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Convolution Transforms of Almost Periodic Functions. (**)

In this paper we present necessary and sufficient conditions that a function be a convolution transform (with kernel in a certain class) of a function almost periodic in the sense of BOHR.

We first recall some facts of convolution transform theory only in such detail as we shall need in the proofs to follow.

Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $\sum_{k=1}^{\infty} 1/a_k^2 < \infty$, let $\{b_n\}_{n=1}^{\infty}$ and b be real numbers with

$$(b - b_n)^2 = O(\sum_{k=n}^{\infty} 1/a_k^2) \qquad \text{as} \qquad n \to \infty,$$

and let

$$E(s) = e^{bs} \prod_{k=1}^{\infty} \{1 - (s/a_k)\} e^{s/a_k}, \qquad P_n(s) = e^{b_n s} \prod_{k=1}^{n} \{1 - (s/a_k)\} e^{s/a_k} \quad (n = 1, 2, \ldots),$$

$$G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{E(s)} \, ds, \qquad G_n(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st} P_n(s)}{E(s)} \, ds \qquad (n = 1, 2, \ldots).$$

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Then the following statements hold:

(1) $$G_n(t) \geqslant 0 \qquad (n = 1, 2, \ldots),$$

(2) $$\int_{-\infty}^{\infty} G_n(t)\, dt = 1 \qquad (n = 1, 2, \ldots),$$

$$P_n(\mathrm{D})\, G(t) = G_n(t) \qquad (n = 1, 2, \ldots; \quad -\infty < t < \infty).$$

Here D stands for differentiation and $e^{a\mathrm{D}}h(x)$ is defined to be $h(x + a)$. Also if

$$f(x) = \int_{-\infty}^{\infty} G(x - t)\, \varphi(t)\, dt$$

for some function $\varphi(t)$ essentially bounded on $-\infty < t < \infty$, then

(3) $$P_n(\mathrm{D})\, f(x) = \int_{-\infty}^{\infty} G_n(x - t)\, \varphi(t)\, dt \qquad (n = 1, 2, \ldots; \quad -\infty < x < \infty)$$

and

(4) $$\lim_{n \to \infty} P_n(\mathrm{D})\, f(x) = \varphi(x) \text{ almost everywhere } (-\infty < x < \infty).$$

All the above may be found in [**3**].

We next recall some definitions and results from Bohr's theory of almost periodic functions (see [**1**]).

(5) Let $\beta(x) \in C\ (-\infty < x < \infty)$. The real number τ is called a translation number for $\beta(x)$ corresponding to $\varepsilon > 0$ if $|\beta(x + \tau) - \beta(x)| \leqslant \varepsilon$ for $-\infty < x < \infty$. Such a τ is denoted $\tau = \tau_\beta(\varepsilon)$.

(9) The function $\beta(x) \in C\ (-\infty < x < \infty)$ is *almost periodic* if for every $\varepsilon > 0$ there exists an $L = L(\varepsilon)$ such that every interval $A \leqslant x \leqslant A + L$ of length L contains a $\tau_\beta(\varepsilon)$.

(7) Thus, if $\beta(x)$ is almost periodic and the continuous function $\gamma(x)$ is such that every $\tau_\beta(\varepsilon)$ is a $\tau_\gamma(\varepsilon)$, then $\gamma(x)$ is almost periodic. We then say that $\gamma(x)$ is *dominated* by $\beta(x)$.

(8) An almost periodic function is uniformly continuous and bounded on $-\infty < x < \infty$.

(9) If $\{\beta_n(x)\}_{n=1}^{\infty}$ are almost periodic and uniformly convergent to $\beta(x)$ on $-\infty < x < \infty$, then $\beta(x)$ is almost periodic.

We now state and prove our representation theorem.

Theorem. *Let $G(t)$, $P_n(t)$ $(n = 1, 2, \dots)$ be as above. Then necessary and sufficient conditions that $f(x) \in C^\infty$ $(-\infty < x < \infty)$ be representable in the form*

$$(10) \qquad f(x) = \int_{-\infty}^{\infty} G(x-t)\,\varphi(t)\,dt \qquad (-\infty < x < \infty),$$

where $\varphi(t)$ is almost periodic are that the functions $\{P_n(D)\,f(x)\}_{n=1}^{\infty}$ be uniformly dominated by some almost periodic function, and be uniformly bounded.

Proof.

Suppose first that $f(x)$ has the form (10) for some almost periodic $\varphi(t)$. Denote $P_n(D)\,f(x)$ by $f_n(x)$. Then by (3), for $n = 1, 2, \dots$ and any x,

$$(11) \qquad f_n(x) = \int_{-\infty}^{\infty} G_n(x-t)\,\varphi(t)\,dt = \int_{-\infty}^{\infty} G_n(t)\,\varphi(x-t)\,dt.$$

Given $\varepsilon > 0$ let $\tau = \tau_\varphi(\varepsilon)$. Then

$$f_n(x+\tau) - f_n(x) = \int_{-\infty}^{\infty} G_n(t)\,[\varphi(x-t+\tau) - \varphi(x-t)]\,dt,$$

so that, using (1), (5) and (2),

$$|f_n(x+\tau) - f_n(x)| \leqslant \int_{-\infty}^{\infty} G_n(t)\,|\varphi(x-t+\tau) - \varphi(x-t)|\,dt \leqslant \varepsilon \int_{-\infty}^{\infty} G_n(t)\,dt = \varepsilon.$$

Hence $\tau = \tau_{f_n}(\varepsilon)$. Thus for each $n = 1, 2, \dots$, $f_n(x)$ is denominated by $\varphi(x)$.

By (8) we may choose M such that $|\varphi(x)| \leqslant M$ for $-\infty < x < \infty$, so that from (11)

$$|f_n(x)| \leqslant M \int_{-\infty}^{\infty} G_n(t)\,dt = M \qquad (n = 1, 2, \dots;\ -\infty < x < \infty).$$

The necessity of our conditions is thus established.

To prove the sufficiency of the conditions we again set $f_n(x) = P_n(\mathrm{D}) f(x)$ and assume that the $f_n(x)$ are uniformly bounded, and uniformly dominated by the almost periodic function $\beta(x)$. [The $f_n(x)$ are then almost periodic by (7).]

We first show that there exists a subsequence of the $f_n(x)$ which converges uniformly on $-\infty < x < \infty$. By (8) $\beta(x)$ is uniformly continuous. Hence given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|\beta(x+\tau) - \beta(x)| \leqslant \varepsilon \quad \text{for} \quad |\tau| \leqslant \delta \qquad (-\infty < x < \infty),$$

so that $|\tau| \leqslant \delta$ implies $\tau = \tau_\beta(\varepsilon)$. Since each $f_n(x)$ is dominated by $\beta(x)$ it follows that $|\tau| \leqslant \delta$ implies $\tau = \tau_{f_n}(\varepsilon)$ for $n = 1, 2, \ldots$. This may be expressed as

$$|f_n(x+\tau) - f_n(x)| \leqslant \varepsilon \quad \text{for} \quad |\tau| \leqslant \delta \quad (n = 1, 2, \ldots; \quad -\infty < x < \infty),$$

which shows that the family $\{f_n(x)\}_{n=1}^{\infty}$ is equi-uniformly continuous on $-\infty < x < \infty$. A well-known theorem (see [**2**, p. 59]) says that if the functions of a sequence are uniformly bounded and equi-uniformly continuous on a *compact interval*, then there exists a subsequence converging uniformly on that interval. Hence there exists a subsequence of the $f_n(x)$, say $\{f_n^{(1)}(x)\}$, which converges uniformly on $-1 \leqslant x \leqslant 1$. Since the $f_n^{(1)}(x)$ are uniformly bounded and equi-uniformly continuous on $-\infty < x < \infty$, and hence on $-2 \leqslant x \leqslant 2$, there exsists a subsequence of the $f_n^{(1)}(x)$, say $\{f_n^{(2)}(x)\}$, converging uniformly on $-2 \leqslant x \leqslant 2$. For each $k = 2, 3, \ldots$ we thus define $\{f_n^{(k)}(x)\}$ as a subsequence of $\{f_n^{(k-1)}(x)\}$ which converges uniformly on $-k \leqslant x \leqslant k$. It is then clear that the sequence $\{f_n^{(n)}(x)\}$ converges uniformly on every *finite* interval. But since each $f_n^{(n)}(x)$ is almost periodic and dominated by $\beta(x)$, uniform convergence on every finite interval implies uniform convergence on $-\infty < x < \infty$. To show this we choose, given $\varepsilon > 0$, an L such that every interval of length L contains a $\tau_\beta(\varepsilon/4)$. [$L$ exists by (8).] Now set $g_n(x) = f_n^{(n)}(x)$. We have shown that the $g_n(x)$ converge uniformly on every finite interval so that we may choose N such that, for $n, m \geqslant N$,

$$|g_n(x) - g_m(x)| \leqslant \varepsilon/2 \qquad (0 \leqslant x \leqslant L).$$

For any real y we may choose $\tau = \tau_\beta(\varepsilon/4)$ such that $-y \leqslant \tau \leqslant -y + L$. But then $0 \leqslant y + \tau \leqslant L$ and so

$$|g_n(y+\tau) - g_m(y+\tau)| \leqslant \varepsilon/2 \qquad (n, m \geqslant N).$$

Since $\beta(x)$ dominates $g_n(x)$ for all n then $\tau = \tau_{g_n}(\varepsilon/4) = \tau_{g_m}(\varepsilon/4)$. Thus

$$|g_n(y+\tau) - g_n(y)| \leqslant \varepsilon/4, \qquad |g_m(y+\tau) - g_m(y)| \leqslant \varepsilon/4 .$$

Combining the last three inequalities we have

$$|g_n(y) - g_m(y)| \leqslant \varepsilon \qquad (n,\ m \geqslant N).$$

Since y was arbitrary we have shown that there exists a function $\varphi(x)$ such that

(12) $$\lim_{n\to\infty} g_n(x) = \varphi(x) \quad \text{uniformly} \quad (-\infty < x < \infty).$$

Moreover, by (9), $\varphi(x)$ is *almost periodic*.

Since the $f_n(x)$ are uniformly bounded, by a theorem in [3] there exists a function $\psi(t)$ essentially bounded on $-\infty < t < \infty$ such that

(13) $$f(x) = \int_{-\infty}^{\infty} G(x-t)\, \psi(t)\, dt \qquad (-\infty < x < \infty).$$

By (4)

(14) $$\lim_{n\to\infty} f_n(x) = \psi(x) \quad \text{almost everywhere} \quad (-\infty < x < \infty).$$

Since the $g_n(x)$ are a subsequence of the $f_n(x)$ (12) and (14) imply

$$\psi(x) = \varphi(x) \quad \text{almost everywhere} \quad (-\infty < x < \infty).$$

But this and (13) then show that

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\, \varphi(t)\, dt \qquad (-\infty < x < \infty).$$

Since $\varphi(t)$ is almost periodic, the proof is complete.

References.

1. H. Bohr, **Almost periodic functions.** Chelsea, New York 1951.

2. R. Courant and D. Hilbert, **Methods of mathematical physics.** Vol. I. Interscience, New York 1953.

3. I. I. Hirschman (jr.) and D. V. Widder, **The convolution transform.** Princeton University Press, Princeton 1955.