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On the Expansion of Solutions of Ordinary Differential Equations According to Powers of the Initial Constants or of Parameters. (**)

1. – In a system of r differential equations with s parameters which is of the form

$$\mathrm{d}x/\mathrm{d}t = f(x,\ \mu;\ t), \tag{1}$$

where $x = (x^1, \ldots, x^r)$, $f = (f^1, \ldots, f^r)$, $\mu = (\mu^1, \ldots, \mu^s)$ and $x' = \mathrm{d}x/\mathrm{d}t$, suppose that

$$f(0,\ 0;\ t) \equiv 0 \tag{2}$$

and that the initial condition assigned for $x = x(t,\ \mu)$ at $t = 0$ is

$$x(0,\ \mu) \equiv 0\ . \tag{3}$$

Here μ is a set of parameters which can be, for instance, integration constants of a system

$$\mathrm{d}x/\mathrm{d}t = f(x;\ t) \tag{1 bis}$$

if, after a change of notation, the latter system and the initial conditions belonging to it are written as (1) and (3) respectively.

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For the sake of simplicity, it will be assumed that $r = 1$ and $s = 1$. But it will be clear from the proofs that nothing is changed when x, f and μ are vectors.

2. – Let $f(x, \mu; t)$ be a (complex-valued) function on a five-dimensional region which is the product set of an interval

$$0 \leqq t \leqq a \qquad (a > 0) \tag{4}$$

on the real t-axis and of a dicylinder about the origin of the complex (x, μ)-space, say of the dicylinder

$$|x| < 1, \qquad |\mu| < 1. \tag{5}$$

Suppose that f is continuous in $(x, \mu; t)$ on

$$|x| < 1, \qquad |\mu| < 1, \qquad 0 \leqq t \leqq a, \tag{6}$$

is regular in (x, μ) on (5) when t is fixed, is bounded in $(x, \mu; t)$, say

$$|f(x, \mu; t)| < 1 \qquad \text{on } (6), \tag{7}$$

and satisfies (2) on (4). Accordingly, f has on (6) an absolutely convergent expansion of the form

$$f(x, \mu; t) = \sum_{i+j>0}\sum f_{ij}(t) \cdot x^i \mu^j, \tag{8}$$

where the functions $f_{ij}(t)$ are continuous on (4) and, in view of (7) and of CAUCHY's coefficient estimate, satisfy the inequalities

$$|f_{ij}(t)| \leqq 1 \qquad \text{on } (4). \tag{9}$$

Under these assumptions, a fundamental result of the *Méthodes Nouvelles* (or, rather, what is between the lines of POINCARÉ's proof; cf. [7], pp. 270-272) can be formulated as follows: No matter how large $a > 0$ may be, there exists a sufficiently small $\alpha = \alpha_a > 0$ in such a way that the (unique) solution $x = x(t, \mu)$ of (1) and (3) can be expanded into a power series

$$x(t, \mu) = \sum_{k=1}^{\infty} \varphi_k(t) \cdot \mu^k \tag{10}$$

which, in terms of a unique sequence of functions $\varphi_1(t)$, $\varphi_2(t)$, ... which are continuous on (4), is valid for

$$0 \leqq t \leqq a, \qquad |\mu| < \alpha . \tag{11}$$

3. – Poincaré's proof consists in three steps: with the aid of (8) and (7), first he majorizes (1) by

$$dx/dt = \sum_{i+j>0}\sum 1 \cdot x^i \mu^j , \tag{12}$$

then he majorizes (12) [where, in view of (5), the double series has the sum $(1-x)^{-1}\ (1-\mu)^{-1} - 1$] by

$$dx/dt = (x+\mu)/(1-x-\mu), \tag{13}$$

and finally he solves (13) (which is possible by a separation of variables). The result is that the solution of (13) and (3) [*hence* the solution of (1) and (3)] possesses an expansion (10) which is valid on (11) if the $\alpha = \alpha_a$ in (11) is chosen as follows:

$$\alpha = 2e^a\left\{1-(1-e^{-a})^{1/2}\right\} - 1 = (e^{-a}/4) + (e^{-2a}/8) + \dots . \tag{14}$$

Since (13) is just a majorant of the differential equation (12) (which, in contrast to (13), cannot be solved by separating variables only), Perron [**6**] investigated the analogue of Poincaré's bound (14) for the case in which (12) is not worsened to (13). The result of Perron's discussions (which depend, among other things, on the Vivanti-Pringsheim theorem concerning positive power series) is as follows: if $\alpha = \alpha_a$ is defined by

$$\alpha = \sum_{n=1}^{\infty} (n!\, e^n/n^{n-1})^{-1} \cdot (a+1)^{n-1}\, e^{-na} = e^{-a}/e + \dots , \tag{15}$$

then the expansion (10) of the solution of (12) and (2) is valid on (11) but becomes divergent at the boundary point $(t, x) = (\alpha, a)$ of (11).

4. – Since (12) is the *best* majorant of (1) by virtue of (8) and (9), this seems to settle the problem of the « best constant » $\alpha = \alpha_a$. Actually, it does not. The trouble is that (9), though necessary, is not sufficient in order that

(8) satisfies (7). Suffice it to say that if F^* denotes the sum of the double series occuring in (12), then (7) becomes violated when F is replaced by F^*, since F^* becomes infinite (in fact, as strongly as a pole) when either $x \to 1$ or $\mu \to 1$.

In what follows, an $\alpha = \alpha_a > 0$ allowable in (11) for (10) will be obtained so as to use the full force of (7), rather than just the corollary (9) of (7). Correspondingly, the proof will have to involve function-theoretical arguments. [In this regard, it should be noted that although PERRON's treatment of (12) depends on a function-theoretical fact, the theorem of VIVANTI-PRINGSHEIM, the latter is not used in order to show that (15) is an allowable $\alpha = \alpha_a$, but merely to show that the value (15) of α cannot be improved in the case (12).] The function-theoretical elements, to be used in obtaining an allowable $\alpha = \alpha_a > 0$ in (7), will be, on the one hand, SCHWARZ's lemma and, on the other hand, the following fact (which is a well-known consequence of the inequalities of JACOBI-JENSEN): If a function of z is regular, and in absolute value less than a constant C, on a circle $|z| < R$, then the absolute value of the derivative of the function at any point z of the circle is less than $CR/(R^2 - |z|^2)$ (and this bound is sharp). Cf. [**1**], pp. 506-507.

5. – Let $f(x, \mu; t)$ be any function satisfying the conditions specified in the first two sentences of Section **2**.

Choose any pair of numbers b, α satisfying

(16) $$0 < b < 1, \qquad 0 < \alpha < 1,$$

put

(17) $$N = N(\alpha) = \text{l.u.b.}\, |f(0, \mu; t)| \quad \text{for} \quad |\mu| < \alpha, \qquad 0 \leqq t \leqq a,$$

where $a > 0$ is the number fixed in (6), and let

(18) $$L = L(b, \alpha) = \text{l.u.b.}\, |f_x(x, \mu; t) \quad \text{for} \quad |x| < b, \ |\mu| < \alpha, \ 0 \leqq t \leqq a$$

[in (18), the subscript of f denotes partial differentiation]. Then, for reasons to be explained in Section **10**, the solution $x = x(t; \mu)$ of (1) and (3) exists, and is a continuous function, on the three-dimensional (t, μ)-region

(19) $$0 \leqq t \leqq \min\{a, \ L^{-1} \cdot \log(1 + bL/N)\}, \qquad |\mu| < \alpha,$$

at least, and, when t has any fixed value on the interval $0 \leqq t \leqq$ min [specified in (19)], the function $x(t, \mu)$ of μ is regular on the circle specified by the last of the inequalities (19). Hence, there exists an expansion (10) which is valid on the region (19). Consequently, (10) will be valid on the region (11) if

$$L^{-1} \cdot \log (1 + bL/N) \geqq a. \tag{20}$$

If $L = 0$, then (19) and (20) are meaningless. But it can be assumed that $L > 0$. In fact, if $L = 0$, then (18) shows that $f(x, \mu; t)$ does not contain x at all, and so (1) becomes trivial. If $L > 0$ but $N = 0$, then the log in (19) must be interpreted to be ∞, hence (19) reduces to (10), and so there is no problem (for any positive $\alpha \leqq 1$).

6. – There will now be discussed the conditions (on b and α) under which the pair of numbers (17), (18) will satisfy (20).

First, if t is fixed on the interval (4), then, since $f(x, \mu; t)$ is regular on (5) and satisfies (7), the function $f(0, \mu; t)$ of μ is regular on the circle $|\mu| < 1$, is less than 1 in absolute value and, in view of (2), vanishes at $\mu = 0$. Hence, by SCHWARZ's lemma, $|f(0, \mu; t)| \leqq |\mu|$. In view of (17), this means that

$$N \leqq \alpha \qquad (0 < \alpha < 1). \tag{21}$$

Next, if the estimate which was referred to at the end of Section **4** is applied to the derivative of the function $f(x, \mu; t)$ of $x = z$, it follows from (6) that, for every t contained in the interval (4),

$$|f_x(x, \mu; t)| (1 - b^2)^{-1} \quad \text{if} \quad |x| < b \quad \text{and} \quad |\mu| < \alpha;$$

cf. (16). In view of (18), this means that

$$L \leqq (1 - b^2)^{-1} \qquad (0 < b < 1). \tag{22}$$

Finally, it is readily seen that the producet L^{-1} log occurring in (20) is a decreasing function not only of $N (> 0)$ but also of L (> 0). It follows therefore from (21) and (22) that condition (20) will be satisfied if the expression

$$\beta_a(b) = (1 - b^2) \cdot \log \left\{ 1 + b \cdot (1 - b^2)^{-1} / \alpha \right\} \tag{23}$$

satisfies the inequality $\beta_a(b) \geqq a$ for *some* pair of numbers $b = b(a)$, $\alpha = \alpha(a)$ which are subject only to (16).

7. – Since $\beta_a(b) = a$ is sufficient for $\beta_a(b) \geqq a$, it now follows (cf. Section **5**) that the expansion (10) of the solution of (1) and (3) is valid on the region (11) whenever $\alpha = \alpha_a$, where

$$0 < \alpha < 1, \tag{24}$$

is so chosen (with reference to the given $a > 0$) that the following condition is satisfied: if $\beta_a(b)$ is defined by (23), then the transcendental equation $\beta_a(b) = a$ has some ro t $b = b(\alpha;\, a)$ on the interval $0 < b < 1$. But it is clear from (23) and (24) that $\beta_a(b)$ is positive for $0 < b < 1$ and tends to 0 when $b \to 0$ or $b \to 1$. Hence the desired root b of $\beta_a(b) = a$ will exist if and only if the maximum of $\beta_a(b)$ on $0 < b < 1$ is not less than a.

In view of (23), this requires that $\gamma(\alpha) \geqq a$, where

$$\gamma(\alpha) = \max_{0<b<1} \left[(1 - b^2)\cdot\log\left\{1 + b\cdot(1 - b^2)^{-1/\alpha}\right\}\right]. \tag{25}$$

Since $\gamma(\alpha) = a$ is sufficient for $\gamma(a) \geqq a$, and since (25) defines γ as a decreasing and continuous function of α, the condition just obtained can be formulated as follows: the transcendental equation

$$\gamma(\alpha) = a \tag{26}$$

possesses a root $\alpha = \alpha_a$ on the interval (24). Accordingly, *an* $\alpha = \alpha_a$ *satisfying* (26) *and* (24) *will have the property that the expansion* (10) *of the solution of* (1) *and* (3) *is valid on the range* (11).

8. – Since $\gamma(\alpha)$ is a positive, increasing and continuous function on the interval (24), and since it is also clear from (25) that

$$\gamma(\alpha) \to \infty \qquad \text{as} \qquad \alpha \to 0, \tag{27}$$

the equation (26) has on the interval (24) a unique root $\alpha = \alpha_a$, provided that a is large enough, say

$$a > \text{const.}, \tag{28}$$

where the const. is a universal positive constant. But the proviso (28) on a can be omitted, since, if (28) is not satisfied, then, by the very nature of the problem, the expansion (10) will be valid on the region (11) if the α occurring in (11) is chosen to be

$$\alpha = 1. \tag{29}$$

Actually, the problem of POINCARÉ concerns the « large », rather than the « small », values of a.

In any case, POINCARÉ's theorem, according to which there exists *some* $\alpha = \alpha_a > 0$ for which (10) is valid on (11), now follows for every $a > 0$. In fact, such an $\alpha = \alpha_a$ is given by (29) or by the (unique) root $\alpha = \alpha_a$ on (24) of the equation (26), defined by (25), according as a is not or is large enough to satisfy (28).

The mere fact that (29) holds for sufficiently small values of $a > 0$ implies that PERRON's result (15) cannot be the « best » in the present problem, that of (6). In fact, (15) represents *the* α-value belonging to (12), whether $a > 0$ be small or large. But (15) is a value which is within the range (24) for every $a > 0$. That (29) must hold for sufficiently small $a > 0$ (in other words, that the const. > 0 of (28) actually exists), is obvious. In fact, when (29) does not hold, then (25) and (24) imply that

$$\gamma(\alpha) < \text{const.} \qquad (0 < \alpha < 1) \tag{30}$$

where

$$\text{const.} = \max_{0<b<1} \left[(1 + b^2)\cdot\log\left\{1 + b\cdot(1 - b^2)^{-1}\right\}\right]. \tag{31}$$

But the constant (31) is positive, and (30) implies that (26) holds when a violates (28).

9. – There remains to be ascertained the assertion of Section **5** concerning the region (19).

Consider first the case in which f is free of a parameter μ, so that (1) and (3) simplify to

$$dx/dt = f(x;\, t), \qquad x(0) = 0. \tag{32}$$

It will be supposed that on the three-dimensional region

$$0 \leqq t \leqq a, \qquad |x| < b, \tag{33}$$

where x is complex, the function $f(x; t)$ is continuous [so that, in particular,

(34) $$|f(0; t)| \leqq N \qquad \text{on } (4),$$

where N is a constant], that the function $f(x; t)$ of x is regular on the circle $|x| < b$ (for every fixed t contained in the interval $0 \leqq t \leqq a$), finally that the partial derivative $f_x(x; t)$ is bounded in x and t together, say

(35) $$|f_x(x; t)| \leqq L \qquad \text{on } (33),$$

where L is a constant. Although f and x are complex, it is readily seen that (35) is equivalent to the LIPSCHITZ condition

(35 bis) $$|f(x'; t) - f(x''; t)| \leqq L \cdot |x' - x''|,$$

where $(x'; t)$, $(x''; t)$ is any pair of points contained in the $(x; t)$-region (33).

Replace the interval (4) by the (possibly) shorter interval

(36) $$0 \leqq t \leqq \min\left\{ a,\ L^{-1} \cdot \log(1 + bL/N) \right\},$$

if neither of the (non-negative) bounds L, N is 0. If $N = 0$, interpret (36) to be (4). If $N > 0$ but $L = 0$, let (36) be interpreted as the interval $0 \leqq t \leqq \min(a, b/N)$, the limit of (36) as $L \to 0$.

According to E. LINDELÖF [4], p. 123, the sequence

(37) $$x_1(t), \ \ldots, \ x_n(t), \ \ldots$$

of the successive approximations

(38) $$x_n(t) = \int_0^t f(x_{n-1}(s), s)\, ds, \qquad \text{where} \qquad x_0(t) \equiv 0,$$

exists on (36), satisfies

(39) $$|x_n(t)| \leqq b \qquad \text{on } (36)$$

and converges, uniformly on (36), to the solution $x = x(t)$ of (32), whenever the function $f(x; t)$ is continuous on (33) and satisfies (34) and (35 bis).

The circumstance that x and f are real-valued in LINDELÖF's paper, does not matter, since his proof holds, without any modification, in the complex field also (in fact, even t could be complex, with (4) replaced by a t-circle). The same remark applies to a result of LIPSCHITZ himself [**5**], pp. 509-514, which is of an earlier date than that of LINDELÖF and leads precisely to the interval (36) when the process (38) in replaced (cf. also a paper of O. HÖLDER [**3**]) by process of the «polygonal» method. Correspondingly, the latter method could (but will not be made to) replace the method of the successive approximations in Section **10**.

Remark. Since the sequence (37) tends to the solution $x(t)$ of (32) if t is in (32), it follows from (39) that

$$|x(t)| \leqq b \tag{40}$$

holds on (36). But the fact that precisely (36) is the interval on which (40) can be concluded from (34), (35) and (32), expresses precisely the estimate which result if HAAR's inequality in the theory of partial differential equations (cf. [**2**]) is applied to LINDELÖF's particular case of an ordinary differential equation.

10. – Consider now the case in which f contains a parameter, μ, which varies on a complex domain, say on the circle $|\mu| < \alpha$. Then (1) and (3) take the place of (32). Suppose that $f(x, \mu; t)$ is continuous on the product set of the region (33) and the circle $|\mu| < \alpha$, that $f(x, \mu; t)$ is regular in (x, μ) on the dicylinder $(|x| < b,\ |\mu| < \alpha)$ when t is fixed on the interval (4), and that there exist two bounds, N and L, which are independent of μ, and satisfy (34) and (35) for every fixed μ, if $|\mu| < \alpha$.

Clearly, the assertion of Section **5**, concerning the region (19), will be proved if it is ascertained that, under the assumptions just mentioned, the solution $x(t, \mu)$ of (1) and (3) exists on the product set of the interval (36) and the circle $|\mu| < \alpha$, and that the function $x(t, \mu)$ of μ is regular on the circle $|\mu| < \alpha$ for every fixed t contained in the interval (36). But this can be concluded as follows (cf. [**8**]):

For a *fixed* μ in $|\mu| < \alpha$, consider on (36) the sequence

$$x_1(t, \mu), \quad \ldots, \quad x_n(t, \mu), \quad \ldots \tag{37 bis}$$

which results in the present case of (38). Then it is clear from the integral defining the function $x_n(t, \mu)$ that the latter is a regular function of μ on the circle $|\mu| < \alpha$ when t is fixed on the interval (36). It follows therefore from (39) and from the maximum principle that

(39 bis) $$|x_n(t, \mu)| < b \qquad \text{on (36) if } |\mu| < \alpha.$$

But the sequence (37 bis) tends on (36) to the solution $x(t, \mu)$ of (1) and (3) if μ is fixed in $|\mu| < \alpha$. On the other hand, the functions (37 bis) of μ are regular and, in view of (39 bis), uniformly bounded on $|\mu| < \alpha$ if t is in (36). Consequently, the convergence of the sequence (37 bis) is uniform on every circle $|\mu| < \alpha - \varepsilon$ $(< \alpha)$ if t is in (36). Since this implies that the limit function, $x(t, \mu)$, is regular on $|\mu| < \alpha$ when t is in (36), the proof is complete.

11. – At the end of Section **7**, the final result was formulated in terms of a root of the transcendental equation (26), defined by (25), rather than in terms of an explicit expression for $\alpha = \alpha_a$. An explicit expression could be obtained by using LAGRANGE's series for the (inverses of the) logarithmic-algebraic functions involved.

Instead of proceeding in this manner, a comparatively favorable estimate of $\alpha = \alpha_a$ (cf. Section **12**) will be obtained, without much formal work, by relaxing the strict result of Section **7**. This relaxation results if the factor b of $(1 - b^2)^{-1}$ in (25) is not exploited. The resulting situation is as follows:

Since (25) implies that

(41) $$\gamma(\alpha) < \max_{0 \leqq b < 1} \left[(1 - b^2) \cdot \log\left\{1 + (1 - b^2)^{-1}/\alpha\right\}\right],$$

it is clear that the transcendental equation (26) will possess a root $\alpha = \alpha_a$ satisfying (24) when (29) is not allowed, and that

(42) $$0 < \alpha_a^* < \alpha_a < 1 \qquad \text{or} \qquad \alpha_a^* = \alpha_a = 1$$

will hold, if $\alpha^* = \alpha_a^*$ can be so chosen that, on the one hand, $0 < \alpha^* \leqq 1$ and, on the other hand, the value of a is equal to what results when α is replaced by α^* on the right of (41). Since $1 \leqq v < \infty$ is equivalent to $0 \leqq b < 1$ if $v = (1 - b^2)^{-1}$, the latter condition can be written as

$$\max_{1 \leqq v < \infty} \left\{v^{-1} \cdot \log(1 + v/\alpha^*)\right\} = a$$

or, if $u = \alpha^*/v$, as

(43) $$\max_{0 \leqq u < \alpha^*} \left\{u \cdot \log(1 + u^{-1}\right\} = a\alpha^*.$$

It will now be shown that the (unique) root $\alpha^* = \alpha^*_a$ of the transcendental equation (43) on the range $0 < \alpha^* \leqq 1$ is

$$\alpha^* = (e^a - 1)^{-1} \qquad \text{if} \qquad \log 2 \leqq a < \infty \tag{44}$$

and that, by virtue of the *disjunctive* alternative (42),

$$\alpha^* = 1 \qquad \text{if} \qquad 0 < a \leqq \log 2 \tag{45}$$

[note that the inequality imposed on a in (44) is equivalent to $(e^a - 1)^{-1} \leqq 1$].

First, a differentiation shows that the derivative of the function

$$u \cdot \log (1 + u^{-1}) \tag{46}$$

vanishes at a $u = u_0 > 0$ if and only if

$$\log (1 + u_0^{-1}) = (1 + u_0)^{-1}. \tag{47}$$

But if u_0 increases from $u_0 = 0$ to $u_0 = 1$, then the value of the functions occurring on the left and on the right on (47) decrease from ∞ to $\log 2$ and from 1 to 1/2, respectively. Since $\log 2 = 0{,}6 \ldots > 1/2$, and since both curves are convex (toward the u_0-axis), this implies that the equation (47) has no root u_0 on the interval $0 < u_0 \leqq 1$. It follows that the function (46), which vanishes at $u = 0$ and is positive for $u > 0$, is increasing at every positive $u \leqq 1$. Consequently, if $0 < \alpha^* \leqq 1$, then (43) can be written as

$$\alpha^* \cdot \log (1 + 1/\alpha^*) = a\alpha^*.$$

This means that $1 + 1/\alpha^* = e^a$, hence $\alpha^* = (e^a - 1)^{-1}$, if $0 < \alpha^* \leqq 1$. In view of the parenthetical remark made after (45), this proves both (44) and (45).

12. – It follows that the expansion (10) is surely valid on the region (11) (determined by any given $a > 0$ and a corresponding $\alpha > 0$) whenever

$$\alpha = (e^a - 1)^{-1} = \sum_{n=1}^{\infty} e^{-na}, \qquad \text{where} \qquad \log 2 \leqq a < \infty, \tag{48}$$

or

$$\alpha = 1, \qquad \text{where} \qquad 0 < a \leqq \log 2 \tag{48}$$

(and it is not claimed that $\alpha = 1$ cannot hold for $0 < a < A$, where $A > \log 2$). In fact, it is seen from (42) that (48) and (49) follow from (44) and (45).

It is now seen that $\log 2$ is a lower bound for the const. defined in connection with (28). But it also follows that

$$\alpha = e^{-a}, \qquad \text{where} \qquad 0 < a < \infty, \tag{50}$$

is allowed in (11). In fact, the α of (50) is less than the α of (48) or of (49). But even (50) improves on PERRON's α, since the first term, e^{-a}/e, of the positive series (15) is less than $e^{-a}/1 = e^{-a}$.

References.

[1] L. BIEBERBACH, Article II C 4 (1920), Encyklopädie der mathematischen Wissenschaften, B. G. Teubner, Leipzig.

[2] A. HAAR, *Über Eindeutigkeit und Analytizität der Lösungen partieller Differentialgleichungen*, Atti del Congresso Internazionale dei Matematici del 1928, Vol. **3**, pp. 5-10, N. Zanichelli, Bologna 1930.

[3] O. HÖLDER, *Abschätzungen in der Theorie der Differentialgleichungen*, SCHWARZ Festschrift (1914), p. 116-132.

[4] E. LINDELÖF, *Sur l'application des méthodes d'approximations successives a l'étude des intégrales réelles des équations differentielles ordinaires*, J. Math. Pures Appl. (4) **10** (1894), 117-128.

[5] R. LIPSCHITZ, *Differential- und Integralrechnung*, Max Cohen und Sohn, Bonn 1880.

[6] O. PERRON, *Über die Entwickelbarkeit der Integrale von Differentialgleichungen nach Potenzen eines Parameters und der Anfangswerte*, Math. Ann. **113** (1936), 292-303.

[7] H. POINCARÉ, *Oeuvres*, Vol. **7**, Gauthier-Villars, Paris 1952.

[8] A. WINTNER, *Remarks to an earlier Note*, Amer. J. Math. **76** (1954), 717-720.