Riv. Mat. Univ. Parma 10 (1959), 85-98

R. E. FULLERTON (*)

Prime Ends for Open Subsets of Two Dimensional Manifolds - I. (**)

1. - Introduction.

The theory of prime ends of simply connected plane domains was introduced by CARATHEODORY and has been extensively studied by various authors [1, 4, 6, 8, 12]. In particular, CESARI [2 a, b, c] and the author [7 a, b] have made use of this theory in the study of surfaces defined over such domains. Certain of these results can also be obtained for more general planar sets, in particular, multiply connected JORDAN regions. CESARI, in his book, *Surface Area* [2 a] has made fundamental use of these notions in developing a CAVALIERI type inequality which has proved to be a powerful tool in studying properties of surfaces. URSELL and YOUNG [12] have investigated in considerable detail the structure of prime ends of plane domains. More recently results have been obtained by CESARI [2 d, e], NEUGEBAUER [10 a, b, c], FLEMING [5], and others concerning surfaces defined over multiply connected plane domains and over two dimensional manifolds. Hence a topological study of prime ends for open subsets of two dimensional manifolds would seem to be of value. One of the purposes of the present paper is to furnish basic results necessary for the study of surfaces defined over two-manifolds. These results will be used in future papers for the investigation of contours for surfaces defined on two-manifolds and, in particular, for proving the CESARI-CAVALIERI type

---

(*) Address: Department of Mathematics, University of Maryland, College Park, Maryland, U.S.A..

(**) This research was supported in part by the Purdue Research Foundation, Contract No. AF 18(600)-1484. The author is indebted to L. CESARI for making this support available to him.

Received September 4, 1958.

inequality for such surfaces. Further extension of known theorems on surfaces over a simply connected planar region to surfaces of this type seems likely.

The principal results are concerned with defining an order on the prime ends, defined in terms of accessibility properties. For open subsets of simply connected planar regions this was done by CESARI [2 a] using his sets $A(Q, \gamma)$. This method fails in the case of a two manifold and we introduce instead the concept of admissible ends and prime ends as defined in section **3**. In the last section it is shown that this reduces to CESARI's method in the case of a simply connected plane region.

2. - Notations and basic definitions.

We shall be concerned throughout the paper with a fixed compact triangulable two dimensional metric manifold M. M may be orientable or non-orientable and may or may not have a boundary. Thus to each point $p \in M$ there exists an open subset $N_p \subset M$ which is homeormophic either to an open disk in the plane whose center is the counter image of p or to an open half disk in the plane plus its bounding diameter with p the image of the mid-point of the diameter. Neighborhoods N_p of the above type will be called *co-ordinate neighborhoods* since, if desired, it is possible to introduce local co-ordinate systems at each point of M. The structure of all such twomanifolds is known [**9**] and, in particular, M is known to be homeomorphic to a closed disk in the plane with a certain finite number of arcs of its circumference identified in pairs. Frequent use will be made of this type of representation of M.

We shall also make considerable use of the theory of prime ends of plane domains in building up locally a prime end theory for open subsets of manifolds. There are several definitions of ends and prime ends [**1**, **2** a]. The definition used here is that used by CESARI [**2** a] which depends upon accessibility properties of boundary points. If A is a simply connected bounded open subset of the euclidean plane, $p \in A^*$, A^* the boundary of A and if C is an arc ending at p and having its other points in A, then C is said to define an *end* of A. If C' is another arc in $A \cup (p)$ with end point p, then C' is equivalent to C if (1) $C, C' \subset A \cup (p)$, (2) $C \cap A^* = C' \cap A^* = (p)$, (3) either for every neighborhood N_p of p, $C_1 \cap C_2 \cap (N_p - (p)) \neq 0$ or for every N_p there exist sub arcs b, b' of C, C' respectively and a third arc b'' such that $b \cap b' = (p)_1$, $b'' \subset N_p \cap A$ and $b \cup b' \cup b''$ form the boundary of a simply connected JORDAN region in A. Thus the set of arcs in A ending at accessible points of A^* can be divided into equivalence classes, $\{\eta\}_A$. Each equivalence class η is defined to be an *end* of A with end point p. If $\eta_1, \eta_2, \eta_3, \eta_4$ are different ends of A, let b_1, b_2, b_3, b_4 be arcs defining these ends and assume that the arcs are distinct except perhaps at their end points on A^*. Then if there exists an arc C such that $b_1 \cup C \cup b_3$ form a cross cut which divides A into two parts, one containing b_2 and the other b_4 then we say that the ends η_1 and η_3 separate η_2 and η_4. If one end is designated as η_∞ then an open *interval* (η_1, η_2) is the set of all ends η such that η_1, η_2 separate η from η_∞. A *prime end* ω of A is defined by a nested sequence $\{ (\eta_n', \eta_n'') \}$ of intervals in $\{ \eta \}$ which has at most one end in common. Thus each end defines a prime end but some prime ends have no corresponding ends. Equivalent sequences of ends can be defined in the obvious manner and a prime end can be defined as a class of equivalent sequences of this type. Evidently to each prime end ω corresponds a connected set of boundary points $E_\omega \subset A^*$. Frequently only connected subsets of A^* will be considered in which case the ordering on $\{ \eta \}_A$ is equivalent to that of a real interval.

A homeomorphic image of the interval $0 \leqslant t < 1$ or $0 < t < 1$ is called an *indefinite arc*. For simply connected plane domain A, CESARI has shown [2 c] that if ω is a prime end of A there exists an indefinite arc b lying entirely in A which has a portion of the boundary set corresponding to ω as a limiting set. He has also shown that the boundary of A can always be approximated in a certain sense by an arc, a simple closed curve or an indefinite arc. We shall establish similar results in the following for open subsets of two-manifolds.

3. - Ends and prime ends of open subsets of M.

Let $Q \subset M$ be a connected open subset of M and denote the boundary components of Q by $\{ \gamma' \}_Q$. If $\gamma' \in \{ \gamma' \}_Q$ let γ be the set $\gamma' - Q$, i.e. γ is the set of points of γ not on the boundary of M. If M is a manifold without boundary, $\gamma' = \gamma$. Let $A(Q, \gamma)$ be the component of $M - \gamma$ which contains Q, ($A(Q, \gamma)$ may equal $M - \gamma$). Then the set $A(Q, \gamma)$ has γ as a portion of its boundary. We shall be concerned with developing a theory of prime ends for A with respect to γ.

Let $p \in \gamma$ be accessible from A. Let N_p be a co-ordinate neighborhood of p. Then as in section **2**, ends of A ending at p can be defined in $N_p \cap A$. It is evident that in the original definition of an end, a neighborhood of p would suffice for the definition. Prime ends of $A \cap N_p$ ending on γ can also be defined but, in general, a prime end will not be defined locally. Thus to every point $p \in A$ accessible from A, ends can be defined.

Definition 3.1. An end $\eta \in \{ \eta \}_{A, \gamma}$ with end point $p \in \gamma$ is said to be *admissible* if there exists an arc C in the equivalence class of arcs defining η

such that for any neighborhood U_p of p, $U_p \cap C \cap Q \neq 0$. A prime end is admissible if it is determined by admissible ends.

Let γ be covered by a set of co-ordinate neighborhoods in M. Let N be one of these neighborhoods. Then $(N - \gamma) \cap A$ will consist of components $\{K\}$ (possibly infinitely many) each having points of γ on its boundary.

Lemma 3.1. If K is a component of $(N - \gamma) \cap A$ then K is an open simply connected planar set or a planar open set of genus one.

Proof:

Evidently K is open. If $\gamma \subset N$ then either $A \subset N$ and $A = K$ is simply connected or the complement of A is in N and the boundary of K consists of γ and N^* and hence K has genus one. If $\gamma \not\subset N$ then K^* consists of portions of γ and portions of N^*. K cannot have more than one boundary component since if it had more than one, one component of K^* would separate the others from the exterior of N and this would imply either that $\gamma \subset K$ or N^* is disconnected, both statements yielding contradictions.

The case in whick K has genus one implies $\gamma \subset N$ and the prime end theory in this case reduces to the planar case which is already hnown. Thus in the following sections we consider only the significant case in which $\gamma \not\subset N$ and hence every K is simply connected and can be considered as a simply connected subset of E_2.

Assume now that γ has been chosen and that γ does not lie completely in any co-ordinate neighborhood. For each point $p \in \gamma$ let N_p be a co-ordinate neighborhood which contains p. Since γ is a closed subset of M, γ is compact and there exists a finite family $\{N^k\}$ $(k = 1, 2, \ldots, n)$ chosen from the family $\{N_p\}$ which covers γ. Let γ^1 be a component of $N^1 \cap \gamma$. Let K^1 be a component of $(N^1 - \gamma) \cap A$ which has γ^1 as part of its boundary. By Lemma 3.1 we can assume that K^1 is simply connected. Since this is true, ends for K^1 can be defined in the ordinary way and assigned a cyclic ordering with end points on K^*. In particular, this ordering defines an ordering of the ends ending on γ^1. It is also possible that prime ends of K may exist here and be defined in the usual way. Thus the ends and prime ends of K^1 ending on γ^1 have a defined ordering. It can be assumed that $\gamma^1 \neq \gamma$ since otherwise the investigation of the prime ends would reduce to the planar case. Thus there exists another neighborhood of the family $\{N^k\}$ which we shall take to be N^2 such that $\gamma^1 \cap N^2 \neq 0$ and $K^1 \cap N^2 \neq 0$. Let $K^2 \subset N^2$ be a component of $(N^2 - \gamma) \cap A$ wich intersects K^1 in a set which has the last component if it exists (in the ordering of ends of K^1 ending on γ^1) of $\gamma^1 \cap N^2$ as a portion of its boundary and which contains arcs in K^1 which define ends of K^1 ending on γ^1. Now let the ends and prime ends of K^2 be cyclically ordered in a way which will be consistent with the order already defined in $K^1 \cap K^2$ for ends ending on γ^1. Let γ^2 be the component of $\gamma \cap N^2$ which coincides with the last component of $\gamma^1 \cap N^2$ in $K^1 \cap N^2$. Thus the ordering of ends of K^2 ending on K^{2*} induces an ordering on the ends and prime ends of K^2 ending on γ^2. Thus a consistent ordering of ends and prime ends of $K^1 \cup K^2$ ending on $\gamma^1 \cup \gamma^2$ has been defined. The same process can be continued to successive components $K^3, K^4, \ldots$ of the various $(N^k - \gamma) \cap A$ and a sequence (possible finite) of components of the $(N^k - \gamma) \cap A$ can be found and an order on the ends and prime ends of $\bigcup_{i=1}^{\infty} K^i$ which end in γ can be constructed. The same process can be continued in the other direction from K^1.

Now let $\mathfrak{N}$ be any finite covering of γ by coordinate neighborhoods where the original neighborhood N_p^1 is in $\mathfrak{N}$ in each case. Thus to each such covering, an order on the ends and prime ends can be defined yielding an ordered set $\sigma_{\mathcal{N}}$. Furthermore, if the same order sense is always used in N_p^1, then if $\mathfrak{N}$, $\mathfrak{M}$ are two such coverings $\sigma_{\mathcal{N}} \subset \sigma_{\mathcal{N} \cup \mathcal{M}}$, $\sigma_{\mathcal{M}} \subset \sigma_{\mathcal{N} \cup \mathcal{M}}$ with order preserved. Let $\sigma = \bigcup \sigma_{\mathcal{N}}$ where the union is taken over all such finite coverings.

Definition 3.2. A maximal set of ends and prime ends of the type σ above will be called a *segment* of ends and prime ends of A ending on γ.

It will be seen that every end or prime end of γ as defined above lies on a unique segment. Thus the set $\{\eta\}_{A,\gamma}$ can be decomposed into segments. It is evident that the ordering on each segment is the same as that for a real interval. It will be noted that different portions of the boundary of any of the sets K may have their ends and prime ends ordered differently, i.e. the ordering as different segments bordering K may not coincide with a cyclic ordering for all ends of K ending on K^*. If M is orientable, the orderings may be chosen to be compatible with a cyclic ordering of all ends of K but this fact is immaterial in the following sections.

Theorem 3.1. *Let σ be a segment of ends and prime ends of A ending on γ. Then for any $\varepsilon > 0$ there exists an arc $C \subset A$ (possible indefinite) such that if η_1, η_2; $\eta_1 < \eta_2$ are elements of σ with defining arcs b_1, b_2 respectively then C_σ has a sub arc beginning on b_1 and ending on b_2 such that every accessible point of γ corresponding to an end between η_1 and η_2 is within ε distance of C_σ. Furthermore, there exists a one to one order preserving correspondence between points of C_σ and elements of $\{\eta\}_{A,\gamma}$ between η_1 and η_2.*

Proof:

Let $\mathfrak{N} = \{N^k\}$ $(k = 1, 2, \ldots, n)$ be a finite set of co-ordinate neighborhoods, each of diameter less than ε which cover γ and let η_1, η_2 be elements of $\sigma_{\mathcal{N}}$. For N^1 and the corresponding γ^1, let η_1 be the end with b_1 as a defining arc, $b_1 \subset K \cup (p)$ and let η_1' be an end of K^1 and K^2 anding on $\gamma^1 \cap N^1 \cap N^2$

with defining arc $b' \subset \overline{K^1} \cap \overline{K^2}$ and also assume that b_1 and b' have no points in common except possibly a point of γ^1. Then every end between η_1 and η_1' can be characterized by an arc lying in K^1. Let C_1 be an arc in K^1 which joins points of b_1 and b'. Then every end between η_1 and η_1' can be defined by an arc in K^1 strating on C_1 and such that no two of these defining arcs for the ends intersect except possibly at points of γ^1 [2 b]. The same process can be continued for the entire segment by defining arcs C_2, C_3, ... which continue from each other in such a way that the initial point of C_i is the end point of C_{i-1} and such that no two of the arcs intersect except at end points. The arc C_1 can be continued also in the opposite direction by arcs C_0, C_{-1}, C_{-2}, ... in such a way that the entire set $\bigcup_{i=-\infty}^{\infty} C_i$ is an indefinite arc C_σ lying within ε distance of the portion γ^1 of the boundary which consists of end points of ends and prime ends in the segment σ. The correspondence of any end η with the point where its defining arc intersects C gives a one to one order preserving correspondence between C_σ and σ.

In case C_σ is an arc or simple closed curve we say that σ is a *complete segment.*

4. - Properties of boundary components of open subsets of M.

This section will be devoted to proving a lemma about the number of components of the boundary of a connected open subset of M which will be useful in the following sections.

Lemma 4.1. Let U be any co-ordinate neighborhood on M which intersects, at most, one boundary component of M. Let Q be a connected non-void open subset of M and let V be any component of $U - Q^*$ which contains no points of Q. Then V^* contains points of only finitely many components of Q^*.

Proof:

Let γ be a component of Q^*. Assume first that $\gamma \cap V^* \neq 0$ and that γ is deformable to a point on M. Since $\gamma \cap V^* \neq 0$, $M - \gamma$ must consist of two components of which one is a planar open set. In case this set contains points of Q it contains all of Q since γ separates M in this case. Then if V^* contains portions of γ it contains points of no other components of Q^* since γ separates all such components from V.

If γ is deformable to a point on M and the planar component of $M - \gamma$ lies in the complement of Q then γ disconnects this component from all other components of Q^*. Thus if $V^* \cap \gamma \neq 0$, V^* intersects no other components of Q^*. Hence, in case γ is deformable to a point on M, if $V^* \cap \gamma \neq 0$, V^* can contain points of no other component of Q^*.

We shall assume now that V^* contains points of a boundary component of Q which is not deformable to a point on M. Then every component $\gamma \in Q^*$ such that $\gamma \cap V^* \neq 0$ is not deformable to a point in M.

Let M be represented as a closed disk D in the plane with certain of its boundary arcs identified in pairs. This can be done in such a way that U is either interior to the disk or intersects one of the boundary arcs which is not identified with any other. Figures 1 and 2 show how this can be done in a

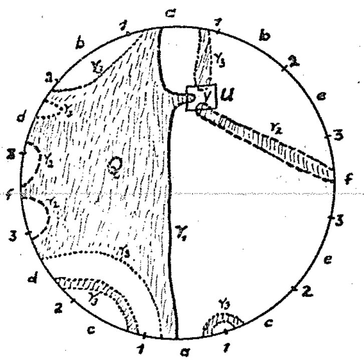

Figure 1.

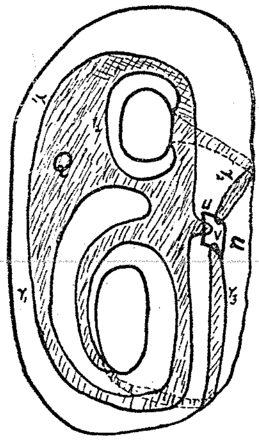

Figure 2.

particular case. Let $\mathfrak{S} = \{ s_1, s_2, s_3, \ldots, s_n \}$ be the finite family of distinguished boundary arcs of the disk D some of which are pair wise identified. Those which correspond to the boundary components of M are not identified with any others.

Let γ_1 be a connected portion of $Q^* \cap D$ which intersects D^* in two distinct arcs $s_{i_1}, s_{i_2} \in \mathfrak{S}$. Then γ_1 divides D into two parts D_1, D_2. Let D_1 be the part containing V. Then if γ_2 is any other component of $Q^* \cap D$ which intersects V^*, γ_2 can behave in several ways. If γ_2 intersects the same two arcs as γ_1, then V lies in the subset of D bounded by $\gamma_1, \gamma_2, s_{i_1}, s_{i_2}$ and no other component of $Q^* \cap D$ which intersects V^* can intersect two different arcs in $\mathfrak{S}$. If γ_2 intersects D^* in points of arcs s_{i_3}, s_{i_4} where at least one of the arcs s_{i_3}, s_{i_4} is distinct from s_{i_1}, s_{i_2}, then the boundary of the component of $D - (\gamma_1 \cup \gamma_2)$ which contains V consists of four pieces $\gamma_1, \gamma_2, a_1, a_2$, where a_1, a_2 are disjoint arcs on D^*. Since both γ_1, γ_2 intersect V^*, if γ_3 is any other component of $Q^* \cap D$ which intersects V^*, γ_3 cannot join a_1 and a_2 since then it would divide V into two disjoint pieces contrary to the connectedness of V. Thus γ_3 must join two elements of $\mathfrak{S}$ which lie on a_1 or two elements of $\mathfrak{S}$ which lie on a_2 or γ_3 can intersect only one element of $\mathfrak{S}$, either on a_1 or on a_2. If γ_3 joins two elements of $\mathfrak{S}$ on a_1 and if γ_4 is a fourth component of $Q^* \cap D$ then γ_4 will have fewer elements of $\mathfrak{S}$ available to intersect than γ_3. If this process is continued, choosing successively components $\gamma_5, \gamma_6, \ldots$, each of which join two elements of $\mathfrak{S}$, then each will have fewer elements available to join than the preceding one. Since $\mathfrak{S}$ contains only a finite number of elements, there can be only a finite number of components of $Q^* \cap D$ which intersect V^* and which join distinct elements of $\mathfrak{S}$.

The only components of $Q^* \cap D$ which intersect V^* and remain to be considered are those which intersect only one element of $\mathfrak{S}$. Let μ_1 be such a component and assume that μ_1 intersects only $s_{j_1} \in \mathfrak{S}$. Let γ be the component of Q^* which contains μ_1. Then γ must also intersect the other element s_{j_2} of $\mathfrak{S}$ which is identified with s_{j_1}. Let Q_1 be the portion of Q bounded by μ_1 and s_{j_1} and let Q_2 be the component of $Q \cap D$ which contains the portion of s_{j_2} corresponding to $\overline{Q}_1 \cap s_{j_1}$. Consider the components of $\gamma \cap Q_2^*$. All these components may intersect only s_{j_2} and, if so, γ encloses a planar region on μ and this case has already been considered. Hence, $\gamma \cap Q_2^*$ must have components which intersect other arcs of $\mathfrak{S}$ than s_{j_2}. Let us assume first that only one other arc s_{j_3} of $\mathfrak{S}$ is intersected by components of $\gamma \cap Q_2^*$. Then if Q_3 is defined with respect to s_{j_3} in the same manner as Q_2 was defined with respect to s_{j_2}, the same argument can be repeated for Q_3. Let us assume that successively $Q_1, Q_2, Q_3, \ldots$ have been defined, the closure of each intersecting only an « initial » and a « terminal » element of $\mathfrak{S}$ as above. We shall show that this situation can occur only a finite number of times for any γ.

Let p_1 be a point of $Q_1 \cap U$ and let p_2 be a point of $U \cap Q$ which is separated from V by some other component of Q^*. Then since Q is an open connected subset of M it is arc wise connected and there exists an arc C joining p_1 and p_2 lying in Q. Since C and γ are closed there is a minimum distance ε between them. Now each arc in the sequence $s_{j_1}, s_{j_2}, s_{j_3}, \ldots$ defined above must intersect both C and γ and the intersections must be at a distance apart of at least ε. Since the boundary curve D^* of D is finite in length, this implies that there can only be a finite number of such intersections. However, no $\overline{Q}_k$ for $k > 1$ can intersect only one arc of $\mathfrak{S}$ since in this case γ will de deformable to a point. If they all intersect two such arcs for $k > 1$ then eventually $s_{i_n} = s_{i_m}$ for some $m < n$ and this implies that $\mu_m \cap \mu_n \neq 0$ contrary to assumption. Thus some $\overline{Q}_k$ must intersect at least three of the arcs in $\mathfrak{S}$.

Now let γ' be another component of Q^* which behaves in the same way as γ. Then again $Q'_1, Q'_2, \ldots, Q'_l$ can be constructed where $\overline{Q}'_l$ intersects at least three of the arcs of $\mathfrak{S}$. However, by an argument similar to that of the first part of the proof, it can be seen that the number of arcs of $\mathfrak{S}$ available for these intersections is less than the number available for the intersections of γ, since these three arcs must be in the complement of $D^* \cap \overline{Q}_k$ and each component of the complement will have fewer arcs to intersect than were available to $\overline{Q}_k$. If successively other components of Q^* which intersect V^* and have the property that the component of $Q \cap D$ which intersects V^* and has its boundary points on the component in question intersects only one element of $\mathfrak{S}$, then eventually one of the sets of the form Q_j must have boundary points on at least three elements of $\mathfrak{S}$. However, since there are finitely many elements of $\mathfrak{S}$, there can be only finitely many such components.

Since we have exhausted all possible types of components of R^* which intersect V^* it can be seen that V^* contains points of only finitely many components of Q^*.

Figures 1 and 2 show a particular case in which for a manifold of genus two, three components of Q^* intersect V^*. It seems probable that at least for an oriented manifold without boundary and with genus n, that the maximum number of components of Q^* which intersect V^* should be $n+1$.

5. - The distribution of admissible ends.

Admissible and inadmissible ends and prime ends were defined in section 3. We prove here that any segment consists entirely either of admissible or of inadmissible ends and prime ends.

Lemma 5.1. Let γ be a boundary component of Q, let N_p be a coordinate neighborhood of a point $p \in \gamma$, let K be a component of $A \cap (N_p - \gamma)$ which has p as an accessible boundary point and suppose K contains a sequence of points $\{p_1, p_2, p_3, \dots\} \subset Q$ such that $\lim_{n\to\infty} p_n = p$. Then any end defined by an arc from K to p is admissible.

Proof:

Assume that there exists in K an arc b with end point p which contains no points of $\overline{Q}$ except p. Let $\{q_1, q_2, q_3, \dots\} \subset b$ be a sequence of points converging to p. By Theorem 3.1 it is possible to construct a sequence of arcs $\{C_i\}$ in K joining each p_i to each q_i and such that all points of C_i are within ε_i of the portions of γ bounding K where $\lim_{i\to\infty} \varepsilon_i = 0$. However, each arc C_i contains a point p_i' which lies on a boundary component of Q since Q is connected, $p_i \in Q$, $q_i \notin Q$. Thus $\lim_{i\to\infty} p_i' = p$. However, by Lemma 4.1, only a finite number of boundary components of Q can intersect K in this manner since $b-(p)$ lies in a component of $K - Q^*$ which contains no points of Q. Thus an infinite sequence of $\{p_{i_k}'\}$ of the $\{p_i'\}$ must belong to the same boundary component $\gamma' \subset Q^*$. However, γ' is closed and hence $\gamma' = \gamma$ since they contain the common point p. However, this implies that each p'_{i_k} is not in K contrary to the construction. Hence the arc b must contain a sequence of points of Q converging to p and b defines an admissible end.

If there exists no arc $b \subset K$ such that b defines the end η and $(b - (p)) \cap \overline{Q} = 0$ $b \cap \overline{Q}^*$ contains an infinite sequence $\{ q_i \}$ of points converging to p such that in the ordering of points on b, $q_1 < q_2 < q_3 < \ldots < p$, none of these points can be points of γ. It is possible then to construct around each point q_i a sphere s_i of radius less than $1/i$ which does not intersect γ or any of the other points of the sequence and lies in K. Each sphere s_i contains a point $p_i \in Q \cap K$. For each i let q'_i be a point of $b \cap s_i$ with $q'_i > q_i$. Then q_i can be joined to p_i by an arc b_i in s_i and p_i can be joined to q'_i by a similar such arc b'_i in such a way that $b_i \cap b'_i = p_i$. Let this be done for each i. Then if the portions of the arc b between q_i and q'_i are removed the union of the portions of b which remain plus the union of all the b_i and b'_i forms an arc in K which is equivalent to b and which defines an admissible end. Thus b itself defines an admissible end.

Lemma 5.2. Let σ be a segment and let $\eta_1, \eta_2 \in \sigma$, $\eta_1 < \eta$ be inadmissible ends of σ. Then every end η in the interval $\eta_1 \leqslant \eta \leqslant \eta_2$ is inadmissible.

Proof:

Assume that there exist ends in the interval which are admissible. Then there exists a set K of the type used to define the segment for which there are defined both admissible and inadmissible ends. Let η' be admissible and determined by an arc $b' \subset K$ and let η'' be inadmissible and determined by an arc $b'' \subset K$. Then all of b'' except its end point on γ lies in a component of $K - Q^*$ which contains no points of Q. By a construction similar to that in the proof of the last lemma, arcs $\{ C_i \}$ can be constructed in K approaching γ and having initial points in Q and terminal points not in Q. Thus there exist a sequence of boundary points $\{ p'_{i_k} \}$ as in the last lemma, all belonging to the same component of Q^* approaching a point of γ. Thus not all C_i are entirely in K contrary to assumption. Thus no K can contain both admissible and inadmissible ends.

Theorem 5.1. *Every segment σ consists entirely of admissible ends and prime ends or entirely of inadmissible ends and prime ends.*

Proof:

By the proof of the above lemma, every set K used in defining σ must consist entirely of admissible ends and prime ends or entirely of inadmissible ones. Since the sets K overlap in the definition of σ, σ must consist entirely of admissible ends or entirely of inadmissible ends.

6. - Prime ends of the second kind.

In section 3, segments were defined and the prime ends which arose locally in defining the segments will be called prime ends of the first kind. In case all the C_σ of the type discussed in Theorem 3.1 were arcs or simply closed curves, the segment σ was called complete. If, however, C_σ is an indefinite arc, then consider the portions of C_σ from some point $q \in C_\sigma$ in the increasing direction. If the set $C_{\sigma,q}$ is an indefinitite arc in the sense that it is an order preserving topological image of $0 \leqslant t < 1$ then C_σ, q will be said to define a *prime end of second kind.* In the ordering this prime end ω will be defined to follow all elements of σ. A similar definition is used to define a prime end of second kind which is less than all elements of σ. If two segments σ, σ' are given and if $C_{\sigma,q}$, $C_{\sigma'.q'}$ are defined as above, then $C_{\sigma.q}$, $C_{\sigma',q'}$ will be said to define the same prime end of second kind if given any ε, there exists a co-ordinate neighborhood N of diameter less than ε such that $C_{\sigma,q}$, $C_{\sigma',q'}$, have points in the same component K of $A \cap (N - \gamma)$. If the ordering on σ is given, then if $C_{\sigma,q}$ defines a prime end which follows all the elements of σ, then, if necessary, let this ordering on σ' be changed to that $C_{\sigma',q'}$, defines a prime end wich precedes all the elements of σ'. Treat the case in which $C_{\sigma,q}$ defines a prime end preceding σ in the same manner. If this process is carried out successively, a number of segments σ can be joined together in a linearly ordered manner. A maximal linearly ordeal set of ends and prime ends formed in this manner will be called a *complete segment* of ends and prime ends of A ending on γ. It will be noted that complete segments may contain prime ends of both first and second kind.

Lemma 6.1. Let the arc $C_{\sigma,q}$ define a prime end of second kind. Let $C_{\sigma,q}$ be divided into sub arcs $l_1, l_2, l_3, \ldots$ by points $q = q_1 < q_2 < q_3 < \ldots$. Then the arcs $l_1, l_2, l_3, \ldots$ have a limiting continuum Λ. Furthermore for any subdivision of $C_{\sigma,q}$ into sub arcs in the above manner, the superior limiting set Λ is the same. We define $\overline{\lim}\ C_{\sigma,q} = \Lambda$.

Proof: The proof is the same as that given in [2 c, p. 7] and will not be repeated here.

Theorem 6.1. *The union of the limiting sets of all arcs defining the same prime end ω is a connected set E_ω.*

Proof: By Lemma 6.1 each limiting set is a continuum. Let $C_{\sigma,q}$, $C_{\sigma',q'}$ be two indefinite arcs defining the same prime end ω of second kind. Then for any integer n there exsists a co-ordinate neighborhood N_n of diameter less than $1/n$ such that $C_{\sigma,q}$, $C_{\sigma',q'}$ can be joined in some component K of $(N-\gamma) \cap A$ by an arc lying in K. This implies that $C_{\sigma,q}$, $C_{\sigma',q'}$ have a limit point in common and hence that their two limiting sets have a non void intersection. Thus the union of the limiting sets of $C_{\sigma,q}$, $C_{\sigma',q'}$ is connected. Hence the union of all such sets is connected.

Further information about the structure of the sets E_ω would be valuable. In particular, information of the type discussed by URSELL and YOUNG [**12**] in the case of a simply connected plane domain concerning the principal points and the wings of E_ω would appear useful. In connection with this the question of whether or not every segment σ was a subset of a unique complete segment would be interesting. Another question of interest is the question of whether or not every complete segment has all its ends either admissible or inadmissible. The answer to these two questions seems likely to be affirmative. The author intends to investigate these and other questions in later papers.

7. - Admissible ends in plane domains.

Let M be a simply connected JORDAN region in the plane and let $Q \subset M$ be an open connected subset with boundary components $\{\gamma\}_Q$. We shall show that the set $A(Q, \gamma)$ defined in section **3** of this paper coincides with the set $A'(Q, \gamma)$ defined by CESARI in [2 a, p. 312] and that in this case, the set of all ends from $A(Q, \gamma)$ ending on γ is admissible. This will prove that the prime end theory developed by CESARI yields the same results as the present theory in case M is a simply connected JORDAN region. It will be recalled that under the definition of $A'(Q, \gamma)$ given in [2 a], $A'(Q, \gamma) = Q \cup U(\gamma' \cup \beta')$ where $\beta' = \beta'(\gamma', \gamma)$ consists of all elements of M which are separated from γ by γ' in M and where $U(\gamma' \cup \beta')$ is taken over all $\gamma' \in \{\gamma\}_Q$, $\gamma' \neq \gamma$.

Evidently $A'(Q, \gamma) \subset A(Q, \gamma)$ since A' is connected, open, contains Q and does not contain γ. Assume that there exists a point $p \in A$, $p \notin A'$. In this case, γ divides M into two components and, by definition, p lies in the component which contains Q. Since $p \notin A'$, $p \notin Q$ and hence p is separated from Q by a boundary component γ' of Q. However, since $p \in A$, $\gamma' \neq \gamma$. However, in the case of a plane region, every boundary component is known to separate the component of its complement not containing Q from all other boundary components of Q. Thus γ' separates p from γ and $p \in A'$. Hence $A = A'$.

It must next be proved that all ends from A ending on γ are admissible. However, it will be noted that in this case M itself is a co-ordinate neighborhood and A is a set of type K. If K is of genus one in this case, a cut can be made by removing an arc joining a point of γ and one of M^* and thus we get a simply connected co-ordinate neighborhood. However, the same proof as that used in Lemma 5.1 shows that the end η must be admissible. Since η was an arbitrary end, all ends of A ending on γ are admissible. Thus in this case the admissible ends of $A(Q, \gamma)$ correspond to all the ends of $A'(Q, \gamma) = A(Q, \gamma)$. Hence also all prime ends are admissible.

To show that there exist situations for more general manifolds M for which not all ends from $A(Q, \gamma)$ are admissible, consider the case in which M is a torus and Q is a band which goes around the torus with two boundary components γ, γ'. It is easily seen that $A(Q, \gamma)$ consists of $M - \gamma$ and that not all ends from $A(Q, \gamma)$ to γ are admissible.

Bibliography.

1. C. Caratheodory: *Über die Begrenzung einfach zusammenhaengender Gebiete*, Math. Ann. **73** (1913), 323-370.

2. L. Cesari: (a) *Surface Area*, Princeton 1956.
   (b) *Contours of Fréchet surfaces*, Riv. Mat. Univ. Parma **4** 1(953), 173-194.
   (c) *Properties of contours*, Rend. Mat. e Appl. Roma (5) **15** (1956), 341-365.
   (d) *A new process of retraction and fine-cyclic elements*, An. Acad. Brasil Ci.
   (e) *Fine-cyclic elements of surfaces of type v*, Riv. Mat. Univ. Parma **7** (1956), 149-185.

3. L. Cesari and R. E. Fullerton: (a) *On regular representations of surfaces*, Riv. Mat. Univ. Parma **2** (1951), 279-288.
   (b) *Smoothing methods for contours*, Ill. J. Math. **1** (1957), 395-405.

4. G. Choquet: *Structure des domaines plane et accessibilité*, C. R. Acad. Sci. Paris **216** (1943), 279-280.

5. W. H. Fleming: *Non-degenerate surfaces and fine-cyclic surfaces*, Duke Math. J. (to appear).

6. F. Frankl: *Topologische Beziehung is sich kompakter Teilmengen euklidischer Raume zu ihren Komplementen sowie Anwendungen Prime-enden Theorie*, Acad. Wiss. Wien **136** (1927), 689-699.

7. R. E. Fullerton: (a) *On the rectifications of contours of a Fréchet surface*, Riv. Mat. Univ. Parma **4** (1953), 207-212.
   (b) *On the subdivision of surfaces into pieces with rectifiable boundaries*, Riv. Mat. Univ. Parma **4** (1953), 289-298.

8. E. Hopf: *A theorem on the accessibility of boundary parts of an open point set*, Proc. Amer. Math. Soc. **1** (1950), 76-79.

9. B. von Kerekjártó: *Topologie*-I, Berlin 1923.

10. C. J. NEUGEBAUER: (a) *B-sets and fine-cyclic elements*, Trans. Am. Math. Soc. (to appear).

(b) *Local A-sets, B-sets and retraction*, Ill. J. Math. (to appear).

11. T. RADÓ: *Lenght and Area*, Amer. Math. Soc. Coll. Pub. 30, 1948.

12. H. D. URSELL and L. C. YOUNG: *Remarks on the theory of prime ends*, Mem. Amer. Math. Soc. 3 (1951).

13. G. T. WHYBURN: *Analytic Topology*, Amer. Math. Soc. Coll. Pub. 28, 1942.