textstyle\sum_{j=1}^{u^m} \mu_m\,(C_{ij} \cap \pi_i M) = \mu_m(\pi_i M) < (1 - \alpha)/u^m \quad (i = 1, \ldots, k)\,.$$

Let $D_i = C_{i1} - \bigcup_{j=1}^{u^m} t_{ij}(C_{ij} \cap \pi_i M)$. Then $\mu_m(D_i) > \mu_m(C_{i1}) - (1 - \alpha)/u^m = \alpha/u^m$, so that $\mu_m(C_1 \cap \pi_i^{-1} D_i) > \alpha/u^n$ $(i = 1, \ldots, k)$. Then by the pigeon hole principal,

$$\mu_m(\textstyle\bigcap_{i=1}^{k} C_1 \cap \pi_i^{-1} D_i) > 1/(2u^n)\,.$$

As this in positive, there is a point, say x_0 in

$$\textstyle\bigcap_{i=1}^{k} (\pi_i^{-1} D_i) \cap (\mathrm{Int}\ C_1).$$

Let $x_{i1} = \pi_i(x_0)$, $x_{ij} = t_{ij}^{-1}(x_{i1})$, and $L_{ij} = \pi_i^{-1}(x_{ij})$, so that $x_{ij} \in \mathrm{Int}\ C_{ij}$ and $x_{ij} \notin \pi_i(M)$, for $i = 1, \ldots, k$; $j = 1, \ldots, u^m$.

The remainder of the construction depends only upon the $(n - m)$-dimensional hyperplanes, L_{ij}, the subdivision $\mathcal{C}$, and the following properties of these elements:

(a) $L_{ij} \cap M = 0 \qquad (i = 1, \ldots, k;\quad j = 1, \ldots, u^m)$;

(b) *if* $C \in \mathcal{C}$ *and* j_i *is that integer such that* $\pi_i(C) = C_{ij_i}$ $(i = 1, \ldots, k)$, *then* $\bigcap_{i=1}^{k} L_{ij_i}$ *is a single point* $w = w(C)$, *which lies in the interior of* C.

(a) follows from the facts $x_{ij} \notin \pi_i(M)$ and $L_{ij} = \pi_i^{-1}(x_{ij})$. To prove (b), let p_z be the integer $\geqslant 0$, such that C is the n-cell $p_z/u \leqslant x^z \leqslant (p_z + 1)/u$ and

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(³) The autor is grateful to L. Cesari for aid in simplifying this argument; see [1, pp. 287, 288, 292] for a similar construction.

note that $w(C)$ is the point with coordinates $p_z/u + x_0^z$, where superscripts denote coordinates, $z = 1, \ldots, n$.

To complete the proof, let (a, b) be one of the pairs (i, j) $(i = 1, \ldots, k;$ $j = 1, \ldots, u^m)$. Suppose the coordinates were so chosen that $\pi_a(x^1, \ldots, x^n) = (x^1, \ldots, x^m, 0, \ldots, 0)$. Define $r'_{ab} : C_{ab} - x_{ab} \to \mathrm{Sk}_{m-1} C_{ab}$ to be the radial retraction; then for each $p \in C_{ab}$ there is a scaler $t = t(p) \geqslant 0$, such that $r'_{ab}(p) = q$, where $q^z = p^z + t\cdot(p^z - x^z_{ab})$ $(z = 1, \ldots, m)$, and $q^z = p^z = 0$ $(z = m+1, \ldots, n)$. Define $r_{ab} : F_a - x_{ab} \to (C_{ab} - \mathrm{Sk}_{m-1} C_{ab})$, by $r_{ab}(p) = r'_{ab}(p)$, if $p \in C_{ab}$, $r_{ab}(p) = p$, otherwise.

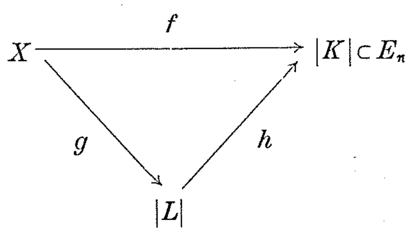
Let r^z_{ab} be the zth coordinate function of r_{ab} $(z = 1, \ldots, m)$, and define $r' : C_0 - L_{ab} \to C_0 - \pi_a^{-1}(C_{ab} - \mathrm{Sk}_{m-1} C_{ab})$ by $r'(x) = (r'_{ab}(x^1), \ldots, r'_{ab}(x^m), x^{m+1}, \ldots, x^n)$. This is possible as $L_{ab} = \pi_a^{-1}(x_{ab})$. Note that r' moves no point out of any n-cell of $\mathfrak{C}$ which contains it. Furthermore,

$$\text{(a}'\text{)} \qquad r'(M) \cap L_{ij} = 0 \qquad (i = 1, \ldots, k; \quad j = 1, \ldots, u^m).$$

For suppose, on the contrary, that $p \in M$ and that $r'(p) \in L_{\alpha\beta}$ for some α, β; note that $p \in \pi_a^{-1}(C_{ab})$. Let C be a cell of $\mathfrak{C}$ which contains both p and $r'(p)$ and let $w = w(C)$. Then $w \in L_{ab} \cap L_{\alpha\beta}$ so that $w^z = x^z_{ab}$ for $z = 1, \ldots, m$. The hyperplane $L_{\alpha\beta}$ satisfies $n - m$ equations $x^z = w^z$, for $z \in Z_0 \subset \{1, \ldots, m\}$. Let $Z_0 = Z_1 \cup Z_2$, where the elements of Z_1 are $\leqslant m$ and those of Z_2, $> m$. Let $q = r'(p)$; then by definition of r', togother with the assumption that $q \in L_{\alpha\beta}$, we have, for all $z \in Z_1$, $q^z = w^z = p^z + t\cdot(p^z - w^z)$, where $t = t(\pi_a(p)) \geqslant 0$, so that $p^z = w^z$. If $z \in Z_2$, then $q^z = p^z = w^z$. Hence $p \in L_{\alpha\beta}$ which contradicts (a).

By induction on (i, j) we obtain a retraction $r : C_0 - \bigcup_{i,j} L_{ij} \to C_0$ such that for any pair (i, j), $\pi_i[r(M)] \cap C_{ij} \subset \mathrm{Sk}_{m-1} C_{ij}$. But this is equivalent to the condition, $r(M) \subset \mathrm{Sk}_{m-1} \mathfrak{C}$, and the proof is complete.

Proof of Theorem 1. (a) implies (b) as $L^p_m(f) \leqslant L^*_m(f)$ is always true. Next, assume (b), let $\varepsilon > 0$, and let C be an n-cell containing $f(X)$. As this proposition is unchanged if the metric for E_n is multiplied by a positive constant, we will suppose that C has edge length 1. By definition of L^p_m there is an $(\varepsilon/2)$-triangle



where $e_m(h) < (\varepsilon/(4n^{1/2})^m/(2k)$. Using the Lemma, we obtain a retraction $r : hg(X) \to K'$, a polyhedron of dimension $m - 1$, lying in E_n. Then $rh\, g(X) \subset K'$, and a slight modification yields a map f', so that $f'(W) \subset K'$ is itself a polyhedron.

(c) clearly implies (d), with the second factor the identity. Now suppose $\varepsilon > 0$ and $\mathfrak{U}$ is an open cover of X. Assuming (d), we have

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & E_n \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & M & \end{array}$$

where $\varrho\,(f,\, hg) < \varepsilon/2$ and $\dim M \leqslant m-1$. For each $\delta > 0$, we have, by the simplicial approximation theorem, the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\quad g \quad} & M & \xrightarrow{\quad h \quad} & E_n \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g'} & \nearrow{\scriptstyle h'} & \\ |X_\alpha| & \xrightarrow{\quad h_1 \quad} & |M_\beta| & & \end{array}$$

in which β is an $(m-1)$-dimensional open cover of M, α is an m-dimensional open cover of X, α star refines $\mathfrak{U}$, X_α and M_β are the nerves of α and β, g_1 and g' are canonical, h_1, h' are simplicial, and « δ-commutativity » holds in the rectangle and triangle.

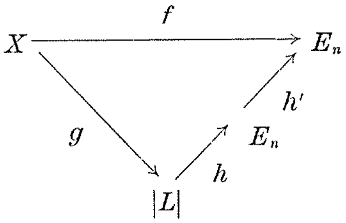
That g_1 a $\mathfrak{U}$-map, follows (as in the appendix) from the fact that α star refines $\mathfrak{U}$. Clearly $e_m\,(h'\,h_1) = 0$, as each simplex in X_α goes into a simplex of dimension $\leqslant m-1$ under $h'\,h_1$. Similarly, $e^*_m\,(g,\, X_\alpha,\, h'\,h_1) = 0$. For sufficiently small δ,

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & E_n \\ & {\scriptstyle g_1}\searrow \quad \nearrow{\scriptstyle h'h_1} & \\ & |X_\alpha| & \end{array}$$

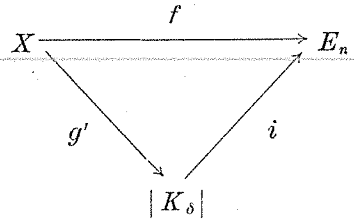
is an ε-triangle, which proves (e). (e) is a strengthened version of the definition of (a).

Suppose now that X is triangulable. As before, (g) implies (f) and (f) implies (a). Finally, (d) implies (g) in the same way that it implies (e).

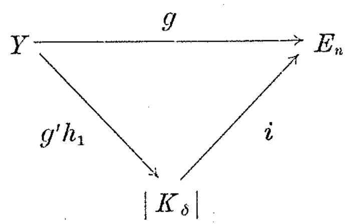
Proof of Corollary 1.1. The zero area case is the substance of parts (e) and (g) of Theorem 1. To prove the positive area case, introduce a contracting map, $h' : E_n \to E_n$ defined by $h'(e^1, \ldots, e^n) = (re^1, \ldots, re^n)$, with r less than but sufficiently close to 1 to obtain the ε-triangle



Proof of Corollary 1.2. Let $\varepsilon > 0$. For each ε', $\delta > 0$, there is, by (1c), a δ-triangle



where K_δ is an $(m-1)$-dimensional complex in E_n and i is the identity. For sufficiently small δ, there is, for each vertex v of K_δ, a point $p(v) \in f(X)$ within ε' of v. Let $j : K_\delta \to E_n$ be the semi-linear map determined by $v \to h_2\, p(v)$. Then for sufficiently small ε',



is commutative within ε, so that, by (1 d), $L^*_m(g) = 0$.

4. - Cesari's fine-cyclic elements and a result of C. J. Neugebauer.

Fine-cyclic elements were introduced by Cesari in [2]. We will use the formulation of Neugebauer [7] and restate in this section a result of this author.

(Neugebauer) *A space X satisfying (#) is of dimension $\geqslant 2$ if and only if it has a fine-cyclic element.*

For a fine-cyclic element Γ of X also satisfies (#) and is separated by no finite set, [7]. It follows that no 0-dimensional set separates Γ and in particular, $\dim \Gamma \geqslant 2$. The other way follows from [8; 3], because if X has no fine-cyclic element, X is the union of a finite number of dendrites and hence of dimension $\leqslant 1$.

5. - Proof of Theorem 2 and the Example.

Theorem 2. As monotone maps preserve (#), the result of the previous section shows that (2) and (3) are equivalent. That (2) implies (1), (often noted by previous authors) follows from part (d) of Theorem 1.

To complete the proof, assume that both $L_m^*(f) = 0$ and $\dim M > 1$. As light closed maps cannot lower dimension, M is of finite dimension and therefore contains a 2-dimensional CANTOR-manifold [5; pp. 91, 94]. Hence there exist two points $b_1, b_2 \in M$ with $h(b_1) \neq h(b_2)$ such that no finite subset of M separates b_1 and b_2 in M. Choose $a_i \in g^{-1}(b_i)$ and let $c_i = h(b_i)$, for $i = 1, 2$, and let $d = \varrho(c_1, c_2)$.

By part (c) of Theorem 1, there exists a sequence $f_1, f_2, \ldots$, of maps of X into E_n, such that for each positive integer i, $\varrho(f_i, f) < d/(4i)$ and $f_i(X)$ is a polyhedron of dimension $\leqslant 1$. For each i, $\varrho[f_i(a_1), f_i(a_2)] < d/2$, and so there exists a finite set $K_i \subset f_i(X)$ such that K_i separates $f_i(a_1)$ and $f_i(a_2)$ in $f_i(X)$ and $\varrho(K_i, f_i(a_j)) < d/4$ $(j = 1, 2)$. Let $L_i = f_i^{-1}(K_i)$; then L_i separates a_1 and a_2 in X. As X is of finite degree of multicoherence, say of degree q, there exist finitely many components $L_{i0}, \ldots, L_{iq_i}$ of L_i whose union separates a_1 and a_2, where $q_i \leqslant q$, $(i = 1, 2, \ldots)$. Hence there is a subsequence $\{i_\alpha\}_{\alpha=1}^{\infty}$ of the index i, such that (a) all the q_{i_α}'s are the same, say $q_{i_\alpha} = r$, and (b) each of the sequence $\{L_{i_\alpha 0}\}_{\alpha=1}^{\infty}, \ldots, \{L_{i_\alpha r}\}_{\alpha=1}^{\infty}$ converges, say to sets $L_0', \ldots, L_r'$, which must be continua.

As $\varrho[f(a_i), L_j] > d/4$ $(i = 1, 2;\ j = 0, \ldots, r)$, and as $\bigcup_{j=0}^{r} L_{i_\alpha j}$ separates a_1 from a_2 $(\alpha = 1, 2, \ldots)$, it follows by arc-wise connectivity that $\bigcup_{j=0}^{r} L_j'$ separates a_1 and a_2. Furthermore, $f(L_j')$ is a single point, as $f(L_{ij})$ is a single point of the finite set K_i $(i = 1, 2, \ldots;\ j = 0, \ldots, r)$.

Inasmuch as each of the continua L_j' has a single point as image under f, each has a single point as image under the monotone factor, g, say $g(L_j') = e_j$ $(j = 0, \ldots, r)$. As the finite set $E = \{e_0, \ldots, e_r\}$ cannot separate b_1 and b_2 in M, there exists an arc β from b_1 to b_2 lying in $M - E$. Then $g^{-1}(\beta)$ is a continuum containing a_1 and a_2 and lying in $g^{-1}(M - E) \subset X - \bigcup_{j=0}^{r} L_j'$, contradicting the fact that $\bigcup_{j=0}^{r} L_j'$ separates a_1 and a_2 in X. This completes the proof.

Example. Let A denote a circular disc in the plane and let J_1, J_2, ... denote a countable collection of mutually exclusive circles in A with mutually exclusive interiors, I_1, I_2, ..., such that $\bigcup_{i=1}^{\infty} I_i$ is dense in A. Let J'_i denote the circle concentric with J_i and half its radius, and I'_i the interior of J'_i, for $i = 1, 2, \ldots$. Let $X = A - \bigcup_{i=1}^{\infty} I'_i$. Then X is a locally connected 2-dimensional CANTOR-manifold.

Define an upper semi-continuous collection G filling up X as follows: for each positive integer i, each circle concentric with J_i and lying in $J_i \cup I_i - I'_i$ is an element of G. The remaining elements of G consist of those single points of $X - \bigcup_{i=1} J_i \cup I_i$. It follows from the MOORE Theorem [6], that the decomposition space, Y, is a dis: D plus a countable collection α_1, α_2, ... of mutually exclusive arcs, each intersecting D in only one end-point. For each i, α_i is the decomposition of $J_i \cup I_i - I'_i$. Let f be the decomposition map; then f is monotone. Y will be thought of as embedded in E_3, D as a solid square in the xy-plane, and the arcs α_1, α_2, ... as being vertical line intervals, lying, except for their end-points, A_1, A_2, ..., above D.

We use the characterization given in part (d) of Theorem 1 to show that $L_2^*(f) = 0$. Let $\varepsilon > 0$ and n be a positive integer such that the diameter of $\alpha_i < \varepsilon$, for all $i \geqslant n$. Let r be the retraction of Y onto $Y - \bigcup_{i=n}^{\infty} (\alpha_i - A_i)$ defined as follows: $r(y) = A_i$, if $y \in \alpha_i$, $i \geqslant n$, and $r(y) = y$, otherwise. Let $f' = rf$; then $\varrho(f, f') < \varepsilon$.

Let G' be the monotone decomposition of X defined as follows: for each $i \geqslant n$, that portion of each radii of J_i which lies in $J_i \cup I_i - I'_i$ is an element of G'; for $i < n$, each circle concentric with J_i and lying in $J_i \cup I_i - I'_i$ is an element of G'; and each individual point of $X - \bigcup_{i=1}^{\infty} (J_i \cup I_i)$ is an element of G'. Then G' is an upper semicontinuous collection of continua filling up X and each element of G' maps into a single point under f'.

Let M be the decomposition space of G' and let $g: X \to M$ be the decomposition map. Then $g: X \to M$, $f' g^{-1}: M \to f'(X)$ is a factorization of f'. Furthermore, M is the sum of a well known (« Swiss cheeze example ») compact, 1-dimensional space and a finite number of arcs, those corresponding to α_1, α_2, ..., α_{n-1}. Hence M is 1-dimensional so that $L_2^*(f) = 0$.

To complete the proof, suppose that $g: X \to M$, $h: M \to f'(X)$ is an arbitrary factorization of f. Let $g_1: X \to M_1$, $g_2: M_1 \to M$ be the monotone, light factors of g, and consider two cases:

Case 1. For some integer i, $g_1(J_i)$ is non-degenerate. Then there is a circle K in $I_i - I'_i$, concentric with and interior to J_i, such that $g_1(K')$ is non-degenerate for all circles K' which lie between J_i and K. Let L denote the annulus ring consisting of J_i, K, and all points of X between J_i and K. For each point $z \in g_1(L)$, $g_1^{-1}(z)$ is a sub-continuum of some circle concentric with J_i, because the images, under f, of each two such circles are mutually exclusive. By the MOORE Theorem, $g_1(L)$ is again an annulus ring, and hence $\dim M \geqslant 2$. Therefore $\dim M \geqslant 2$, as light, closed maps cannot lower dimension.

Case 2. For each integer i, $g_1(J_i)$ is a single point. But then, by the MOORE Theorem, $g_1(X - \bigcup_{i=1}^{\infty} I_i)$ is a disc, so that, again, $\dim M \geqslant 2$.

6. - Appendix.

The definition of L_m^p given in [**11**] differs from that given in section 1 only in that the condition

1 p) g is a $\mathfrak{U}$-map

is replace l with

1 p′) there exists an open cover α of X which refines $\mathfrak{U}$ such that $|L| = |X_\alpha|$, where X_α is the nerve of α and g is canonical relative to α.

Call a map g, $\mathfrak{U}$-*canonical* if and only if it satisfies 1 p′). To show that these two definitions of L_m^p are equivalent, it will suffice to prove:

Suppose X is a compact Hausdorff space, L is a complex, $\mathfrak{U}$ is an open cover of X, and $g: X \to |L|$. Then

a) *if g is a $\mathfrak{U}$-map, then g is $\mathfrak{U}$-canonical;*

b) *there exists an open cover $\mathfrak{V}$ of X such that if g is $\mathfrak{V}$-canonical, then g is a $\mathfrak{U}$-map.*

Proof.

Suppose that g is a $\mathfrak{U}$-map. As X is compact, there exists an open cover $\mathfrak{W}$ of X such that for each $W \in \mathfrak{W}$, $g^{-1}(W)$ lies in some element of $\mathfrak{U}$. There is a subdivision L' of L, such that for each vertex v' of L', $\mathrm{St}'(v')$ (the open star about v' in L'), lies in some element of $\mathfrak{W}$, [**4**, p. 63]. Let $\alpha = g^{-1}(\mathrm{St}'(v'))$: v' is a vertex of L'. Then α refines $\mathfrak{U}$ and X_α is isomorphic to L'', where L'' is the sub-complex of L' consisting of simplex σ of L' such that $|\sigma| \cap g(X) \neq 0$, together with all faces of such simplexes. Identifying these two, $g: X \to |X_\alpha| = |L''| \subset |L|$ is canonical, so that g is $\mathfrak{U}$-canonical.

Tho prove b), let $\mathfrak{V}$ be an open cover of X which star refines $\mathfrak{U}$, suppose α refines $\mathfrak{V}$, and that $g: X \to |X_\alpha| \subset |L|$ is canonical. Let $V_1, \ldots, V_n$ denote the elements of α and let $v_1, \ldots, v_n$ denote the corresponding vertices of X_α. Now suppose $p \in |X_\alpha|$ and let $(v_{i_1} \ldots v_{i_m})$ denote the unique open simples of X_α which contains p. Then $V_{i_1} \cap \ldots \cap V_{i_m} \neq 0$, so that, as α star refines $\mathfrak{U}$, here exists an element $U_0 \in \mathfrak{U}$ such that $V_{i_1} \cup \ldots \cup V_{i_m} \subset U_0$. Then, as g is canonical relative to α, $g^{-1}(p) \subset U_0$.

Analogu; arguments show that the definition of L_m^* given in [**11**] is equivalent to that given in this paper.

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