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On Evaluating the Stokes' Stream Function by Means of a Symmetric Difference Analogue. (**)

1. - Introduction.

In fluid dynamics, the STOKES' stream function, that is, the function which satisfies

$$u_{xx} + u_{yy} - (1/y)\, u_y = 0 \qquad (y \neq 0) \tag{1.1}$$

is of considerable interest. This function is constant on the streamlines. The problem to be considered is a DIRICHLET type problem. Let G be a closed, bounded, simply connected plane region whose interior is denoted by R and whose boundary curve is denoted by S. Let G not contain any point where $y = 0$. Let $g(x, y)$ be defined and continuous on S. The problem then is to produce a function $u(x, y)$ such that

(a) $u(x, y) \equiv g(x, y)$ on S,

and

(b) $u(x, y)$ satisfies (1.1) in R.

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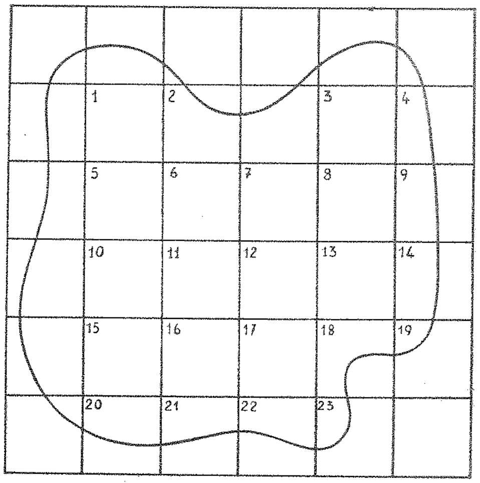
Under general conditions, there exists a unique solution [5] and only such cases will be considered. However, the analytical determination of $u(x, y)$ is quite another story from that of its existence and usually offers what are at present insurmountable problems. The approach here, then, will be from a numerical analysis point of view.

2. - General Method.

Since G is closed, and bounded, and no y value of G is zero, then there exists a positive number h such that $0 < h < |y|$, for all y in G. Fix such a value h for the mesh size. Let (x_0, y_0) be an arbitrary, but fixed point of G and denote by G_h the set of all points of the form $(x_0 + mh, y_0 + nh)$ contained in G, where m and n are integers. Two points (x_1, y_1) and (x_2, y_2) are called adjacent if and only if $(x_2 - x_1)^2 + (y_2 - y_1)^2 = h^2$ and the straight line segment joining them is contained in G. The interior of G_h, denoted by R_h, is the set of all points of G_h whose four adjacent points belong to G_h. The boundary of G_h, denoted by S_h, and called the lattice boundary, is defined by $S_h = G_h - R_h$. It is also assumed that any pair of points of R_h can be joined by a connected polygonal arc consisting of straight line segments which join adjacent points of R_h.



Now, the technique proposed will involve the application of a difference equation to yield a system of linear equations whose solution is an approximation to the solution of the formulated DIRICHLET problem at the points of G_h. The question of solving the linear system will not be considered but may be approached by means of CRAMER's rule, GAUSS' elimination procedure, matrix inversion, relaxation, iteration, gradient methods, and other techniques.

Suppose then that G_h consists of n points. Number the points as in the diagram, so that the numbers increase from left to right and also increase if one reads down any column of points. Denote the coordinates of the point numbered k by (x_k, y_k) and the unknown stream function at (x_k, y_k) by $u(x_k, y_k) \equiv u_k$, for $k = 1, 2, \ldots, n$. Let (x_i, y_i) be an arbitrary point of S_h, the lattice boundary. Approximate u_i by $g(x', y')$, where (x', y') is the nearest point of S to (x_i, y_i).

If (x', y') is not unique, choose any one of the set of nearest points and use it. The problem of finding numerical approximations to $u(x, y)$ on the lattice boundary is, though crudely done, adequate for present purposes. For the diagram, this means that at the points 1, 2, 3, 4, 5, 7, 9, 10, 14, 15, 19, 20, 21, 22, 23 the values of the stream function have been approximated by values at nearby points of S.

Now apply the difference equation:

$$(2.1) \qquad 2\, u(x, y)\left(\frac{1}{y} + \frac{1}{2y+h} + \frac{1}{2y-h}\right) - \frac{1}{y}\left[u(x+h, y) + u(x-h, y)\right] - \frac{2\, u(x, y+h)}{2y+h} - \frac{2\, u(x, y-h)}{2y-h} = 0,$$

to the points of R_h. Start with the first, or highest, row of points; traverse the row from left to right; proceed to the next row; traverse the row from left to right; proceed to the next row, etc.; continue the process until all the points of R_h have been used. The process yields a linear system of n equations in n unknowns which has a symmetric coefficient matrix (see [7]).

Finally, one should note the following three points:

(a) Details for constructing (2.1) are described in [3], pp. 153-155.

(b) In actual practice, the subscript notation is most convenient. For example, application of (2.1) at the point numbered 12 in the diagram would yield:

$$2\, u_{12}\left(\frac{1}{y_{12}} + \frac{1}{2\, y_{12}+h} + \frac{1}{2\, y_{12}-h}\right) - \frac{1}{y_{12}}(u_{13} + u_{11}) - \frac{2\, u_7}{2\, y_{12}+h} - \frac{2\, u_{17}}{2\, y_{12}-h} = 0\,.$$

In this equation u_{11}, u_{12}, u_{13} and u_{17} are unknowns while u_7 is a constant determined by the method for points of S_h.

(c) No denominator in (2.1) is zero since $0 < h < |y| < |2y|$.

3. – A Theorem on an Error Bound.

In this section, let $u(x, y)$ be the solution of the DIRICHLET problem being considered and $U(x, y)$ the solution of the numerical method described in Section **2**. This means that at any point (x, y) of R_h:

$$(3.1) \qquad 2\, U(x, y)\left(\frac{1}{y} + \frac{1}{2\,y+h} + \frac{1}{2\,y-h}\right) - \frac{1}{y}\left[U(x+h, y) + U(x-h, y)\right] - \frac{2\, U(x, y+h)}{2\,y+h} - \frac{2\, U(x, y-h)}{2\,y-h} = 0.$$

Theorem 1. *The solution of the system of linear equations which results by application of the process described in Section 2 is unique.*

Proof. It is sufficient to show that the determinant of the system of linear equations is not zero and this is done by demonstrating that the only solution of the homogeneous system which results by considering $g(x, y) \equiv 0$ on S is the zero vector. Suppose then there exists a non-trivial solution for the homogeneous system. For some point of R_h, $U \neq 0$. Suppose $U > 0$. Let the largest value M occur at (x_0, y_0) and let $U_0 = U(x_0, y_0)$, $U_1 = U(x_0 + h, y_0)$, $U_2 = U(x_0, y_0 + h)$, $U_3 = U(x_0 - h, y_0)$, $U_4 = U(x_0, y_0 - h)$. Then:

$$2\, U_0 \left(\frac{1}{y_0} + \frac{1}{2\, y_0 + h} + \frac{1}{2 y_0 - h}\right) - \frac{1}{y_0} (U_1 + U_3) - \frac{2\, U_2}{2\, y_0 + h} - \frac{2\, U_4}{2\, y_0 - h} = 0,$$

or

$$U_0 = \frac{4\, y_0^2 - h^2}{2\,(8\, y_0^2 - h^2)} (U_1 + U_3) + \frac{2\, y_0^2 - h\, y_0}{8\, y_0^2 - h^2}\, U_2 + \frac{2\, y_0^2 + h\, y_0}{8\, y_0^2 - h^2}\, U_4 . \tag{3.2}$$

Case 1. Suppose $U_1 = U_2 = U_3 = U_4$. Then, from (3.2) it follows that $U_0 = U_1$ and hence that $U_0 = U_1 = U_2 = U_3 = U_4 = M$.

Case 2. Suppose not all of U_1, U_2, U_3, U_4 are equal. Then there exists a maximum. Without loss of generality let it be U_1. Then $U_2 = U_1 - k_2$, $U_3 = U_1 - k_3$, $U_4 = U_1 - k_4$, where k_2, k_3, k_4 are non-negative and at least one is positive. Substitution of these values into (3.2) yields:

$$U_0 = \frac{4\, y_0^2 - h^2}{2(8\, y_0^2 - h^2)} (2\, U_1 - k_3) + \frac{2\, y_0^2 - h y_0}{8\, y_0^2 - h^2} (U_1 - k_2) + \frac{2\, y_0^2 + h\, y_0}{8\, y_0^2 - h^2} (U_1 - k_4)$$

or

$$U_0 = U_1 - k_3 \frac{4\, y_0^2 - h^2}{2(8\, y_0^2 - h^2)} - k_2 \frac{2\, y_0^2 - h\, y_0}{8\, y_0^2 - h^2} - k_4 \frac{2\, y_0^2 + h\, y_0}{8\, y_0^2 - h^2} . \tag{3.3}$$

Now since $0 < h < |\, y\,|$, it follows that $2\, y_0^2 \pm h\, y_0$, $8\, y_0^2 - h^2$ and $4\, y_0^2 - h^2$ are positive. Also, since at least one of k_2, k_3, k_4 is positive, from (3.3): $U_0 = U_1 - K$, where $K > 0$. Hence, $U_1 > U_0$, which is a contradiction. Hence $U_0 = U_1 = U_2 = U_3 = U_4 = M$.

Using, then, say, $(x + h, y)$ in place of (x, y), the same argument as presented above may be applied and continued in a finite number of steps to show that U at a boundary point is equal to M. But this is a contradiction since $g(x, y) \equiv 0$

on S and the method described for points of S_h implies U at all lattice boundary points is zero.

A contradiction is similarly reached if U is assumed to be less than zero at (x_0, y_0). Hence, the only solution is the trivial solution, which proves the theorem.

Now let:

$$\lambda[v(x, y)] \equiv -\frac{1}{h^2}\left\{2\, v(x, y)\left(\frac{1}{y} + \frac{1}{2\,y+h} + \frac{1}{2\,y-h}\right) - \right. \tag{3.4}$$

$$\left. -\frac{1}{y}\,[v(x+h, y) + v(x-h, y)] - \frac{2\,v(x, y+h)}{2\,y+h} - \frac{2\,v(x, y-h)}{2\,y-h}\right\}.$$

Lemma 1. If $\lambda[v] \leqslant 0$ on R_h and $v \geqslant 0$ on S_h, then $v \geqslant 0$ on R_h.
(The proof follows as in [**3**], p. 156.)

Lemma 2. If $-\,|\,\lambda[v_1]\,| \geqslant \lambda[v_2]$ on R_h and $|\,v_1\,| \leqslant v_2$ on S_h, then $|\,v_1\,| \leqslant v_2$ on R_h.
(The proof follows as in [**3**], p. 156.)

Lemma 3. If $|\,\lambda[v]\,| \leqslant A$ on R_h and if $|\,v\,| \leqslant B$ on S_h, and r is the radius of a circle which contains G and which is properly selected, as described below, then $|\,v\,| \leqslant A\,r^2/4 + B$, on R_h.

Proof.

Let $w(x, y) = \left[\frac{A\,r^2}{4}\left\{1 - \frac{(x-a)^2 + (y-b)^2}{r^2}\right\} + B\right]$, where $(x-a)^2 + (y-b)^2 = r^2$ is the equation of any circle containing G for which $b \geqslant (2\,y-1)(4\,y^2-h^2)/(6y)$, for all y in G. Note that this latter inequality is equivalent to: $1/(2y) + 3b/(4\,y^2-h^2) \geqslant 1$. At least one such value b exists since G is closed and bounded and $(2y-1)(4\,y^2-h^2)/(6y)$ is continuous on G.

Direct calculation yields $\lambda[w] = -\,A[1/(2y) + 3b/(4y^2-h^2)] \leqslant -\,A$, by the above choice of b. Also, $w \geqslant B$ on S_h.

Now since $|\,\lambda[v]\,| \leqslant A$ on R_h, by assumption, and it has been shown that $\lambda[w] \leqslant -\,A$, it follows that $-\,\lambda[w] \geqslant |\,\lambda[v]\,|$ on R_h. Since $|\,v\,| \leqslant B$ on S_h and $w \geqslant B$ on S_h, $w \geqslant |\,v\,|$. Hence: $-\,|\,\lambda[v]\,| \geqslant \lambda[w]$ on R_h and $w \geqslant |\,v\,|$ on S_h. By Lemma 2, $w \geqslant |\,v\,|$ on R_h, or $|\,v\,| \leqslant w \leqslant A\,r^2/4 + B$. This completes the proof.

Theorem 2. *If $u(x, y)$ is of class C^4 in the closed region G, $y \neq 0$ in G, $0 < h < |\,y\,|$ for all y in G, u denotes the solution of the Dirichlet type problem associated with equation* (1.1), *while U denotes the solution of the linear system which results from application of* (2.1) *according to the method of Section* **2**, *then*

$$|\,U - u\,| \leqslant (r^2/4)\left\{h^2\,M_2/\bar{y}^3 + h^2\,M_3/\bar{y}^2 + h^2\,M_4/\bar{y}\right\} + 2h\,M_1, \tag{3.5}$$

where: $M_i = \max | \partial^{m+n} u / \partial x^m \partial y^n |$, *over* G *for* $m + n = i$, $i = 2, 3, 4$; $\bar{y} = \text{GLB} | y |$ *for all* y *in* G; r *is the radius of any circle of the type described in Lemma* 3.

Proof.

Let $Q = \lambda[u] - (u_{xx} - u_y/y + u_{yy})$. Substitution of the finite TAYLOR series expansions for u_1, u_2, u_3, u_4 into the expression for Q yields:

$$Q = -\frac{1}{h^2}\left\{ \frac{u_{2,0}\, h^4}{(4y^2 - h^2)y} - \frac{u_{4,0}(\xi_1, y)\, h^4}{24\, y} - \frac{u_{4,0}(\xi_2, y)\, h^4}{24\, y} + \right.$$

$$\left. + \frac{2\, u_{0,3}\, h^4}{3(4y^2 - h^2)} - \frac{u_{0,4}(x, \eta_1)\, h^4}{12(2y + h)} - \frac{u_{0,4}(x, \eta_2)\, h^4}{12(2y - h)} \right\}.$$

Hence, since $0 < h < |y|$, it readily follows that $|2y + h| \geqslant |2y| - |h| > |y|$, and $|2y - h| \geqslant |2y| - |h| > |y|$, so that:

$$|Q| < M_2 h^2/\bar{y}^3 + M_4 h^2/\bar{y} + M_3 h^2/\bar{y}^2, \qquad \bar{y} = \text{GLB}\, |y| \qquad \text{in } G.$$

Now, since $u_{xx} - u_y/y + u_{yy} = 0$, it follows that

$$(3.6) \qquad |\lambda[u] - (u_{xx} - u_y/y + u_{yy})| = |\lambda[u]| = |Q| \leqslant$$

$$\leqslant M_2 h^2/\bar{y}^3 + M_4 h^2/\bar{y} + M_3 h^2/\bar{y}^2.$$

Also, for any point of S_h, U was chosen as the value $g(x', y')$ at the nearest point (x', y') on the boundary S, and $g(x', y') = u(x', y')$ on S. Thereby, for any point (x, y) of S_h, as shown in [3], p. 158,

$$(3.7) \qquad |U(x, y) - u(x, y)| \leqslant 2h\, M_1.$$

Also, it must be noted that: $\lambda[U] = 0$, by (3.1). Hence

$$(3.8) \qquad |\lambda[u - U]| = |\lambda[u] - \lambda[U]| = |\lambda[u]|.$$

Applying then Lemma 3 to (3.6), (3.7) and (3.8), one finds that, on R_h,

$$|U - u| \leqslant (r^2/4)\{ M_2 h^2/\bar{y}^3 + M_4 h^2/\bar{y} + M_3 h^2/\bar{y}^2 \} + 2h\, M_1.$$

Hence the theorem is proved.

Note that (3.5) is an error bound for the numerical solution U, though a rather crude one. One may easily refine it, but its chief use here, for which it is more than adequate, is the establishment of the following basic theorem.

Theorem 3. *Under the conditions of Theorem 2, the numerical solution U converges to the analytic solution u as the grid size approaches zero, that is,* $\lim_{h \to 0} U = u$.

The proof is an immediate consequence of inequality (3.5).

Bibliography.

1. L. Collatz, *Bemerkugen zur Fehlerabschatzung für das Differenzenverfahren bei partiellen Differentialgleichungen*, Z. Angew. Math. Mech. **13** (1933), 56-57.

2. S. Gerschgorin, *Fehlerabschatzung für das Differenzenverfahren zur Losung partieller Differentialgleichungen*, Z. Angew. Math. Mech. **10** (1930), 373-382.

3. D. Greenspan, *On the numerical evaluation of the Stokes' stream function*, Math. Tables Aids Comput. **11** (1957), 150-160.

4. W. E. Milne, **Numerical solution of differential equations**, John Wiley, New York 1953.

5. I. G. Petrovsky **Lectures on partial differential equations** (English edition), Interscience, New York 1954 (cf. p. 232).

6. G. Shortley, R. Weller, P. Darby and E. H. Gamble, *Numerical solution of axissymmetrical problems with applications to electrostatics and torsion*, J. Appl. Phys. **18** (1947), 116-129.

7. D. M. Young, *Iterative methods for solving partial difference equations of elliptic type*, Trans. Amer. Math. Soc. **76** (1954), 92-111.

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