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III. - A Further Extension of a Cyclic Additivity Theorem of a Surface Integral. (**)

Introduction.

This is the third and last paper in a series of three papers and deals with an application of the theory developed in [7] to a surface integral. The papers [7], [8] will be referred to with Roman numerals **I, II**, respectively, followed, if necessary, by Arabic numerals indicating the specific section in **I** or **II**.

Let A be an admissible set of E_2 (see [**2**; 5.1]), and let (T, A) be a continuous mapping from A into E_3, where E_2, E_3 are the Euclidean plane and the Euclidean three space, respectively. If $L(T, A)$ denotes the LEBESGUE area of (T, A) (see [**2**]), then for $L(T, A) < \infty$, L. CESARI [**4, 2**] has introduced a surface integral $J(T, A) = \int F\, d\sigma$ as a WEIERSTRASS integral. In this paper a cyclic additivity theorem for $J(T, A)$ will be established similar to the corresponding theorem for the LEBESGUE area $L(T, A)$ (see **II.12**).

The question of cyclic additivity of $J(T, A)$ was first studied by J. CECCONI [**1**], who has proved the following theorem. If $A = Q$ is the unit square, and if $(T, Q) = lm$, $m: Q \Longrightarrow \mathfrak{M}$, $l: \mathfrak{M} \to E_3$ is a monotone-light factorization of (T, Q), then $J(T, Q)$ is *weakly cyclicly additive*, i. e.,

$$(1) \qquad J(T, Q) = \sum J(l\, r_C\, m, Q), \qquad C \subset \mathfrak{M},$$

where r_C is the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element C of $\mathfrak{M}$.

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If $(T, Q) = sf$, $f: Q \to \mathfrak{M}$, $s: \mathfrak{M} \to E_3$ is an unrestricted factorization of (T, Q) (see [**6**]), then $J(T, Q)$ is also *strongly cyclicly additive*, i. e.,

$$(2) \qquad J(T, Q) = \sum{}' J(s\, r_C f, Q)$$

where $\sum'$ extends over all proper cyclic elements C of $\mathfrak{M}$ for which $C \cap f(Q) \neq 0$. The formula (2) has been proved in [**9**].

Both formulas (1) and (2) are concerned with the unit square Q. In this paper Q is replaced by an admissible set, and the following further extension of (2) is proved. If $(T, A) = sf$, $f: A \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$, is an unrestricted factorization of (T, A) in the sense of **I.9**, then

$$(3) \qquad J(T, A) = \sum J(s\, r_C f, G_C), \qquad C \in \mathfrak{K},$$

where $\mathfrak{K}$ is the class of proper cyclic elements associated with $(T, A) = sf$, and where G_C is the set associated with $C \in \mathfrak{K}$. For the above terminology the reader is referred to **I.12**.

The treatment of cyclic additivity of the surface integral $J(T, A)$ and the LEBESGUE area $L(T, A)$ are somewhat different. The difference stems from the fact that $J(T, A)$ need not be non-negative. From this point of view, the present paper is not a direct application of **I**. As will be seen, however, many proofs given in **I** can readily be modified to apply to $J(T, A)$.

A Cyclic Additivity Theorem.

III.1. – In this paragraph we shall give the definition of an admissible set and those properties of the LEBESGUE area which are needed in the sequel.

Definition. A subset A of the Euclidean plane E_2 will be termed *admissible* provided one of the following cases holds. (a) A is a simply connected JORDAN region; (b) A is a finitely connected JORDAN region; (c) A is a finite union of disjoint regions of the type (a) or (b); (d) A is any open set in E_2; (e) A is any set open in a set of the type (a), (b), or (c). In particular A may be a *figure* F, i. e., a finite union of disjoint finitely connected polygonal regions. The reader is referred to [**2**; 5.1].

For (T, A) a continuous mapping from an admissible set into E_3 one can define the LEBESGUE area $L(T, A)$ as in L. CESARI [**2**; 5.8].

(i) If (T, A) is a continuous mapping from an admissible set A into E_3, and if A_n $(n = 1, 2, \ldots)$ is any sequence of admissible sets such that $A_n \subset A_{n+1} \subset A$, $A_n^0 \uparrow A^0$, then $L(T, A_n) \to L(T, A)$ as $n \to \infty$ (L. CESARI [**2**; 5.14 (iv)]).

(ii) If an admissible set A can be written as the union A_i $(i = 1, 2, \ldots)$ of disjoint admissible sets with the property that each interior point of A is interior to one A_i, then $L(T, A) = \sum L(T, A_i)$ (L. CESARI [**2**; 5.14 (ii)]).

III.2. – Let A be an admissible set in E_2 and let (T, A) be a continuous mapping from A into E_3 such that $L(T, A) < \infty$. Then T: $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $(u, v) \in A$. Let us introduce the plane mappings T_1: $y = y(u, v)$, $z = z(u, v)$; T_2: $z = z(u, v)$, $x = x(u, v)$; T_3: $x = x(u, v)$, $y = y(u, v)$, $(u, v) \in A$. The image of A under T_r lies in a Euclidean plane which we designate by E_{2r} $(r = 1, 2, 3)$. For π a polygonal region in A, let us denote by π^* the counterclockwise oriented boundary curve of π. Let $O(p_r; T_r, \pi)$ be the topological index of a point $p_r \in E_{2r}$ with respect to $T_r(\pi^*)$ if $p_r \notin T_r(\pi^*)$. We set $O(p_r; T_r, \pi) = 0$ if $p_r \in T_r(\pi^*)$. We define according to L. CESARI [**2**] the following quantities:

$$v(T_r, \pi) = \iint | O(p_r; T_r, \pi) |, \quad u(T_r, \pi) = \iint O(p_r; T_r, \pi), \qquad (r = 1, 2, 3),$$

$$u(T, \pi) = [u(T_1, \pi)^2 + u(T_2, \pi)^2 + u(T_3, \pi)^2]^{1/2},$$

where the integration in the above expressions is performed over the plane E_{2r}. Let $V(T, A)$, $V(T_r, A)$ $(r = 1, 2, 3)$ be the GEÖCZE area of the mappings (T, A), (T_r, A) $(r = 1, 2, 3)$ (see [**2**; 9.1]). By [**2**; 24.1 (i)] we have the equality $L(T, A) = V(T, A)$.

III.3. – Let X be a compact subset of E_3, and let $F(x, y, z, u, v, w)$ be a function defined for each $(x, y, z) \in X$ and for each triple $(u, v, w) \neq (0, 0, 0)$ satisfying, moreover, the following conditions:

(1) $F(x, y, z, u, v, w)$ is continuous for each $(x, y, z) \in X$ and for each triple $(u, v, w) \neq (0, 0, 0)$.

(2) $F(x, y, z, u, v, w)$ is positively homogeneous of degree one with respect to u, v, w, i. e.,

$$F(x, y, z, ku, kv, kw) = k \cdot F(x, y, z, u, v, w)$$

for each $k > 0$.

If we put $F(x, y, z, 0, 0, 0) = 0$ for each $(x, y, z) \in X$ we have as a consequence of (2) that F is also continuous at each point $(x, y, z, 0, 0, 0)$ where $(x, y, z) \in X$.

Let (T, A) be a continuous mapping from an admissible set $A \subset E_2$ into E_3, and assume that $T(A) \subset X$ and $L(T, A) < \infty$. Let T_r $(r = 1, 2, 3)$ be the plane mappings introduced in **III.2.** For $[\pi_k \ (k = 1, \dots, n)]$ a finite system of non-overlapping polygonal regions in A let us consider the following non-negative indices with respect to (T, A):

$$m = \max_{r=1,2,3} \Big| \sum_{k=1}^{n} T_r(\pi_k^*) \Big|, \qquad \delta = \max_{k=1,\dots,n} \operatorname{diam}[T(\pi_k)],$$

$$\mu = \max \Big[V(T, A) - \sum_{k=1}^{n} u(T, \pi_k), \quad V(T_r, A) - \sum_{k=1}^{n} |u(T_r, \pi_k)| \quad (r = 1, 2, 3)\Big]$$

In view of the hypothesis $L(T, A) < \infty$, it is possible to determine for $\varepsilon > 0$ given, a system of non-overlapping polygonal regions $[\pi_k \ (k = 1, \dots, n)]$ in A with indices less than ε with respect to (T, A) (L. CESARI [**2**; 22.4 (i)]).

Select a point (u_k, v_k) from each π_k and consider the sum

$$\sum_{k=1}^{n} F[x(u_k, v_k), \ y(u_k, v_k), \ z(u_k, v_k), \ u(T_1, \pi_k), \ u(T_2, \pi_k), \ u(T_3, \pi_k)] .$$

L. CESARI [**4**, **2**] has shown that

$$\lim_{m,\delta,\mu \to 0} \ \sum_{k=1}^{n} F[\ \dots \ ] ,$$

exists and is finite, and we shall denote this limit by $J(T, A)$ or $\int_{(T, A)} F \, d\sigma$.

III.4. – In the following three paragraphs we will discuss three lemmas for $J(T, A)$, in nature similar to those for the LEBESGUE area mentioned in **III.1.**

Lemma. Let (T, A) be a continuous mapping from an admissible set $A \subset E_2$ into E_3 such that $L(T, A) < \infty$ and $T(A) \subset X$. Assume there is a finite number of disjoint admissible sets A_i $(i = 1, \dots, t)$ with $A_i \subset A$. If $L(T, A) = \sum_{i=1}^{t} L(T, A_i)$ and $L(T_r, A) = \sum_{i=1}^{t} L(T_r, A_i)$ $(r = 1, 2, 3)$, where T_r $(r = 1, 2, 3)$ are the plane mappings introduced in **III.2**, then $J(T, A) = \sum_{i=1}^{t} J(T, A_i)$.

Proof. For each i, let $\{S_n^i\}$ be a sequence defined as follows. For each i, S_n^i is a system of a finite number of non-overlapping polygonal regions $\pi \subset A_i$ with indices m_n^i, δ_n^i, μ_n^i with respect to (T, A_i) such that m_n^i, δ_n^i, $\mu_n^i \to 0$ as $n \to \infty$. For each n, let $S_n = \bigcup_{i=1}^{t} S_n^i$. Then S_n is a finite system of non-overlapping polygonal regions in A. In view of the hypothesis $L(T, A) = \sum_{i=1}^{t} L(T, A_i)$, $L(T_r, A) = \sum_{i=1}^{t} L(T_r, A_i)$ $(r = 1, 2, 3)$, and the fact that $L(T, A) = V(T, A)$, we conclude that the indices m_n, δ_n, μ_n of S_n with respect to (T, A) satisfy the following relation:

$$(1) \qquad m_n \leqslant \sum_{i=1}^{t} m_n^i, \qquad \delta_n = \max[\delta_n^i \quad (i = 1, \ldots, t)], \qquad \mu_n \leqslant \sum_{i=1}^{t} \mu_n^i .$$

Hence:

$$(2) \qquad m_n, \; \delta_n, \; \mu_n \to 0 \quad \text{as} \quad n \to \infty .$$

Select now a point in each polygonal region $\pi \in S_n$ and form

$$(3) \qquad \sum_{\pi \in S_n} F[\quad \ldots \quad] = \sum_{i=1}^{t} \sum_{\pi \in S_n^i} F[\quad \ldots \quad],$$

where the argument within the brackets is given by **III.3**. Letting $n \to \infty$ we finally infer

$$(4) \qquad \begin{cases} \sum_{\pi \in S_n} F[\quad \ldots \quad] \to J(T, A), \\ \sum_{\pi \in S_n^i} F[\quad \ldots \quad] \to J(T, A_i) \qquad (i = 1, \ldots, t), \end{cases}$$

and hence, in view of (3),

$$(5) \qquad J(T, A) = \sum_{i=1}^{t} J(T, A_i) .$$

III.5. – Lemma. Let (T, A) be a continuous mapping from an admissible set $A \subset E_2$ into E_3 such that $T(A) \subset X$ and $L(T, A) < \infty$. Assume there is a sequence of admissible sets A_j $(j = 1, 2, \ldots)$ with the property that $A_j \subset A_{j+1} \subset A$, $A_j^0 \uparrow A^0$. Then $J(T, A_j) \to J(T, A)$ as $j \to \infty$.

Proof. From **III.1** (i):

$$(1)\quad \begin{cases} L(T,\ A_j)\ \to L(T,\ A) \quad \text{as } j\to\infty\,, \\ L(T_r,\ A_j) \to L(T_r,\ A) \quad \text{as } j\to\infty \quad (r=1,\ 2,\ 3)\,, \\ L(T,\ A_j)\ \leqslant L(T,\ A) \quad (j=1,\ 2,\ldots)\,, \\ L(T_r,\ A_j) \leqslant L(T_r,\ A) \quad (r=1,\ 2,\ 3;\ j=1,\ 2,\ldots)\,. \end{cases}$$

For each j, choose a finite system S_j of non-overlapping simple polygonal regions $\pi\subset A_j$ with indices $m_j',\ \delta_j',\ \mu_j'$ with respect to $(T,\ A_j)$ subject to the following conditions:

$$(2)\qquad m_j',\ \delta_j',\ \mu_j'\to 0 \quad \text{as} \quad j\to\infty\,.$$

(3) If we select a point in each $\pi\in S_j$ $(j=1,\ 2,\ldots)$ and form the sum $\sum_j=\sum_{\pi\in S_j} F[\ \ldots\ ]$ as in **III.3**, then $|\,J(T,\ A_j)-\sum_j|\to 0$ as $j\to\infty$.

Now, for each j, S_j is also a finite system of non-overlapping polygonal regions in A. Hence there is associated with each S_j a set of indices $m_j,\ \delta_j,\ \mu_j$ with respect to $(T,\ A)$. We assert that

$$(4)\qquad m_j,\ \delta_j,\ \mu_j\to 0 \quad \text{as} \quad j\to\infty\,.$$

Since $m_j=m_j'$, $\delta_j=\delta_j'$, we have $m_j,\ \delta_j\to 0$ as $j\to\infty$. To prove that $\mu_j\to 0$, let $\varepsilon>0$ be given. From (1) we have an integer $I'>0$ such that

$$(5)\quad \begin{cases} 0\leqslant L(T,\ A)-L(T,\ A_j)=V(T,\ A)-V(T,\ A_j)<\varepsilon/2\,, \\ 0\leqslant L(T_r,\ A)-L(T_r,\ A_j)=V(T_r,\ A)-V(T_r,\ A_j)<\varepsilon/2 \quad (r=1,\ 2,\ 3) \end{cases}$$

for all $j>I'$. In view of (2) there is an integer $I''>0$ with the property that

$$(6)\qquad \mu_j'<\varepsilon/2 \qquad \text{for all} \qquad j>I''.$$

Let $I=\max\,[I',\ I'']$. Let j be any integer greater than I. By definition (see **III.3**) we have:

$$\mu_j'=\max\,[C_j,\ C_{jr}\ (r=1,\ 2,\ 3)], \qquad\qquad \mu_j=\max\,[C,\ C_r\ \ (r=1,\ 2,\ 3)]\,,$$

where $C_j = V(T, A_j) - \sum_{\pi \in S_j} u(T, \pi)$, and similar identifications for C_{jr} $(r = 1, 2, 3)$, C, and C_r $(r = 1, 2, 3)$.

Since $C \geqslant C_j$, $C_r \geqslant C_{jr}$ $(r = 1, 2, 3)$ we have $\mu_j \geqslant \mu_j'$. Suppose now that

$$\mu_j - \mu_j' \geqslant \varepsilon/2. \tag{7}$$

The cases where $\mu_j = C$, $\mu_j' = C_j$; $\mu_j = C_r$, $\mu_j' = C_{jr}$, $(r = 1, 2, 3)$ are excluded, since in each case by (5) $\mu_j - \mu_j' < \varepsilon/2$. Let us define τ_j as follows. If $\mu_j = C$, let $\tau_j = C_j$; if $\mu_j = C_r$, let $\tau_j = C_{jr}$ $(r = 1, 2, 3)$. Then from (5)

$$0 \leqslant \mu_j - \tau_j < \varepsilon/2 . \tag{8}$$

Hence subtracting (8) from (7), $\mu_j - \mu_j' - \mu_j + \tau_j > 0$ or $\tau_j > \mu_j'$, a contradiction. Hence $\mu_j - \mu_j' < \varepsilon/2$, and since $\mu_j' < \varepsilon/2$ from (6), we have $\mu_j < \varepsilon$ for all $j > I$. Since $\varepsilon > 0$ was arbitrary, (4) follows.

From the definition (see **III.3**), we have $\sum_j \to J(T, A)$ as $j \to \infty$. From (3) we finally infer $J(T, A_j) \to J(T, A)$ as $j \to \infty$. This completes the proof of the lemma.

III.6. - Lemma: There is a constant $M > 0$ such that $|J(T, A)| \leqslant \leqslant M \cdot L(T, A)$, where T is any continuous mapping from an admissible set $A \subset E_2$ into E_3 for which $L(T, A) < \infty$ and $T(A) \subset X$.

Proof. Since $F(x, y, z, u, v, w)$ is continuous on X and since X is compact, we have a constant $M > 0$ such that

(1) $|F(x, y, z, u, v, w)| < M$ for $(x, y, z) \in X$ and for all (u, v, w) for which $u^2 + v^2 + w^2 = 1$. Let S_n be a finite system of non-overlapping simple polygonal regions $\pi \subset A$ with indices m_n, δ_n, μ_n with respect to (T, A). In view of L. Cesari [**2**; 21.3 (i), 22.4 (i)] there is a sequence $\{S_n\}$ such that

$$m_n, \ \delta_n, \ \mu_n \to 0 \quad \text{as} \quad n \to \infty , \tag{2}$$

$$\lim_{n \to \infty} \sum_{\pi \in S_n} u(T, \pi) = L(T, A) . \tag{3}$$

Let us select a point (u, v) from each $\pi \in S_n$ and consider the sum (see **III.3**)

$$\sum_{\pi \in S_n} F[x(u, v), y(u, v), z(u, v), u_1, u_2, u_3] = T_n , \tag{4}$$

where $u_r = u(T_r, \pi)$ $(r = 1, 2, 3)$. We may assume that $(u_1, u_2, u_3) \neq (0, 0, 0)$. In view of the homogeneity of F, (4) becomes

$$(5) \qquad T_n = \sum_{\pi \in S_n} u(T, \pi) \cdot F[x(u, v), y(u, v), z(u, v), a_1, a_2, a_3] ,$$

where $a_r = u_r/u$ $(r = 1, 2, 3)$. Since $a_1^2 + a_2^2 + a_3^2 = 1$, we infer from (1)

$$(6) \qquad | T_n | \leqslant M \cdot \sum_{\pi \in S_n} u(T, \pi) .$$

From (3) and the definition of $J(T, A)$, $| J(T, A) | \leqslant M \cdot L(T, A)$.

III.7. – Let (T, A) be a continuous mapping from an admissible set $A \subset E_2$ into E_3 such that $L(T, A) < \infty$ and $T(A) \subset X$. The approach followed to show that $J(T, A)$ is cyclicly additive is similar to the approach used in **II.** However, as already noted in the introduction, the functional $J(T, A)$ need not be non-negative and hence one cannot directly apply the result in **I.15.** The cyclic additivity of $L(T, A)$ (see **II.12**) in conjunction with the lemmas **III.4, 5, 6** will lead to a cyclic additivity formula of $J(T, A)$.

III.8. – Let T be a continuous mapping from a PEANO space P into a metric space P^*, written $T : P \to P^*$.

Definition: An unrestricted factorization of T consists of a PEANO space $\mathfrak{M}$ and two continuous mappings s, f such that $f : P \to \mathfrak{M}$, $s : \mathfrak{M} \to P^*$, $T = sf$.

III.9. – In accordance with the observation made in **III.7,** we proceed to establish the following theorem.

Theorem. *Let (T, R) be a continuous mapping from a finitely connected Jordan region $R \subset E_2$ into E_3 such that $L(T, R) < \infty$ and $T(R) \subset X$. Let $(T, R) = sf$, $f : R \to \mathfrak{M}$, $s : \mathfrak{M} \to E_3$ be an unrestricted factorization of (T, R). If for C a proper cyclic element of $\mathfrak{M}$, we denote by r_C the monotone retraction from $\mathfrak{M}$ onto C, then*

$$(1) \qquad J(T, R) = \sum{}' J(s\, r_C f, R) ,$$

where $\sum'$ denotes the summation extended over all $C \subset \mathfrak{M}$ for which $C \cap f(R) \neq 0$.

Proof. From **II.8**, we have

$$L(T,\ R) = \sum L(s\, r_C f,\ R), \qquad C \subset \mathfrak{M} . \tag{2}$$

With this additional formula, the proof of (1) can be carried out by an entirely analogous procedure as in [**9**].

III.10. – The definition of an unrestricted factorization of a mapping given in **III.8** does not apply to continuous mappings $(T,\ A)$ from an admissible set $A \subset E_2$ into E_3, since A need not be a PEANO space. Hence we use the definition already stated in **I.9** and **II.10**.

Definition: An unrestricted factorization of a mapping $(T,\ A)$ as above consists of a PEANO space $\mathfrak{M}$, a subset $\mathfrak{M}^*$ of $\mathfrak{M}$, and two continuous mappings s, f such that

$$f : A \to \mathfrak{M}^* , \tag{1}$$

$$s : \mathfrak{M}^* \to E_3 , \tag{2}$$

$$T = sf . \tag{3}$$

We shall write $(T,\ A) = sf, \quad f : A \to \mathfrak{M}^*, \quad s : \mathfrak{M}^* \to E_3, \quad \mathfrak{M}^* \subset \mathfrak{M}$.

The remarks in **I.9** as well as the Remark in **II.11** should be observed. In particular, as we have seen in **II.11**, there is always a *trivial* unrestricted factorization of a mapping $(T,\ A)$.

III.11. – The following lemma corresponds to the lemma in **I.10** and is proved by the same method.

Lemma. Let $(T,\ R)$ be a continuous mapping from a finitely connected JORDAN region R into E_3 such that $L(T,\ R) < \infty$ and $T(R) \subset X$. Let $(T,\ R) = sf$, $f : R \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$, be an unrestricted factorization of $(T,\ R)$. Then, if $s(\mathfrak{M}^*) \subset X$, we have

$$J(T,\ R) = \sum{}^* J(s\, r_C f,\ R) , \tag{1}$$

where $\sum^*$ denotes the summation over all proper cyclic elements C of $\mathfrak{M}$ for which $r_C \cdot f(R) \subset \mathfrak{M}^*$.

Proof. In view of **III.9**, the proof is entirely analogous to the proof in **I.10**. However, some care should be taken concerning an interchange in the order of summation which was justified in **I.10** since the functional Φ considered there is non-negative. Since the surface integral $J(T, A)$ need not be non-negative, the following observation is in order.

Remark. Let (T_i, A) $(i = 0, 1, 2, ...)$ be a sequence of continuous mappings from an admissible set $A \subset E_2$ into E_3 such that for each i, $L(T_i, A) < \infty$ and $T_i(A) \subset X$. Assume we have the following additivity formulas

$$J(T_0, A) = \sum_{i=1}^{\infty} J(T_i, A), \qquad L(T_0, A) = \sum_{i=1}^{\infty} L(T_i, A).$$

Then the series $\sum_{i=1}^{\infty} J(T_i, A)$ converges absolutely. This is a simple consequence of **III.6**; namely, by **III.6**, there is a constant $M > 0$ such that, for each i, $|J(T_i, A)| \leqslant M \cdot L(T_i, A)$. Hence

$$\sum_{i=1}^{\infty} |J(T_i, A)| \leqslant M \cdot \sum_{i=1}^{\infty} L(T_i, A) = M \cdot L(T_0, A) < \infty.$$

III.12. – Let A be an admissible set in E_2, and let G be a component of A. Then $G \cap A^0 \neq 0$. This follows readily from **III.1**. If G is a component of A, then G is open in A, and hence G is admissible. But then $G^0 \neq 0$, and since $G^0 \subset A^0$, the result follows.

Let (T, A) be a continuous mapping from an admissible set $A \subset E_2$ into E_3, and let $(T, A) = sf$, $f : A \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$, be an unrestricted factorization of (T, A) (see **III.10**). From **I.11** we have that for a component G of A either $r_C \cdot f(G)$ is disjoint with $\mathfrak{M}^*$ or else lies entirely in $\mathfrak{M}^*$, where r_C is the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element C of $\mathfrak{M}$.

According to **I.12**, we introduce the following terminology. Let $\mathfrak{K}$ be the class of proper cyclic elements C of $\mathfrak{M}$ for which there exists at least one component G of A such that $r_C \cdot f(G) \subset \mathfrak{M}^*$. For each $C \in \mathfrak{K}$, we denote by G_C the union of all components G of A satisfying $r_C \cdot f(G) \subset \mathfrak{M}^*$. Since G_C is open in A, G_C is an admissible set.

We shall term $\mathfrak{K}$ the class of proper cyclic elements associated with $(T, A) = sf$, and we shall term G_C the set associated with $C \in \mathfrak{K}$.

In **I.13** a series of lemmas were proved concerning the class $\mathfrak{K}$ and the set G_C. For convenient reference we restate the results.

Lemma 1. Let A' be an admissible subset of A. Then (T, A') admits of an unrestricted factorization of the form $(T, A') = sf$, $f: A' \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$. Let $\mathcal{K}'$ be the class of proper cyclic elements associated with $(T, A') = sf$. Then a set G'_C is associated with $C \in \mathcal{K}'$ if and only if G'_C is of the form $G_C \cap A'$.

Lemma 2. Let A_i $(i = 1, 2, \ldots)$ be admissible subsets of A such that $\bigcup_{i \geq 1} A_i^0 = A^0$. For each i, (T, A_i) admits of an unrestricted factorization $(T, A_i) = sf$, $f: A_i \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$. Let $\mathcal{K}_i$ be the class of proper cyclic elements associated with $(T, A_i) = sf$. Then $\bigcup_{i \geq 1} \mathcal{K}_i = \mathcal{K}$. Moreover, if $A_i \subset A_{i+1}$ $(i = 1, 2, \ldots)$, then $\mathcal{K}_i \subset \mathcal{K}_{i+1}$.

III.13. – Let (T, A) be a continuous mapping from an admissible set $A \subset E_2$ into E_3 such that $T(A) \subset X$ and $L(T, A) < \infty$. Let $(T, A) = sf$, $f: A \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$ be an unrestricted factorization of (T, A) (see **III.10**). Then, if we denote by s' the mapping s restricted to $\mathfrak{M}' = f(A)$, we have $(T, A) = s'f$, $f: A \to \mathfrak{M}'$, $s': \mathfrak{M}' \to E_3$, $\mathfrak{M}' \subset \mathfrak{M}$ and $s'(\mathfrak{M}') \subset X$. We can therefore assume in the sequel that the original unrestricted factorization satisfies the property $s(\mathfrak{M}^*) \subset X$.

III.14. – The following lemma is similar to the lemma proved in **I.14.**

Lemma. Let A be a finite union of disjoint finitely connected JORDAN regions $R_1, \ldots, R_n$, and let $(T, A) = sf$, $f: A \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$ be an unrestricted factorization of (T, A). If for C a proper cyclic element of $\mathfrak{M}$, r_C denotes the monotone retraction from $\mathfrak{M}$ onto C, then

$$(1) \qquad J(T, A) = \sum J(s\, r_C f, G_C), \qquad C \in \mathcal{K},$$

where $\mathcal{K}$ is the class of proper cyclic elements associated with $(T, A) = sf$, and where G_C is the set associated with $C \in \mathcal{K}$.

Proof. We first assume that $\mathcal{K} \neq 0$. For each i, the mapping (T, R_i) admits of an unrestricted factorization $(T, R_i) = sf$, $f: R_i \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$. If we denote by $\mathcal{K}_i$ the class of proper cyclic elements associated with $(T, R_i) = sf$, then by **III.12** (Lemma 2), $\bigcup_{i=1}^{n} \mathcal{K}_i = \mathcal{K}$. For each $C \in \mathcal{K}$, let $n(C)$ be the integers among $i = 1, \ldots, n$ for which $C \in \mathcal{K}_i$. If we set for $C \in \mathcal{K}_i$, $G_C^i = R_i \cap G_C$, then by **III.12** (Lemma 1), G_C^i is the set associated with $C \in \mathcal{K}_i$. Since R_i is connected, $G_C^i = R_i$, and since

$G_C = \bigcup_{i \in n(C)} G_C^i$, G_C is a finite union of disjoint finitely connected JORDAN regions. In view of L. CESARI [**2**; 5.14 (ii)] and by **III.4**,

$$(2)\qquad \begin{cases} J(T,\ A) = \sum_{i=1}^{n} J(T,\ R_i)\,, \\ J(s\,r_C f,\ G_C) = \sum_{i \in n(C)} J(s\,r_C f,\ G_C^i)\,. \end{cases}$$

From **III.11** we have now for each i, $1 \leqslant i \leqslant n$,

$$(3)\qquad J(T,\ R_i) = \sum\nolimits^* J(s\,r_C f,\ R_i),$$

where $\sum^*$ denotes the summation over all proper cyclic elements C of $\mathfrak{M}$ for which $r_C \cdot f(R_i) \subset \mathfrak{M}^*$. Using the terminology introduced in **III.12**, (3), becomes

$$(4)\qquad J(T,\ R) = \sum J(s\,r_C f,\ G_C^i), \qquad C \in \mathcal{K}_i\,.$$

From (4) and (2) we obtain now:

$$(5)\qquad J(T,\ A) = \sum_{i=1}^{n} \sum_{C \in \mathcal{K}_i} J(s\,r_C f,\ G_C^i)\,.$$

Since, for a given $C \in \mathcal{K}$, $C \in \mathcal{K}_i$ if and only if $i \in n(C)$, we can rewrite (5) in the form (see also the Remark in **III.11**):

$$(6)\qquad J(T,\ A) = \sum_{C \in \mathcal{K}} \sum_{i \in n(C)} J(s\,r_C f,\ G_C^i)\,.$$

From (2) we infer (1).

The above proof was carried out under the assumption that $\mathcal{K} \neq 0$. If $\mathcal{K} = 0$, then it follows from (3) that $J(T, R_i) = 0$ $(i = 1, \ldots, n)$ and from (2) that $J(T,\ A) = 0$. This completes the proof of the Lemma.

III.15. – We are now ready to state and prove our main result.

Theorem. *Let $(T,\ A)$ be a continuous mapping from an admissible set $A \subset E_2$ into E_3 such that $L(T,\ A) < \infty$ and $T(A) \subset X$* (see **III.3**). *Let $(T,\ A) = sf$, $f : A \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$ be an unrestricted factorization of $(T,\ A)$ for which $s(\mathfrak{M}^*) \subset X$* (see **III.13**). *If for C a proper cyclic element of $\mathfrak{M}$,* *we denote by r_C the monotone retraction from $\mathfrak{M}$ onto C, then we have the following cyclic additivity formula* :

$$J(T,\ A) = \sum J(s\, r_C f,\ G_C), \qquad C \in \mathcal{K}, \tag{1}$$

where $\mathcal{K}$ is the class of proper cyclic elements associated with $(T,\ A) = sf$, and where G_C is the set associated with $C \in \mathcal{K}$ (see **III.12**).

Proof. We first assume that $\mathcal{K} \neq 0$. Let F_n $(n = 1, 2, \ldots)$ be a sequence of figures (**III.1**) such that $F_n \subset F_{n+1} \subset A$, $F_n^0 \uparrow A^0$ (see L. CESARI [**2**; 5.6]). The mapping $(T,\ F_n)$ admits of an unrestricted factorization $(T,\ F_n) = sf$, $f : F_n \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to E_3$, $\mathfrak{M}^* \subset \mathfrak{M}$. Let $\mathcal{K}_n$ be the class of proper cyclic elements associated with $(T,\ F_n) = sf$. Then by **III.12** (Lemma 2), $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ $(n = 1, 2, \ldots)$ and $\bigcup_{n \geq 1} \mathcal{K}_n = \mathcal{K}$. Hence for each $C \in \mathcal{K}$, there is an integer $N(C) > 0$ such that $C \in \mathcal{K}_n$, $n > N(C)$. By **III.12** (Lemma 1), $G_C^n = G_C \cap F_n$ is the set associated with $C \in \mathcal{K}_n$, $n > N(C)$. The sequence G_C^n, $n > N(C)$, satisfies the property that $G_C^n \subset G_C^{n+1} \subset G_C$, $G_C^{n^0} \uparrow G_C^0$. Hence from **III.5** we have the following relations:

$$\lim_{n\to\infty} J(T,\ F_n) = J(T,\ A)\,, \tag{2}$$

$$\lim_{n\to\infty} J(s\, r_C f,\ G_C^n) = J(s\, r_C f,\ G_C), \qquad n > N(C) \text{ for each } C \in \mathcal{K}. \tag{3}$$

Let us first assume that there is an infinite number of proper cyclic elements $C_1, \ldots, C_j, \ldots$ in $\mathcal{K}$. From **II.12**,

$$L(T,\ A) = \sum_{i=1}^{\infty} L(s\, r_{C_i} f,\ G_{C_i}) < \infty\,. \tag{4}$$

Hence, for $\varepsilon > 0$ given, there is an integer $I > 0$ such that

$$\sum_{i \geq j} L(s\, r_{C_i} f,\ G_{C_i}) < \varepsilon/(4M), \qquad \text{for all } j > I\,, \tag{5}$$

where M is the constant of **III.6**. Let us fix $j_0 > I$.

There exists now an integer $N > 0$ such that

$$(6)\quad \begin{cases} |J(T, A) - J(T, F_n)| < \varepsilon/4 & (n > N), \\ C_i \in \mathcal{K}_n & (i = 1, \dots, j_0;\ n > N), \\ |\sum_{i=1}^{j_0} J(s\, r_{C_i} f, G^n_{C_i}) - \sum_{i=1}^{j_0} J(s\, r_{C_i} f, G_{C_i})| \leqslant \\ \qquad \leqslant \sum_{i=1}^{j_0} | J(s\, r_{C_i} f, G^n_{C_i}) - J(s\, r_{C_i} f, G_{C_i}) | < \varepsilon/4 & (n > N). \end{cases}$$

For $n > N$, let $j_0(n)$ be the integers i greater than j_0 for which $C_i \in \mathcal{K}_n$. Then from (5):

$$(7)\quad \sum_{i \in j_0(n)} L(s\, r_{C_i} f, G^n_{C_i}) \leqslant \sum_{i \geq j_0} L(s\, r_{C_i} f, G_{C_i}) < \varepsilon/(4M).$$

Now, from (6), (7), **III.6**, and from **III.14**, for $n > N$,

$$|J(T, A) - \sum_{i=1}^{\infty} J(s\, r_{C_i} f, G_{C_i})| \leqslant | J(T, A) - J(T, F_n)| +$$

$$+ | J(T, F_n) - \sum_{C \in \mathcal{K}_n} J(s\, r_C f, G^n_C)| + |\sum_{i=1}^{j_0} J(s\, r_{C_i} f, G^n_{C_i}) - \sum_{i=1}^{j_0} J(s\, r_{C_i} f, G_{C_i})| +$$

$$+ \sum_{i \in j_0(n)} | J(s\, r_{C_i} f, G^n_{C_i})| + \sum_{i > j_0} | J(s\, r_{C_i} f, G_{C_i})| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $J(T, A) = \sum J(s\, r_C f, G_C)$, $C \in \mathcal{K}$.

Now let us assume that the number of proper cyclic elements in $\mathcal{K}$ is finite. Then, as above, $J(T, F_n) = \sum J(s\, r_C f, G^n_C)$, $C \in \mathcal{K}_n$, and for n large we have $\mathcal{K}_n = \mathcal{K}$. From (3), $J(T, A) = \sum J(s\, r_C f, G_C)$, $C \in \mathcal{K}$. Similarly, if $\mathcal{K} = 0$, $J(T, F_n) = 0$ for all n, and hence $J(T, A) = 0$. This completes the proof.

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