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Some Theorems Connected with Generalised Hankel-Transform. (**)

1. – AGARWAL [1] gave a generalisation of the well-known HANKEL-transform, viz.,

$$f(x) = \int_0^\infty \sqrt{xy} \cdot J_\nu(xy) \cdot g(y) \, dy \tag{1}$$

by means of the integral equation

$$f(x) = (1/2)^\lambda \int_0^\infty (xy)^{\lambda+(1/2)} \cdot J_\lambda^\mu(x^2y^2/4) \cdot g(y) \, dy, \tag{2}$$

where $J_\lambda^\mu(x)$ is BESSEL-MAITLAND function defined by the series

$$J_\lambda^\mu(x) = \sum_{r=0}^\infty \frac{(-x)^r}{r! \, \Gamma(1+\lambda+\mu r)} \qquad (\mu > 0). \tag{3}$$

For $\mu = 1$ **1**(2) reduces to **1**(1).

He gave the inversion formula (AGARWAL [1]) and some theorems (AGARWAL [2]) for this generalised transform. Writing **1**(2) in the convenient form

$$f(x) = \int_0^\infty (xy)^\nu \cdot J_\lambda^\mu(xy) \cdot g(y) \, dy \tag{4}$$

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we have (AGARWAL [1]) that, under suitable conditions,

$$(5)\qquad g(y)=(1/\mu)\int_0^\infty (xy)^{[(\lambda+1)/\mu]-\nu-1}\cdot J^{1/\mu}_{[(\lambda+1)/\mu]-1}(xy)^{1/\mu}\cdot f(x)\,dx\,.$$

The object of this paper is to give two theorems connected with this generalised transform **1**(4). The importance of these theorems lies in their application to evaluate integrals involving BESSEL-MAITLAND function easily.

2. – Theorem 1. *If*

$$f(x)=\int_0^\infty (xy)^\nu\cdot J^\mu_\lambda(xy)\cdot g(y)\,dy$$

and

$$y^{-[(\lambda+1)/\mu]+\nu+1}\cdot g(y)=\int_0^\infty t^n\,e^{-(yt)^{1/\mu}}\,\psi(t)\,dt,$$

then

$$f(x)=\mu x^\nu\int_0^\infty t^{n-[(\lambda+1)/\mu]}\,e^{-x/t}\,\psi(t)\,dt,$$

provided $\Re(\lambda)>-1$ *and the integrals involved converge absolutely.*

Proof. We have

$$(1)\qquad f(x)=\int_0^\infty (xy)^\nu\cdot J^\mu_\lambda(xy)\cdot g(y)\,dy=$$

$$=\int_0^\infty (xy)^\nu\cdot J^\mu_\lambda(xy)\cdot y^{[(\lambda+1)/\mu]-\nu-1}\int_0^\infty t^n\,e^{-(yt)^{1/\mu}}\cdot\psi(t)\,dt\;dy=$$

$$=x^\nu\int_0^\infty t^n\cdot\psi(t)\,dt\int_0^\infty y^{[(\lambda+1)/\mu]-1}e^{-(yt)^{1/\mu}}\cdot J^\mu_\lambda(xy)\,dy=$$

$$=x^\nu\cdot\mu\int_0^\infty t^{-[(\lambda+1)/\mu]+n}\,e^{-x/t}\cdot\psi(t)\,dt$$

on expanding $J^\mu_\lambda(xy)$ in the form of the infinite series and integrating term by term, a process easily justifiable.

It only remains to justify the change of the order of integration in **2**(1). Now, we know that (WRIGHT [**3**])

(i) as $|x| \to \infty$,

$$J_\lambda^\mu(x) = O[x^{-k\cdot[\lambda+(1/2)]}\cdot\exp\{\cos(\pi k)\cdot(\mu x)^k/(\mu k)\}\qquad (k = 1/(1+\mu)),$$

and

(ii) as $x \to 0$,

$$J_\lambda^\mu(x) = O(1)\,.$$

Therefore, both the t- and y- integrals in **2**(1) converge absolutely and the double integral exists under the conditions stated and hence by virtue of DE LA VALLÉE POUSSIN's Theorem the change in the order of integration is justified.

Hence the proposition.

3. – Evaluation of Integrals.

The theorem established above can be written in the following form which is more useful for practical applications:

If

$$f(x) = \int_0^\infty (xy)^\nu\cdot J_\lambda^\mu(xy)\cdot g(y)\,\mathrm{d}y \tag{1}$$

and

$$y^{-(\lambda+1)+\mu\cdot(\nu+1)}\cdot g(y^\mu) = \mu\int_0^\infty t^{(n+1)\mu-1}\,e^{-yt}\cdot\psi(t^\mu)\,\mathrm{d}t, \tag{2}$$

then

$$f(x) = \mu x^\nu\int_0^\infty t^{[(\lambda+1)/\mu]-n-2}\,e^{-xt}\cdot\psi(1/t)\,\mathrm{d}t, \tag{3}$$

provided $\Re(\lambda) > -1$ *and the integrals involved converge absolutely.*

From **3**(2) and **3**(3), it is evident that knowing the operational forms of $t^{(n+1)\mu-1}\cdot\psi(t^\mu)$ and $t^{[(\lambda+1)/\mu]-n-2}\cdot\psi(1/t)$, we can find $g(y)$ and $f(x)$ and hence the value of **3**(1).

We shall now evaluate some integrals by the help of the above theorem.

Example 1. Let us take $\psi(t) = e^{-t}$ and $\mu = 1/2$; then, from **3**(2),

$$g(\sqrt{y}) = (1/2)y^{\lambda-(\nu/2)+(1/2)} \int_0^\infty e^{-yt}\, t^{n/2}\, (e^{-\sqrt{t}}/\sqrt{t})\, \mathrm{d}t$$

$$= \frac{\sqrt{\pi}}{2}(-1)^{n/2}\, y^{\lambda-(\nu/2)+(1/2)} \frac{\mathrm{d}^{n/2}}{\mathrm{d}y^{n/2}} \left[y^{-1/2} \cdot \left\{ 1 - \Phi\left(\frac{1}{2\sqrt{y}}\right) \right\} e^{1/(4y)} \right],$$

where n is any positive even integer (including zero) and $\Phi(x)$ is the error function [1], and from **3**(3)

$$f(x) = (1/2)x^{\nu} \int_0^\infty t^{2\lambda-n}\, e^{-xt}\, e^{-1/t}\, dt = x^{\nu-\lambda+(n/2)-(1/2)} \cdot K_{-2\lambda+n-1}(2\sqrt{x}),$$

provided $\Re(2\lambda - n) > -1$ (M. 128). Therefore, from **3**(1),

$$\int_0^\infty y^{2\lambda+1} \cdot J_\lambda^{1/2}(xy) \cdot \left(\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\right)^{n/2} \left[y^{-1} \cdot \left\{ 1 - \Phi\left(\frac{1}{2y}\right) \right\} e^{1/(4y^2)} \right] \mathrm{d}y =$$

$$= (-1)^{n/2} \frac{2^{(n/2)+1}}{\sqrt{\pi}}\, x^{(n/2)-\lambda-(1/2)} \cdot K_{-2\lambda+n-1}(2\sqrt{x}),$$

provided $\Re(\lambda) > -1$ and n is any even positive integer (including zero).

Example 2. Let us take $\psi(t) = \Phi(1/t)$ and $\mu = 1/2$; then, from **3**(3),

$$f(x) = (1/2)x^{\nu} \int_0^\infty t^{2\lambda-n}\, \Phi(t) \cdot e^{-xt}\, \mathrm{d}t = \frac{(-1)^{2\lambda-n}}{2}\, x^{\nu} \frac{\mathrm{d}^{2\lambda-n}}{\mathrm{d}x^{2\lambda-n}} \left[\frac{e^{x^2/4}}{x} \left\{ 1 - \Phi\left(\frac{x}{2}\right) \right\} \right]$$

(M. 128), where $2\lambda - n$ is any positive integer (including zero), and from **3**(2)

$$g(\sqrt{y}) = \frac{1}{2}\, y^{\lambda-(\nu/2)+(1/2)} \int_0^\infty e^{-yt}\, t^{(n-1)/2} \cdot \Phi(1/\sqrt{t})\, \mathrm{d}t =$$

$$= \frac{1}{2}\, y^{\lambda-(\nu/2)+(1/2)} (-1)^{(n-1)/2} \frac{\mathrm{d}^{(n-1)/2}}{\mathrm{d}y^{(n-1)/2}} \{ (1 - e^{-2\sqrt{y}})/y \}$$

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[1] Cf.: W. MAGNUS and F. OBERHETTINGER, **Formulas and theorems for the special functions of mathematical physics** (translated by J. WERMER), Chelsea Publishing Company, New York 1949; cf. p. 128.

It shall be hereafter referred to as (M. 128).

(M. 128), n being any positive odd integer. Therefore, from **3**(1),

$$\int_0^\infty y^{2\lambda+1}\cdot J_\lambda^{1/2}(xy)\cdot\left(\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\right)^{(n-1)/2}\{(1-e^{-2y})/y^2\}\,\mathrm{d}y =$$

$$= 2^{(n-1)/2}(-1)^{2\lambda-(3n-1)/2}\frac{\mathrm{d}^{2\lambda-n}}{\mathrm{d}x^{2\lambda-n}}\left[e^{x^2/4}\left\{1-\Phi\left(\frac{x}{2}\right)\right\}/x\right].$$

Example 3. Let us take $\psi(t) = 1/(1+t^2)$ and $\mu = 1/2$, then, from **3**(3),

$$f(x) = (1/2)x^\nu\int_0^\infty t^{2\lambda-n+2}\{e^{-xt}/(1+t)^2\}\,\mathrm{d}t =$$

$$= \frac{(-1)^{2\lambda-n+2}}{2}\,x^\nu\,\frac{\mathrm{d}^{2\lambda-n+2}}{\mathrm{d}x^{2\lambda-n+2}}(\sin x\cdot\mathrm{Ci}\,x-\cos x\cdot\mathrm{si}\,x),$$

where $2\lambda-n+2$ is any positive integer (including zero) (M. 124) and $\mathrm{Ci}\,x$ and $\mathrm{si}\,x$ are the cosine- and sine-integrals (M. 97). Also, from **3**(2),

$$g(\sqrt{y}) = (1/2)y^{\lambda-(\nu/2)+(1/2)}\int_0^\infty e^{-yt}\{t^{(n-1)/2}/(1+t)\}\,\mathrm{d}t =$$

$$= (1/2)\cdot\Gamma((n+1)/2)\cdot y^{\lambda-(\nu/2)-(n-1)/4}\,e^{y/2}\cdot W_{-(n+1)/4,(n-1)/4}(y),$$

provided $\mathfrak{R}(n) > -1$ (M. 124). Therefore, from **3**(1), we get

$$\int_0^\infty y^{2\lambda-(n-1)/2}\cdot J_\lambda^{1/2}(xy)\cdot e^{y^2/2}\cdot W_{-(n+1)/4,(n-1)/4}(y^2)\,\mathrm{d}y =$$

$$= \frac{(-1)^{2\lambda-n+2}}{\Gamma((n+1)/2)}\frac{\mathrm{d}^{2\lambda-n+2}}{\mathrm{d}x^{2\lambda-n+2}}(\sin x\cdot\mathrm{Ci}\,x-\cos x\cdot\mathrm{si}\,x)$$

provided $\mathfrak{R}(\lambda) > -1$ and $2\lambda-n+2$ is any positive integer including zero. In particular,

(i) when $2\lambda-n+2=0$, i.e., $n=2\lambda+2$, we get

$$\int_0^\infty y^{\lambda-(1/2)}\cdot J_\lambda^{1/2}(xy)\cdot e^{y^2/2}\cdot W_{-(2\lambda+3)/4,(2\lambda+1)/4}(y^2)\,\mathrm{d}y =$$

$$= (\sin x\cdot\mathrm{Ci}\,x-\cos x\cdot\mathrm{si}\,x)/\Gamma(\lambda+(3/2))$$

provided $\Re(\lambda) > -1$, and

(ii) when $2\lambda - n + 2 = 1$, i.e., $n = 2\lambda + 1$, we get

$$\int_0^\infty y^\lambda \cdot J_\lambda^{1/2}(xy) \cdot e^{y^2/2} e \cdot W_{-(\lambda+1)/2, \lambda/2}(y^2/2)\, dy = -(\cos x \cdot \mathrm{Ci}\, x + \sin x \cdot \mathrm{si}\, x)/\Gamma(\lambda+1),$$

provided $\Re(\lambda) > -1$.

Example 4. Let us take $\psi(t) = \cos t$ and $\mu = 1/2$, then, from **3**(3),

$$f(x) = \frac{x^\nu}{2} \int_0^\infty e^{-xt}\, t^{2\lambda - n + (1/2)} \frac{\cos(1/t)}{\sqrt{t}}\, dt =$$

$$= \frac{\sqrt{\pi}}{2} (-1)^{2\lambda - n + (1/2)} x^\nu \frac{d^{2\lambda - n + (1/2)}}{dx^{2\lambda - n + (1/2)}} \left(\frac{1}{\sqrt{x}} e^{-\sqrt{2x}} \cdot \cos \sqrt{2x} \right)$$

(M. 127), provided $2\lambda - n + (1/2)$ is zero or any positive integer. Also, from **3**(2),

$$g(\sqrt{y}) = \frac{1}{2} y^{\lambda - (\nu/2) + (1/2)} \int_0^\infty e^{-yt}\, t^{n/2} \frac{\cos \sqrt{t}}{\sqrt{t}}\, dt =$$

$$= \frac{\sqrt{\pi}}{2} (-1)^{n/2} y^{\lambda - (\nu/2) + (1/2)} \frac{d^{n/2}}{dy^{n/2}} (y^{-1/2} e^{-1/(4y)}),$$

where n is any positive even integer including zero (M. 127),

$$= \frac{\pi}{2\sqrt{2}} y^{\lambda - (\nu/2)} e^{-1/(8y)} \cdot I_{-1/4}(1/(8y))$$

when $n = -1/2$ (M. 126). Therefore, from **3**(1),

$$\int_0^\infty y^{2\lambda+1} \cdot J_\lambda^{1/2}(xy) \cdot \left(\frac{1}{y} \frac{d}{dy} \right)^{n/2} [e^{-1/(4y^2)}/y]\, dy =$$

$$= (-1)^{2\lambda - (n-1)/2}\, 2^{n/2} \frac{d^{2\lambda - n + (1/2)}}{dx^{2\lambda - n + (1/2)}} \left(\frac{1}{\sqrt{x}} e^{-\sqrt{2x}} \cdot \cos \sqrt{2x} \right),$$

provided $2\lambda - n + (1/2)$ is zero or a positive integer and n is zero or an even positive integer, and

$$\int_0^\infty y^{2\lambda} \cdot J_\lambda^{1/2}(xy) \cdot e^{-1/(8y^2)} \cdot I_{-1/4}(1/(8y^2))\, \mathrm{d}y = \sqrt{\frac{2}{\pi}}\,(-1)^{2\lambda+1} \frac{\mathrm{d}^{2\lambda+1}}{\mathrm{d}x^{2\lambda+1}} \left(\frac{1}{\sqrt{x}}\, e^{-\sqrt{2x}} \cdot \cos\sqrt{2x}\right),$$

provided $2\lambda + 1$ is zero or a positive integer.

Example 5. Let us take $\psi(t) = \sin t$ and $\mu = 1/2$, then, from **3**(3),

$$f(x) = \frac{x^\nu}{2} \int_0^\infty e^{-xt}\, t^{2\lambda - n + (1/2)} \frac{\sin(1/t)}{\sqrt{t}}\, \mathrm{d}t =$$

$$= \frac{\sqrt{\pi}}{2} (-1)^{2\lambda - n + (1/2)} x^\nu \frac{\mathrm{d}^{2\lambda - n + (1/2)}}{\mathrm{d}x^{2\lambda - n + (1/2)}} \left(\frac{1}{\sqrt{x}}\, e^{-\sqrt{2x}} \cdot \sin\sqrt{2x}\right),$$

provided $2\lambda - n + (1/2)$ is zero or a positive integer (M. 127). Also, from **3**(2),

$$g(\sqrt{y}) = \frac{1}{2}\, y^{\lambda - (\nu - 1)/2} \int_0^\infty e^{-yt}\, t^{(n-1)/2} \cdot \sin\sqrt{t}\;\; \mathrm{d}t =$$

$$= \frac{\sqrt{\pi}}{4} (-1)^{(n-1)/2}\, y^{\lambda - (\nu - 1)/2} \frac{\mathrm{d}^{(n-1)/2}}{\mathrm{d}y^{(n-1)/2}} \left(y^{-3/2}\, e^{-1/(4y)}\right),$$

where n is any positive odd integer (M. 126),

$$= \frac{\pi}{2\sqrt{2}}\, y^{\lambda - (\nu/2)}\, e^{-1/(8y)} \cdot I_{1/4}(1/(8y))$$

when $n = -1/2$ (M. 126). Therefore, from **3**(1),

$$\int_0^\infty y^{2\lambda+1} \cdot J_\lambda^{1/2}(xy) \cdot \left(\frac{1}{y} \frac{\mathrm{d}}{\mathrm{d}y}\right)^{(n-1)/2} [y^{-3}\, e^{-1/(4y^2)}]\, \mathrm{d}y =$$

$$= 2^{(n+1)/2} (-1)^{2\lambda + (n/2) + 1} \frac{\mathrm{d}^{2\lambda - n + (1/2)}}{\mathrm{d}x^{2\lambda - n + (1/2)}} \left(\frac{1}{\sqrt{x}}\, e^{-\sqrt{2x}} \cdot \sin\sqrt{2x}\right),$$

provided n is any positive odd integer and $2\lambda - n + (1/2)$ any positive integer including zero, and

$$\int_0^\infty y^{2\lambda}\cdot J_\lambda^{1/2}(xy)\cdot e^{-1/(8y^2)}\cdot I_{1/4}(1/(8y^2))\,\mathrm{d}y =$$

$$= \sqrt{\frac{2}{\pi}}\,(-1)^{2\lambda+1}\cdot\frac{\mathrm{d}^{2\lambda+1}}{\mathrm{d}x^{2\lambda+1}}\left(\frac{1}{\sqrt{x}}\,e^{-\sqrt{2x}}\cdot\sin\sqrt{2x}\right),$$

provided $2\lambda + 1$ is any positive integer including zero.

4. – Theorem 2. *If*

$$g(y) = \frac{1}{\mu}\int_0^\infty (xy)^{[(\lambda+1)/\mu]-(\nu+1)}\cdot J^{1/\mu}_{[(\lambda+1)/\mu]-1}(xy)^{1/\mu}\cdot f(x)\,\mathrm{d}x \tag{1}$$

and

$$x^{-\nu}\cdot f(x) = \int_0^\infty t^n\,e^{-xt}\cdot\psi(t)\,\mathrm{d}t, \tag{2}$$

then

$$g(y) = \frac{1}{\mu}\,y^{[(\lambda+1)/\mu]-(\nu+1)}\int_0^\infty t^{n-[(\lambda+1)/\mu]}\,e^{-(y/t)^{1/\mu}}\cdot\psi(t)\,\mathrm{d}t, \tag{3}$$

provided $\Re(\lambda) > -1$ *and the integrals involved converge absolutely.*

The proof of this follows easily, if we proceed on the same lines as in Theorem 1.

5. – Evaluation of Integrals.

As before, this Theorem also can be used to evaluate a large number of integrals, but the form convenient for pratical application is as follows:

If

$$g(y) = \frac{1}{\mu}\int_0^\infty (xy)^{[(\lambda+1)/\mu]-(\nu+1)}\cdot J^{1/\mu}_{[(\lambda+1)/\mu]-1}(xy)^{1/\mu}\cdot f(x)\,\mathrm{d}x \tag{1}$$

and

$$(2) \qquad x^{-\nu}\cdot f(x) = \int_0^\infty t^n\, e^{-xt}\cdot \psi(t)\ \mathrm{d}t,$$

then

$$(3) \qquad g(y^\mu) = y^{\lambda+1-\mu\cdot(\nu+1)} \int_0^\infty t^{\lambda-\mu\cdot(\nu+1)}\, e^{-yt}\cdot \psi(t^{-\mu})\ \mathrm{d}t,$$

provided $\Re(\lambda) > -1$ *and the integrals involved converge absolutely.*

Example 1. Let us take $\psi(t) = (1+t)^{-1/2}$ and $\mu = 2$, then, from **5**(2),

$$f(x) = x^\nu \int_0^\infty e^{-xt}\cdot (t^n/\sqrt{1+t}\,)\ \mathrm{d}t = \sqrt{\pi}\,(-1)^n x^\nu \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n [x^{-1/2}\, e^x\cdot \{1-\Phi(\sqrt{x})\}],$$

provided n is any positive integer including zero (M. 124), and from **5**(3)

$$g(y^2) = y^{\lambda-2\nu-1} \int_0^\infty e^{-yt}\, (t^{\lambda-2n-1}/\sqrt{1+t^2})\ \mathrm{d}t =$$

$$= \frac{\pi}{2}\,(-1)^{\lambda-2n-1}\, y^{\lambda-2\nu-1}\cdot \frac{\mathrm{d}^{\lambda-2n-1}}{\mathrm{d}y^{\lambda-2n-1}}\,[\boldsymbol{H}_0(y) - N_0(y)]$$

provided $\lambda - 2n - 1$ is any positive integer including zero (M. 124) and $\boldsymbol{H}_\nu(z)$ and $N_n(z)$ are STRUVE (M. 40) and NEUMANN (M. 16) functions. Therefore, from **5**(1),

$$(4) \qquad \int_0^\infty x^\lambda\cdot J^{1/2}_{(\lambda-1)/2}(xy)\cdot \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n [x^{-1}\, e^{x^2}\cdot \{1-\Phi(x)\}]\ \mathrm{d}x =$$

$$= \sqrt{\pi}\,(-1)^{\lambda-n-1}\, 2^{n-1}\cdot \frac{\mathrm{d}^{\lambda-2n-1}}{\mathrm{d}x^{\lambda-2n-1}}\,\{\boldsymbol{H}_0(y) - N_0(y)\}$$

provided that n and $\lambda - 2n - 1$ are any positive integers including zero.

Example 2. Let us take $\psi(t) = \cos(1/t)$ and $\mu = 2$, then, from **5**(2),

$$f(x) = x^{\nu}\int_0^{\infty} e^{-xt}\, t^{n+(1/2)} \frac{\cos(1/t)}{\sqrt{t}}\,dt =$$

$$= \sqrt{\pi}\,(-1)^{n+(1/2)}\, x^{\nu}\cdot \frac{d^{n+(1/2)}}{dx^{n+(1/2)}}\left(x^{-1/2}\, e^{-\sqrt{2x}}\cdot\cos\sqrt{2x}\right),$$

where $n + (1/2)$ is any positive integer including zero (M. 127), and from **5**(3)

$$g(y^2) = y^{\lambda-2\nu-1}\int_0^{\infty} e^{-yt}\, t^{\lambda-2n-2}\cdot\cos t^2\,dt =$$

$$= \sqrt{\frac{\pi}{2}}(-1)^{\lambda-2n-2}\cdot y^{\lambda-2\nu-1}\frac{d^{\lambda-2n-2}}{dy^{\lambda-2n-2}}\left[\cos\frac{y^2}{4}\cdot\left\{\frac{1}{2}-S\left(\frac{y}{\sqrt{2\pi}}\right)\right\}-\sin\frac{y^2}{4}\cdot\left\{\frac{1}{2}-C\left(\frac{y}{\sqrt{2\pi}}\right)\right\}\right],$$

where $\lambda - 2n - 2$ is any positive integer including zero (M. 126) and $C(x)$ and $S(x)$ are « FRESNEL Integrals » (M. 96). Therefore, from **5**(1),

$$(5)\qquad \int_0^{\infty} x^{\lambda}\cdot J^{1/2}_{(\lambda-1)/2}(xy)\cdot\left(\frac{1}{x}\frac{d}{dx}\right)^{n+(1/2)}\left(\frac{1}{x}\,e^{-\sqrt{2x}}\cdot\cos\sqrt{2x}\right)dx =$$

$$= 2^{n}\cdot(-1)^{\lambda-n-(3/2)}\cdot\frac{d^{\lambda-2n-2}}{dy^{\lambda-2n-2}}\left[\cos\frac{y^2}{4}\cdot\left\{\frac{1}{2}-S\left(\frac{y}{\sqrt{2\pi}}\right)\right\}-\sin\frac{y^2}{4}\cdot\left\{\frac{1}{2}-C\left(\frac{y}{\sqrt{2\pi}}\right)\right\}\right],$$

provided $\lambda - 2n - 2$ and $n + (1/2)$ are any positive integers (including zero).

Example 3. Let us take $\psi(t) = \sin(1/t)$ and $\mu = 2$, then, from **5**(2),

$$f(x) = x^{\nu}\int_0^{\infty} e^{-xt}\, t^{n+(1/2)}\frac{\sin(1/t)}{\sqrt{t}}\,dt =$$

$$= \sqrt{\pi}\,(-1)^{n+(1/2)}\, x^{\nu}\cdot\frac{d^{n+(1/2)}}{dx^{n+(1/2)}}\left(x^{-1/2}\, e^{-\sqrt{2x}}\cdot\sin\sqrt{2x}\right),$$

where $n + (1/2)$ is any positive integer including zero (M. 126), and from **5**(3)

$$g(y^2) = y^{\lambda-2\nu-1} \int_0^\infty e^{-yt}\, t^{\lambda-2n-2} \cdot \sin t^2 \,\mathrm{d}t =$$

$$= \frac{1}{2}\, y^{\lambda-2\nu-1} \cdot D_{-1}\left(\frac{1+i}{2}\, y\right) \cdot D_{-1}\left(\frac{1-i}{2}\, y\right)$$

when $\lambda - 2n - 2 = -1$, i.e., $n + (1/2) = \lambda/2$ (M. 125),

$$= \sqrt{\frac{\pi}{2}}\, (-1)^{\lambda-2n-2}\, y^{\lambda-2\nu-1} \cdot$$

$$\cdot \frac{\mathrm{d}^{\lambda-2n-2}}{\mathrm{d}y^{\lambda-2n-2}} \left[\cos \frac{y^2}{4} \cdot \left\{\frac{1}{2} - C\left(\frac{y}{\sqrt{2\pi}}\right)\right\} + \sin \frac{y^2}{4} \cdot \left\{\frac{1}{2} - S\left(\frac{y}{\sqrt{2\pi}}\right)\right\}\right]$$

when $\lambda - 2n - 2 = 0, 1, 2, 3, \ldots$ (M. 126). Therefore, from **5**(1),

$$(6) \qquad \int_0^\infty x^\lambda \cdot J^{1/2}_{(\lambda-1)/2}(xy) \cdot \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\lambda/2} \left(\frac{1}{x}\, e^{-\sqrt{2}x} \cdot \sin(\sqrt{2}x)\right) \mathrm{d}x =$$

$$= \frac{2^{(\lambda/2)-1}}{\sqrt{\pi}}\, (-1)^{\lambda/2} \cdot D_{-1}\left(\frac{1+i}{2}\, y\right) \cdot D_{-1}\left(\frac{1-i}{2}\, y\right),$$

provided λ is any even positive integer including zero, and

$$(7) \qquad \int_0^\infty x^\lambda \cdot J^{1/2}_{(\lambda-1)/2}(xy) \cdot \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n+(1/2)} \left(\frac{1}{x}\, e^{-\sqrt{2}x} \cdot \sin(\sqrt{2}x)\right) \mathrm{d}x =$$

$$= 2^n\, (-1)^{\lambda-n-(3/2)} \cdot \frac{\mathrm{d}^{\lambda-2n-2}}{\mathrm{d}y^{\lambda-2n-2}} \left[\cos \frac{y^2}{4} \cdot \left\{\frac{1}{2} - C\left(\frac{y}{\sqrt{2\pi}}\right)\right\} + \sin \frac{y^2}{4} \cdot \left\{\frac{1}{2} - S\left(\frac{y}{\sqrt{2\pi}}\right)\right\}\right],$$

provided $n + (1/2)$ and $\lambda - 2n - 2$ are any positive integers including zero.

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References.

[1] R. P. Agarwal, *Sur une généralisation de la transformation de Hankel*, Ann. Soc. Sci. Bruxelles, (1) **64** (1950) 164-168.

[2] R. P. Agarwal, *Some properties of generalised Hankel transform*, Bull. Calcutta Math. Soc. **43** (1951), 153-167.

[3] E. M. Wright, *The asymptotic expansion of the generalised Bessel function*, Proc. London Math. Soc. **38** (1934), 257-270.