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# Weak and Strong Cyclic Additivity. (**)

## Introduction.

The purpose of this paragraph is to give the reader some of the historical background of the problem which is considered in this paper.

Let $Q \equiv [0 \leqslant u,v \leqslant 1]$ be the unit square in the $uv$-plane and let $E_3$ be the Euclidean $xyz$-space. For $T$ a continuous mapping from $Q$ into $E_3$, written $T: Q \rightarrow E_3$ (the symbol $\Longrightarrow$ will be reserved for a mapping onto), let us denote by $A(T)$ the LEBESGUE area of the surface represented by $T$ (T. RADÓ [4]). The LEBESGUE area $A(T)$ possesses remarkable cyclic additivity properties; namely, if $T = lm$, $m: Q \Longrightarrow \mathfrak{M}$, $l: \mathfrak{M} \rightarrow E_3$ is a monotone-light factorization of $T$ (G. T. WHYBURN [5]), then

(1) $$A(T) = A(lr_C m), \qquad C \subset \mathfrak{M},$$

where $r_C$ is the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element $C$ of $\mathfrak{M}$ and where the summation in (1) is extended over all proper cyclic elements of $\mathfrak{M}$ (T. RADÓ [4]).

For the proof of (1), T. RADÓ [4] considered a real-valued, non-negative functional $\Phi(T)$ defined for each continuous mapping $T$ from a fixed PEANO space $P$ into a fixed metric space $P^*$. Under certain additional hypotheses on $\Phi(T)$, T. RADÓ established

(2) $$\Phi(T) = \sum \Phi(l r_C m), \qquad C \subset \mathfrak{M},$$

for every continuous mapping $T$ from $P$ into $P^*$.

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An important generalization of (2) was obtained by E. J. MICKLE and T. RADÓ [**3**]. In their paper, the writers considered *unrestricted factorizations* of a continuous mapping $T$ from $P$ into $P^*$, i.e., $T = sf$, $f : P \to \mathfrak{M}$, $s : \mathfrak{M} \to P^*$, where $\mathfrak{M}$ is a PEANO space and $f$ and $s$ are continuous mappings from $P$ into $\mathfrak{M}$ and from $\mathfrak{M}$ into $P^*$, respectively. With certain restrictions on the functional $\Phi(T)$, E. J. MICKLE and T. RADÓ proved the following result:

$$(3) \qquad \Phi(T) = \sum \Phi(s\, r_C f), \qquad C \subset \mathfrak{M},$$

for all continuous mappings $T$ from $P$ into $P^*$ and for all unrestricted factorizations $s$, $f$ of $T$.

In order to show that the LEBESGUE area $A(T)$ is also cyclicly additive under unrestricted factorizations of $T$, two approaches suggest themselves. The first one consists of establishing that $A(T)$ satisfies the conditions of E. J. MICKLE and T. RADÓ to ensure the application of their theorem to $A(T)$. This can indeed be done by a method similar to that used in T. RADÓ [**4**] to prove (1). The second approach, the one followed by E. J. MICKLE and T. RADÓ [**3**], consists of making use of the fact that $A(T)$ is already known to be cyclicly additive under monotone-light factorizations of $T$. These considerations led then to the study of the problems outlined below.

Let $\mathfrak{T}$ be the class of all continuous mappings $T$ from a PEANO space $P$ into a metric space $P^*$, and let $\Phi$ be a real-valued non-negative functional defined for each $T \in \mathfrak{T}$. We shall say that $\Phi(T)$ is *weakly additive* if the following condition holds. For $T = lm$, $m : P \Rightarrow \mathfrak{M}$, $l : \mathfrak{M} \to P^*$ a monotone-light factorization of $T \in \mathfrak{T}$, we have

$$(4) \qquad \Phi(T) = \sum \Phi(l\, r_C m), \qquad C \subset \mathfrak{M},$$

where $r_C$ is the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element $C$ of $\mathfrak{M}$.

If $T = sf$, $f : P \to \mathfrak{M}$, $s : \mathfrak{M} \to P^*$ is an unrestricted factorization of $T \in \mathfrak{T}$ and if $\Phi(T)$ satisfies

$$(5) \qquad \Phi(T) = \sum \Phi(s\, r_C f), \qquad C \subset \mathfrak{M},$$

for every $T \in \mathfrak{T}$ and all unrestricted factorizations of $T$, then we shall term $\Phi(T)$ *strongly additive*. The problem now is to find conditions in order that weak additivity implies strong additivity. In this connection, E. J. MICKLE and T. RADÓ [**3**] obtained the following result:

Theorem. Under the assumption that $P$ is a unicoherent PEANO space, a functional $\Phi(T)$ which is weakly additive and lower semi-continuous is also strongly additive.

Since the above theorem arose from the study of the LEBESGUE area, the condition of lower semicontinuity imposed upon the functional $\Phi(T)$ is not surprising. The main result of this paper is the following: The condition of lower semi-continuity of $\Phi(T)$ can be deleted, i.e., the conclusion of the above theorem remains valid if $\Phi(T)$ is only weakly additive. It is also shown that the condition that $P$ be unicoherent cannot in general be omitted.

Since extensive use is made of the theory of $A$-sets and proper cyclic elements of a PEANO space, it seemed advisable to include in the first two parts of this paper the essential properties of $A$-sets and proper cyclic elements.

## I. – $A$-sets and Proper Cyclic Elements.

**I.1.** – A metric space which is a continuous image of the unit interval $0 \leqslant t \leqslant 1$ is termed a *Peano space*. The following definition is due to G. T. WHYBURN [**5**].

Definition. Let $A$ be a non-degenerate subset of a PEANO space $P$, i.e., $A$ consists of more than one point. Then $A$ will be called an *$A$-set* of $P$, provided (i) $A$ is closed in $P$, (ii) if $P - A \neq 0$, and $G$ is a component of $P - A$, then the frontier of $G$ is a single point (in $A$).

It should be noted that the whole space $P$ is also an $A$-set of $P$.

**I.2.** – In this paragraph we shall state those properties which are needed in the sequel. For the proofs the reader is referred to G. T. WHYBURN [**5**]. Let $P$ a PEANO space.

(i) Let $E$ be a cyclic subset of $P$, i.e., $E$ and, for every $x \in E$, $E - x$ is connected. If $A$ is an $A$-set of $P$ such that $A \cap E$ is non-degenerate, then $E \subset A$.

(ii) Let $\mathfrak{A}$ be a collection of $A$-sets of $P$. If $H = \cap A$, $A \in \mathfrak{A}$ is non-degenerate, then $H$ is an $A$-set of $P$.

(iii) Let $\mathfrak{A}$ be a family of $A$-sets of $P$ such that the intersection of any two distinct $A$-sets of $\mathfrak{A}$ is either empty or else a single point. Then the collection $\mathfrak{A}$ is denumerable.

(iv) Let $P^*$ be a PEANO subspace of $P$, and let $A$ be an $A$-set of $P$. If $P^* \cap A$ is non-degenerate, then $A^* = P^* \cap A$ is an $A$-set of $P^*$. If $P^*$ is an $A$-set of $P$ and if $A^*$ is an $A$-set of $P^*$, then $A^*$ is also an $A$-set of $P$.

(v) There exists a unique continuous and monotone retraction $r_A$ from $P$ onto an $A$-set $A$ of $P$, i.e., $r_A : P \Longrightarrow A$ and $r_A(x) = x$ for $x \in A$. Hence $A$ is a PEANO subspace of $P$.

Remark. Let $A$ be an $A$-set of $P$ and let $A'$ be an $A$-set of $A$. If $r_A$, $r'_{A'}$, $r_{A'}$ denote the monotone retractions from $P$ onto $A$, $A$ onto $A'$, and $P$ onto $A'$, respectively, then $r_{A'} = r'_{A'} \, r_A$.

**I.3.** – Definition: A non-degenerate subset $C$ of a PEANO space $P$ will be termed a *proper cyclic element* of $P$ if and only if $C$ is a cyclic $A$-set of $P$.

Since the above definition is slightly different from the one given in G. T. WHYBURN [**5**], we shall prove in the following three paragraphs the properties of proper cyclic elements needed in the sequel.

**I.4.** – Lemma. Let $E$ be a non-degenerate cyclic subset of a PEANO space $P$. Then there exists a unique proper cyclic element $C$ of $P$ such that $E \subset C$.

Proof. Let $\mathfrak{A}$ be the class of all $A$-sets of $P$ containing $E$. Since $P \in \mathfrak{A}$, the class $\mathfrak{A}$ is not empty. Let $A_0 = \cap\, A$, $A \in \mathfrak{A}$. Since $A_0$ is closed and non-degenerate, $A_0$ is an $A$-set of $P$ (**I.2** (ii)). We assert that $A_0$ is cyclic. If we deny this, there is a point $x \in A_0$ such that $A_0 - x$ is not connected. Since $E - x$ is connected, let $G$ be the component of $A_0 - x$ containing $E - x$. Denote by $\mathrm{Fr}_0$ the frontier operation with respect to $A_0$. Then $\mathrm{Fr}_0(G) = x$, and $A = G \cup x$ is a proper closed subset of $A_0$. We now assert that $A$ is an $A$-set of $A_0$. Let $Q$ be a component $A_0 - A$. Then $\mathrm{Fr}_0\,(Q) \subset A$ and $\mathrm{Fr}_0\,(Q) \cap G = 0$. Therefore, $\mathrm{Fr}_0\,(Q) = x$, and by **I.1**, $A$ is an $A$-set of $A_0$. Since $A_0$ is an $A$-set of $P$, there follows from **I.2** (iv) that $A$ is also an $A$-set of $P$. Since $E \subset A$ and $A$ is a proper subset of $A_0$, we have a contradiction to $A_0 = \cap\, A$, $A \in \mathfrak{A}$. Hence, $A_0$ is a cyclic $A$-set containing $E$, and hence $A_0$ is a proper cyclic element. The uniqueness follows from **I.2** (i).

**I.5** – Lemma. Let $A$ be an $A$-set of a PEANO space $P$. Then the proper cyclic elements of $A$ coincide with the proper cyclic elements of $P$ which are subsets of $A$.

Proof: The proof is an immediate consequence of **I.2** (iv).

**I.6** – From **I.2** (i) and **I.2** (iii) we infer the following further properties of proper cyclic elements.

(i) Two distinct proper cyclic elements of a PEANO space $P$ are either disjoint or else have a single point in common.

(ii) There is at most a denumerable number of proper cyclic elements of a PEANO space $P$.

**I.7.** – The definition of an $A$-set and a proper cyclic element given in **I.1** and **I.3** is easily seen to be equivalent to the definition of an $A$-set and a proper cyclic element of T. RADÓ [**4**].

## II. – Mappings of $A$-sets and Proper Cyclic Elements.

**II.1.** – For the proofs of the results in this paragraph the reader is referred to E. J. MICKLE and T. RADÓ [**3**].

(i) Let $m : P \Longrightarrow \mathfrak{M}$ be a monotone mapping from a PEANO space $P$ onto a PEANO space $\mathfrak{M}$. For every $A$-set $A$ of $P$ the following conditions are satisfied.

(1) $m$ is monotone on $A$.

(2) $\mathfrak{A} = m(A)$ is either an $A$-set or a single point.

(3) For the monotone retraction $r_A$ from $P$ onto $A$ and $r_{\mathfrak{A}}$ from $\mathfrak{M}$ onto $\mathfrak{A}$, $m\, r_A = r_{\mathfrak{A}}\, m$.

(ii) Assume that $P$ is a dendrite, i.e., $P$ is a PEANO space with no proper cyclic elements. Let $m : P \Longrightarrow \mathfrak{M}$ be a monotone mapping. Then $\mathfrak{M}$ is also a dendrite.

(iii) Let $m : P \Longrightarrow \mathfrak{M}$ be a monotone mapping from a PEANO space $P$ onto a PEANO space $\mathfrak{M}$. If $\mathfrak{C}$ is a proper cyclic element of $\mathfrak{M}$, then there exists a unique proper cyclic element $C$ of $P$ such that $m(C) \supset \mathfrak{C}$.

**II.2.** – Let $l : P \Longrightarrow \mathfrak{M}$ be a light mapping from a PEANO space $P$ onto a PEANO space $\mathfrak{M}$. For $C$ a proper cyclic element of $P$, the set $l(C)$ need not be cyclic and hence need not lie in a proper cyclic element of $\mathfrak{M}$. However, if $P$ is *unicoherent*, i.e., if for any two continua (connected closed sets) $F_1$, $F_2$

whose union is $P$, $F_1 \cap F_2$ is also a continuum, then we have the following results.

(1) There is a unique proper cyclic element $\mathfrak{C}$ of $\mathfrak{M}$ such that $l(C) \subset \mathfrak{C}$.

(2) If $r_c$ is the monotone retraction from $P$ onto $C$, and if $r_{\mathfrak{C}}$ is the monotone retraction from $\mathfrak{M}$ onto $\mathfrak{C} \supset l(C)$, then $lr_c = r_{\mathfrak{C}} lr_c$.

For the proof of the above two statements, the reader is referred to E. J. MICKLE [2].

**II.3.** – Concerning non-unicoherent PEANO spaces we have the following

Lemma. Let $P$ be a non-unicoherent PEANO space and let $\gamma$ be a simple arc. Then there exists a continuous mapping $T: P \Longrightarrow \gamma$ such that, if $T = lm$, $m: P \Longrightarrow \mathfrak{M}$, $l: \mathfrak{M} \Longrightarrow \gamma$ is a monotone-light factorization of $T$, $\mathfrak{M}$ contains at least one proper cyclic element (see LIEBERKNECHT [1]).

Proof. There is no loss of generality by assuming that $\gamma$ coincides with the unit interval $I \equiv [0 \leqslant x \leqslant 1]$. Since $P$ is not unicoherent, there are two continua $A$, $B$ of $P$ such that $A \cup B = P$ and $A \cap B = H \cup K$, where $H$, $K$ are non-empty disjoint closed sets. Define $T$ by

$$T(p) = \frac{\varrho(p, H)}{\varrho(p, H) + \varrho(p, K)}, \qquad p \in P,$$

where, e.g., $\varrho(p, H)$ denotes the distance from $p$ to $H$. It follows that $T$ is a continuous mapping from $P$ onto $I$, and $T(p) = 0$ if and only if $p \in H$, $T(p) = 1$ if and only if $p \in K$. Let $T = lm$, $m: P \Longrightarrow \mathfrak{M}$, $l: \mathfrak{M} \Longrightarrow I$ be a monotone-light factorization of $T$. We assert that $\mathfrak{M}$ is not unicoherent. For that purpose we will establish the following relation:

$$(1) \qquad m(A) \cap m(B) = m(A \cap B).$$

Since $m(A) \cap m(B) \supset m(A \cap B)$ is obvious, let $y \in m(A) \cap m(B)$. Since $m$ is monotone, $m^{-1}(y)$ is connected and $m^{-1}(y) \cap A \neq 0$, $m^{-1}(y) \cap B \neq 0$. Since, finally, $P = A \cup B$, there follows $m^{-1}(y) \cap A \cap B \neq 0$. Thus $y \in m(A \cap B)$, and (1) is proved. Therefore, $m(A) \cap m(B) = m(A \cap B) = m(H) \cup m(K)$, and $m(H) \cap m(K) = 0$. Since $m(A)$, $m(B)$ are two continua whose union is $\mathfrak{M}$, and since $m(H)$, $m(K)$ are closed, we have that $\mathfrak{M}$ is not unicoherent. However, every dendrite is unicoherent, and hence $\mathfrak{M}$ contains at least one proper cyclic element.

**II.4.** – Since a monotone image of a unicoherent PEANO space is again a unicoherent PEANO space, we have in view of **II.2** (1) and **II.3** the following characterization of unicoherent PEANO spaces (see also LIEBERKNECHT [1]).

Theorem. Let $P$ be a PEANO space and let $P^*$ be a non-degenerate dendrite. Denote by $\mathfrak{T}$ the class of all continuous mappings from $P$ into $P^*$. Then $P$ is unicoherent if and only for every $T \in \mathfrak{T}$, the middle space $\mathfrak{M}$ in a monotone-light factorization of $T$ is a dendrite.

## III. – Weak and Strong Cyclic Additivity.

**III.1.** – Let $P$ be a fixed PEANO space and let $P^*$ be a fixed metric space. Denote by $\mathfrak{T}$ the class of all continuous mappings from $P$ into $P^*$.

Definition: An *unrestricted factorization* of a continuous mapping $T \in \mathfrak{T}$ consists of a PEANO space $\mathfrak{M}$, called *middle space*, and two continuous mappings $s$, $f$ such that $f : P \to \mathfrak{M}$, $s : \mathfrak{M} \to P^*$, $T = sf$.

**III.2.** – Let $\Phi(T)$ be a functional defined for every $T \in \mathfrak{T}$ such that $\Phi(T)$ is real-valued and non-negative. For certain $T \in \mathfrak{T}$ we may have $\Phi(T) = +\infty$.

We shall say that $\Phi$ satisfies the *property* $\alpha_1$ or that $\Phi$ is *weakly additive* if for every $T \in \mathfrak{T}$, $\Phi(T) = \sum \Phi(l r_C m)$, $C \subset \mathfrak{M}$, where $T = lm$, $m : P \Longrightarrow \mathfrak{M}$, $l : \mathfrak{M} \to P^*$ is a monotone-light factorization of $T$, and where $r_C$ is the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element $C$ of $\mathfrak{M}$.

Let $T = lf$, $f : P \to \mathfrak{M}$, $l : \mathfrak{M} \to P^*$ be an *arbitrary-light* factorization of $T \in \mathfrak{T}$, i.e., $f$ is a continuous mapping from $P$ into a PEANO space $\mathfrak{M}$ and $l$ is a light mapping from $\mathfrak{M}$ into $P^*$ such that $T = lf$. We shall say that $\Phi$ satisfies the *property* $\alpha_2$ if $\Phi(T) = \sum \Phi(l r_C f)$, $C \subset \mathfrak{M}$, for every $T \in \mathfrak{T}$ and for all arbitrary-light factorizations of $T$.

In case $\Phi(T) = \sum \Phi(s\, r_C\, f)$, $C \subset \mathfrak{M}$, for every $T \in \mathfrak{T}$ and for all unrestricted factorizations $T = sf$, $f : P \to \mathfrak{M}$, $s : \mathfrak{M} \to P^*$, we shall say that $\Phi$ satisfies the *property* $\alpha_3$ or that $\Phi$ is *strongly additive.*

Remark. Assume that $\Phi$ satisfies either one of the three conditions $\alpha_1$, $\alpha_2$, $\alpha_3$. If the middle space $\mathfrak{M}$ contains no proper cyclic elements, we agree that $\Phi(T) = 0$. In particular, if $T \in \mathfrak{T}$ is constant, then $\Phi(T) = 0$. For then $T$ admits of a monotone-light factorization whose middle space reduces to a single point, and hence contains no proper cyclic elements.

It is obvious that the property $\alpha_3$ implies the property $\alpha_2$ which in turn implies the property $\alpha_1$, i.e., $\alpha_3 \to \alpha_2 \to \alpha_1$. Concerning the converse implications, we will establish the following results: always $\alpha_2 \to \alpha_3$, and in case $P$ is unicoherent, $\alpha_1 \to \alpha_2$.

**III.3.** – Lemma. Assume that $\Phi$ satisfies either one of the three conditions $\alpha_1$, $\alpha_2$, $\alpha_3$, say $\alpha_2$. Let $T = lf$, $f: P \to \mathfrak{M}$, $l: \mathfrak{M} \to P^*$ be an arbitrary-light factorization of $T \in \mathfrak{T}$. Assume that there is a denumerable number of subsets $\{A_i\}$ of $\mathfrak{M}$ such that (1) for each $i$, $A_i$ is either an $A$-set or a single point of $\mathfrak{M}$, (2) for each proper cyclic element $C$ of $\mathfrak{M}$ there is one and only one $A_i$ containing $C$. Then, if $r_i$ denotes the monotone retraction from $\mathfrak{M}$ onto $A_i$, we have $\Phi(T) = \sum_i \Phi(l\, r_i\, f)$.

Proof. Let $K_i$ be the class of proper cyclic elements of $\mathfrak{M}$ contained in $A_i$. Let for $C \in K_i$, $\bar{r}_c$, $r_c$ be the monotone retractions from $A_i$ onto $C$, and from $\mathfrak{M}$ onto $C$, respectively. From **I.2** (v) we have $\bar{r}_c\, r_i = r_c$. Since for each $i$, $l r_i f$ admits of an arbitrary-light factorization $r_i f: P \to A_i$, $l: A_i \to P^*$, we have (see also **I.5**),

$$\sum_i \Phi(l r_i f) = \sum_i \sum_{C \in K_i} \Phi(l \bar{r}_c r_i f) = \sum_i \sum_{C \in K_i} \Phi(l r_c f) = \sum_{C \subset \mathfrak{M}} \Phi(l r_c f) = \Phi(T).$$

**III.4.** – Theorem: *If $\Phi$ satisfies the property $\alpha_2$, then $\Phi$ also satisfies the property $\alpha_3$.*

Proof. Let $T = sf$, $f: P \to \mathfrak{M}$, $s: \mathfrak{M} \to P^*$ be an unrestricted factorization of a mapping $T \in \mathfrak{T}$, and let $s = l_1 m_1$, $m_1: \mathfrak{M} \Longrightarrow \mathfrak{M}_1$, $l_1: \mathfrak{M}_1 \to P^*$ be a monotone-light factorization of $s$. Then $T = l_1 m_1 f$, $m_1 f: P \to \mathfrak{M}_1$, $l_1: \mathfrak{M}_1 \to P^*$ is an arbitrary-light factorization of $T$. We will show that

$$\Phi(T) = \sum \Phi(s\, r_c\, f), \qquad C \subset \mathfrak{M}, \tag{1}$$

where $r_c$ is the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element $C$ of $\mathfrak{M}$.

Case 1. $\mathfrak{M}$ is a dendrite. Since $m_1: \mathfrak{M} \Longrightarrow \mathfrak{M}_1$ is monotone, we have as a consequence of **II.1** (ii) that $\mathfrak{M}_1$ is also a dendrite. Since $\Phi$ satisfies the property $\alpha_2$, $\Phi(T) = 0$. The right side of (1) is also zero, and hence (1) follows in this case.

Case 2. If $\mathfrak{M}$ is not a dendrite, let $\{C_i\}$ be the sequence of proper cyclic elements of $\mathfrak{M}$. For each $i$, let $m_1(C_i) = A_i$. By **II.1** (i), $A_i$ is either a single point or an $A$-set of $\mathfrak{M}_1$. If $r_i$ denotes the monotone retraction from $\mathfrak{M}_1$ onto $A_i$, we have from **II.1** (i), $m_1\, r_{c_i} = r_i\, m_1$.

Hence $s r_{c_i} f = l_1 m_1 r_{c_i} f = l_1 r_i m_1 f$. Since each proper cyclic element $C_1$ of $\mathfrak{M}_1$ is contained in one and only one $A_i$ (see **II.1** (iii)), the collection of sets $\{A_i\}$ satisfies the conditions of **III.3**. Hence, since $\Phi$ satisfies the property $\alpha_2$,

$$\Phi(T) = \sum_i \Phi(l_1 r_i m_1 f) = \sum_i \Phi(s\, r_{c_i} f).$$

**III.5.** – Let $T = lm$, $m : P \Longrightarrow \mathfrak{M}$, $l : \mathfrak{M} \to P^*$ be a monotone-light factorization of $T \in \mathfrak{T}$. Assume that there is a PEANO space $\mathfrak{M}_1$ and two monotone mappings $m_1$, $m_2$ such that $m = m_2 m_1$ and $m_1 : P \Longrightarrow \mathfrak{M}_1$, $m_2 : \mathfrak{M}_1 \Longrightarrow \mathfrak{M}$. Let $C$, $C_1$ be the generic notation for a proper cyclic element of $\mathfrak{M}$, $\mathfrak{M}_1$, respectively, and denote by $r_C$, $r_{C_1}$ the respective monotone retractions.

Lemma. Under the above conditions, let $K_1$ be the class of proper cyclic elements $C_1$ of $\mathfrak{M}_1$ for which $m_2(C_1)$ is not a single point. If $\Phi$ satisfies the property $\alpha_1$, then

$$\Phi(T) = \sum \Phi(l m_2 r_{C_1} m_1), \qquad C_1 \in K_1 . \tag{1}$$

Proof.

Case 1. $\mathfrak{M}_1$ is a dendrite. Since $m_2 : \mathfrak{M}_1 \Longrightarrow \mathfrak{M}$ is monotone, it follows from **II.1** (ii) that $\mathfrak{M}$ is also a dendrite. Consequently, $\Phi(T) = 0$. The right side of (1) is also zero, and (1) follows in this case.

Case 2. If $\mathfrak{M}_1$ is not a dendrite, let $\{ C_1^i \}$ be the sequence of proper cyclic elements of $\mathfrak{M}_1$. The set $A_i = m_2(C_1^i)$ is either a single point or an $A$-set of $\mathfrak{M}$ (see **II.1** (i)). Moreover, if $r_i$ is the monotone retraction from $\mathfrak{M}$ onto $A_i$, we have $m_2 r_{C_1^i} = r_i m_2$. Since in view of **II.1** (iii), the sequence $\{ A_i \}$ satisfies the conditions of **III.3**, we conclude that $\Phi(T) = \sum_i \Phi(l\, r_i\, m)$. However, for each $i$, $l r_i m = l r_i m_2 m_1 = l m_2 r_{C_1^i} m_1$, and hence

$$\Phi(T) = \sum \Phi(l m_2 r_{C_1} m_1), \qquad C_1 \subset \mathfrak{M}_1 . \tag{2}$$

The formula (1) follows now from (2) by making the observation that for $C_1 \notin K_1$, $l m_2 r_{C_1} m_1$ is constant and hence $\Phi(l m_2 r_{C_1} m_1) = 0$.

**III.6.** – In the following we will have to restrict $P$ to be a unicoherent PEANO space. Let $T = lf$, $f : P \to \mathfrak{M}$, $l : \mathfrak{M} \to P^*$ be an arbitrary-light factorization of a mapping $T \in \mathfrak{T}$, and let $f = l_1 m_1$, $m_1 : P \Longrightarrow \mathfrak{M}_1$, $l_1 : \mathfrak{M}_1 \to \mathfrak{M}$ be a monotone-light factorization of $f$. Then $T = l\, l_1 m_1$, $m_1 : P \Longrightarrow \mathfrak{M}_1$, $l\, l_1 : \mathfrak{M}_1 \to P^*$ is a monotone-light factorization of $T$. $\mathfrak{M}_1$, being a monotone image of a unicoherent PEANO space, is itself a unicoherent PEANO space. If $C_1$ is a proper cyclic element of $\mathfrak{M}_1$, then by **II.2** there is a unique proper cyclic element $C$ of $\mathfrak{M}$ such that $C \supset l_1(C_1)$. For a given proper cyclic element $C$ of $\mathfrak{M}$, let $K_C$ be the class of proper cyclic elements $C_1$ of $\mathfrak{M}_1$ such that $l_1\,(C_1) \subset C$. We observe that $K_C$ may be empty.

**III.7.** – Under the same conditions as **III.6**, let $C$ be a fixed proper cyclic element of $\mathfrak{M}$, and let $r_c$ be the monotone retraction from $\mathfrak{M}$ onto $C$. Consider the mapping $r_c l_1 : \mathfrak{M}_1 \to C$, and let $r_c l_1 = l_2 m_2$, $m_2 : \mathfrak{M}_1 \Longrightarrow \mathfrak{M}_2$, $l_2 : \mathfrak{M}_2 \to C$ be a monotone-light factorization of $r_c l_1$. For $C_1$ a proper cyclic element of $\mathfrak{M}_1$, let $r_{c_1}$ denote the monotone retraction from $\mathfrak{M}_1$ onto $C_1$.

Lemma: If $\Phi$ satisfies the property $\alpha_1$, then

$$\Phi(l r_c f) = \sum \Phi(l l_1 r_{c_1} m_1), \qquad C_1 \in K_c . \tag{1}$$

Proof. Since for $C_1 \in K_c$, $l_1(C_1)$ is a non-degenerate subset of $C$, $K_c$ is the class of proper cyclic elements $C_1$ of $\mathfrak{M}_1$ for which $m_2\,(C_1)$ is not a single point. Since $l\, r_c\, f = l\, r_c\, l_1\, m_1 = l\, l_2\, m_2\, m_1$, we infer from **III.5**,

$$\Phi(l r_c f) = \sum \Phi(l l_2 m_2 r_{c_1} m_1), \qquad C_1 \in K_c . \tag{2}$$

Since on $C_1 \in K_c$, $l_2\, m_2 = r_c\, l_1 = l_1$, the formula (1) follows from (2).

**III.8.** – Theorem. *Let $P$ be a unicoherent Peano space. If $\Phi$ satisfies the property $\alpha_1$, then $\Phi$ satisfies also the property $\alpha_2$.*

Proof. Let $T = lf$, $f : P \to \mathfrak{M}$, $l : \mathfrak{M} \to P^*$ be an arbitrary-light factorization of $T \in \mathfrak{T}$, and let $f = l_1\, m_1$, $m_1 : P \Longrightarrow \mathfrak{M}_1$, $l_1 : \mathfrak{M}_1 \to \mathfrak{M}$ be a monotone-light factorization of $f$. We will show that

$$\Phi(T) = \sum \Phi(l r_c f), \qquad C \subset \mathfrak{M}. \tag{1}$$

Case 1. $\mathfrak{M}$ is a dendrite. Since $P$ is unicoherent, and since $f : P \to \mathfrak{M}$ is continuous, we have from **II.4** that $\mathfrak{M}_1$ is also a dendrite. Since $\Phi$ satisfies the property $\alpha_1$, $\Phi(T) = 0$. The right side of (1) is also zero, and hence (1) follows in this case.

Case 2. If $\mathfrak{M}$ is not a dendrite, let $C$ be a proper cyclic element of $\mathfrak{M}$ and let us form the class $K_c$. Since every proper cyclic element $C_1$ of $\mathfrak{M}_1$ is in one and only one class $K_c$, and since $\Phi$ satisfies the property $\alpha_1$, we have by **III.7**,

$$\sum_{C \subset \mathfrak{M}} \Phi(l r_c f) = \sum_{C \subset \mathfrak{M}} \sum_{C_1 \in K_c} \Phi(l l_1 r_{c_1} m_1) = \sum_{C_1 \subset \mathfrak{M}_1} \Phi(l l_1 r_{c_1} m_1) = \Phi(T).$$

**III.9.** – Theorem. *Let $P$ be a unicoherent Peano space. If $\Phi$ is weakly additive, then $\Phi$ is also strongly additive.*

Proof: The proof follows from **III.8** and **III.4**.

**III.10.** – Theorem: *Under the assumption that $P$ is a non-unicoherent Peano space and that $P^*$ contains a non-degenerate Peano space, there exists a functional $\Phi$ which is weakly additive but not strongly additive.*

Proof. Define a functional $\Phi$ as follows. Let $T = lm$, $m : P \Longrightarrow \mathfrak{M}$, $l : \mathfrak{M} \to P^*$ be a monotone-light factorization of $T \in \mathfrak{T}$. Define $\Phi(T)$ to be the number of proper cyclic elements in $\mathfrak{M}$. We observe that $\Phi$ is well-defined, since for every other monotone-light factorization of $T$, the middle spaces are homeomorphic (see T. Radó [**3**]). Since $P^*$ contains a non-degenerate Peano space, we have a simple arc $\gamma^*$ in $P^*$. By **II.3** there is a continuous mapping $T \in \mathfrak{T}$ such that $T : P \Longrightarrow \gamma^*$, and the middle space $\mathfrak{M}$ in a monotone-light factorization of $T$ contains a proper cyclic element. By definition, $\Phi(T) > 0$. However, $T$ admits of an unrestricted factorization whose middle space is the simple arc $\gamma^*$. Thus, if $\Phi$ were strongly additive, we should have $\Phi(T) = 0$. Therefore, $\Phi$ is not strongly additive.

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