Riv. Mat. Univ. Parma 7 (1956), 211-233

L. CARLITZ (*)

A Special Functional Equation. (**)

1. – Introduction.

GLAISCHER [5], [6] showed how the convergence of certain familiar series can be improved by making use of simple algebraic identities. BRADSHAW [3] indicated how results of this kind can obtained by means of the identity

$$2\,f(x+1)\,f(x-1) - x\,f(x+1)\,g(x-1) - x\,f(x-1)\,g(x+1) = 2C, \tag{1.1}$$

where

$$\begin{cases} f(x) = x^{2n} + a_1\,x^{2n-2} + a_2\,x^{2n-4} + \ldots + a_n, \\ g(x) = x^{2n-1} + b_1\,x^{2n-3} + \ldots + b_{n-1}\,x \end{cases} \tag{1.2}$$

and $C = f^2(1)$. He found $f(x)$ for $n \leqslant 4$ and also stated the value of $f(0)$, $f(1)$, $f(2)$.

In the present Note we find the general polynomial solution of (1.1) and discuss various properties of the polynomials $f(x)$ and $g(x)$. Since the numerical coefficients occurring in (1.2) are positive integers, it is of interest to seek arithmetic properties of $f(x)$ and $g(x)$. We find in particular that they satisfy congruences similar to those satisfied by the polynomials of HERMITE and LAGUERRE [4].

2. – Since it is no more difficult we shall assume in place of (1.2) that

$$\begin{cases} f(x) = f_n(x) = x^n + a_1\,x^{n-2} + a_2\,x^{n-4} + \ldots, \\ g(x) = g_n(x) = x^{n-1} + b_1\,x^{n-3} + b_2\,x^{n-5} + \ldots. \end{cases} \tag{2.1}$$

(*) Address: Department of Mathematics, Duke University, Durham, North Carolina, U.S.A..

(**) Received February 8, 1956.

In (1.1) replace x by $x+2$ and subtract corresponding members. We get

$$(2.2) \qquad f(x+1)\{2f(x-1)-2f(x+3)-x\,g(x-1)+(x+2)\,g(x+3)\}+$$
$$+g(x+1)\{(x+2)f(x+3)-x f(x-1)\}=0\,.$$

We shall assume $C\neq 0$; then clearly (1.1) implies that $f(x+1)$ and $g(x+1)$ are relatively prime polynomials. Consequently (2.2) implies that $(x+2)f(x+3)-x f(x-1)$ is divisible by $f(x+1)$; examining the coefficient of x^n in the former we get

$$(x+2)f(x+3)-x f(x-1)=(4n+2)f(x+1),$$

or what is the same thing

$$(2.3) \qquad (x+1)f(x+2)-(x-1)f(x-2)=(4n+2)f(x).$$

We also get

$$(2.4) \qquad (x+1)\,g(x+2)-(x-1)\,g(x-2)-2f(x+2)+2f(x-2)=$$
$$=(4n+2)\,g(x)\,.$$

Now (2.3) implies

$$f(3)=(2n+1)f(1), \qquad 4f(5)=2f(1)+(4n+2)f(3),$$

and so on. Thus $f(3)$, $f(5)$, ... are evaluated in terms of $f(1)$. Since $f(x)$ is a polynomial it is accordingly determined, say by the LAGRANGE interpolation formula, as a multiple of $f(1)$. Since the highest coefficient in $f(x)$ is 1, the value of $f(1)$ is uniquely determined and hence $f(x)$ is fixed. Returning to (1.1) it is readily seen from the equation

$$f(x+1)\{f(x-1)-x\,g(x-1)\}+f(x-1)\{f(x+1)-x\,g(x+1)\}=2C$$

that for given $f(x)$, there exists at most one polynomial $g(x)$ of the required degree. Hence there is at most one pair of polynomials of the form (2.1) that satisfy (1.1).

3. – To find an explicit solution of (1.1) we consider first the polynomial $A_n(x)$ defined by

(3.1) $$A(t) = \frac{(1+t)^u}{(1-t)^{u+1}} = \sum_{n=0}^{\infty} A_n(u)\, t^n.$$

(BATEMAN [**2**] has discussed briefly the polynomial $g_n(z, r)$ defined by

$$(1+t)^{z+r}\,(1-t)^{-z} = \sum_{n=0}^{\infty} t^n\, g_n(z, r)$$

and in particular the case $r = 0$. See also the additional references in [**2**]).
It is clear from (3.1) that

(3.2) $$A_n(u) = \sum_{r+s=n} \binom{u}{r}\binom{u+s}{s},$$

which can also be given the form

(3.3) $$A_n(u) = \sum_{s=0}^{n} \binom{n}{s}\binom{u+s}{n}.$$

It follows from (3.1) that

(3.4) $$\sum_{0}^{\infty} (n+1)\, A_{n+1}(u)\, t^n = A'(t) = \frac{(2u+1+t)\,(1+t)^{u-1}}{(1-t)^{u+2}},$$

which implies the recurrence

(3.5) $$(n+1)\, A_{n+1}(u) = (2u+1)\, A_n(u) + n\, A_{n-1}(u).$$

Since

$$A(t) = \frac{1}{1-t}\left(1 + \frac{2t}{1-t}\right)^u = \sum_{r=0}^{\infty} \binom{u}{r} \frac{2^r\, t^r}{(1-t)^{r+1}},$$

we have also

(3.6) $$A_n(u) = \sum_{r=0}^{n} 2^r \binom{n}{r}\binom{u}{r};$$

this implies the following relation useful later

$$A_n(m) = A_m(n), \tag{3.7}$$

provided m is an integer $\geqslant 0$.

A formula similar to (3.6) is

$$A_m(2u) = \sum_{r=0}^{m} 4^r \binom{u}{r}\binom{m+r}{2r},$$

which is a consequence of the identity

$$\sum_0^\infty A_m(2u)\, t^m = \frac{1}{1-t}\left\{1 + \frac{4t}{(1-t)^2}\right\}^u.$$

We remark also that (3.6) is equivalent to

$$2^n\binom{u}{n} = \sum_{r=0}^{n} (-1)^{n-r}\binom{n}{r} A_r(u), \tag{3.8}$$

while (3.3) is equivalent to

$$2^n\binom{u+n}{n} = \sum_{r=0}^{n} \binom{n}{r} A_r(u). \tag{3.9}$$

Since (3.1) implies

$$A_n(-u-1) = (-1)^n A_n(u), \tag{3.10}$$

(3.8) and (3.9) are equivalent.

The recurrence (3.5) evidently determines $A_n(u)$ uniquely, given the initial conditions $A_0(u) = 1$, $A_1(u) = 2u + 1$. Note that it follows from (3.7) that

$$A_n(1) = A_1(n) = 2n + 1. \tag{3.11}$$

We now seek a second solution $C_n(u)$ of (3.5) such that $C_0(u) = 0$, $C_1(u) = 1$. (For a discussion of the polynomials *associated* with the classical orthogonal polynomials, see TOSCANO [11].) Put

$$C(t) = \sum_0^\infty C_n(u)\, t^n = A(t)\, F(t), \tag{3.12}$$

so that

$$C'(t) = \sum_0^\infty (n+1)\, C_{n+1}(u)\, t^n =$$

$$= 1 + \sum_1^\infty \left\{ (2u+1)\, C_n(u) + n\, C_{n-1}(u) \right\} t^n = 1 + (2u+1)\, C(t) + t \left\{ t\, C(t) \right\}'.$$

This yields

$$(1-t^2)\, C'(t) = 1 + (2u+1+t)\, C(t).$$

But by (3.4) and (3.12)

$$C'(t) = A(t)\, F'(t) + \frac{2u+1+t}{1-t^2} A(t)\, F(t)$$

and therefore

$$F'(t) = \frac{1}{(1-t^2)\, A(t)} = \frac{(1-t)^u}{(1+t)^{u+1}}. \tag{3.13}$$

Since $F(0) = 0$, $F'(0) = 1$, comparison with (3.1) yields

$$F(t) = \sum_{n=1}^\infty (-1)^{n-1} \frac{A_{n-1}(u)}{n} t^n. \tag{3.14}$$

Consequently (3.12) gives

$$C_n(u) = \sum_{r=1}^n \frac{(-1)^{r-1}}{r} A_{r-1}(u)\, A_{n-r}(u). \tag{3.15}$$

In (3.13) take $u = m$, a positive integer. Then repeated integration by parts leads to

$$F(t) = \sum_{r=0}^{m-1} \frac{(-1)^{r+1}}{m-r} \left(\frac{1-t}{1+t}\right)^{m-r} + (-1)^m \log(1+t) + \sum_{r=0}^{m-1} \frac{(-1)^r}{m-r};$$

hence by (3.12)

$$C(t) = \sum_{r=0}^{m-1} \frac{(-1)^{r+1}}{m-r} \frac{(1+t)^r}{(1-t)^{r+1}} + (-1)^m \frac{(1+t)^m}{(1-t)^{m+1}} \cdot \log(1+t) - (-1)^m \tau_m \frac{(1+t)^m}{(1-t)^{m+1}},$$

where $\tau_m = 1 - \frac{1}{2} + \dots + \frac{(-1)^{m-1}}{m}$.

Thus, using (3.1), we get

$$C_n(m) = \sum_{r=0}^{m-1} \frac{(-1)^{r+1}}{m-r} A_n(r) - \sum_{0}^{n-1} \frac{(-1)^{n-r+m}}{n-r} A_r(m) - (-1)^m \tau_m A_n(m). \tag{3.16}$$

Hence in view of (3.7), 3.16) implies

$$(-1)^m C_n(m) - (-1)^n C_m(n) = (\tau_n - \tau_m) A_n(m), \tag{3.17}$$

where m and n are arbitrary non-negative integers. Applying (3.10) to (3.15) we get also

$$C_n(-u-1) = (-1)^{n-1} C_n(u). \tag{3.18}$$

In particular (3.10) and (3.18) imply

$$A_{2n+1}(-1/2) = C_{2n}(-1/2) = 0. \tag{3.19}$$

We have also from (3.1)

$$A_{2n}\left(-\frac{1}{2}\right) = \frac{(2n)!}{2^{2n}(n!)^2}. \tag{3.20}$$

As for $C_{2n+1}(-1/2)$, it is evident from (3.5) that

$$(2n+1) C_{2n+1}(-1/2) = 2n\, C_{2n-1}(-1/2),$$

and this yields

$$C_{2n+1}\left(-\frac{1}{2}\right) = \frac{2n}{2n+1} \frac{2n-2}{2n-1} \cdots \frac{2}{3}. \tag{3.21}$$

We note also that

$$A_n(0) = 1, \qquad C_n(0) = 1 - \frac{1}{2} + \dots + \frac{(-1)^{n-1}}{n}. \tag{3.22}$$

4. – Now put

$$x = 2u + 1 \tag{4.1}$$

and define

$$f_n(x) = n!\, A_n(u), \qquad g_n(x) = n!\, C_n(u)\,. \tag{4.2}$$

It is clear from (3.11), (3.18), (4.1) and (4.2) that

$$f_n(-x) = (-1)^n f_n(x), \qquad g_n(-x) = (-1)^{n-1} g_n(x), \tag{4.3}$$

so that $f_n(x)$ and $g_n(x)$ are of the form (2.1).

In the next place, since $A_n(u)$, $C_n(u)$ both satisfy (3.5), it follows that

$$A_m(u)\, C_{m-1}(u) - A_{m-1}(u)\, C_m(u) = (-1)^m/m\,. \tag{4.4}$$

If we take $u = n$ and make use of (3.7), (4.4) becomes

$$A_n(m)\, C_{m-1}(n) - A_n(m-1)\, C_m(n) = (-1)^m/m.$$

Next applying (3.14) this becomes

$$A_n(m)\left\{-C_n(m-1) - (-1)^m(\tau_n - \tau_{m-1})\, A_n(m-1)\right\} +$$
$$+ A_n(m-1)\left\{-C_n(m) - (-1)^m(\tau_n - \tau_m)\, A_n(m)\right\} = (-1)^n/m,$$

which is the same as

$$A_n(m)\left\{A_n(m-1) - 2m\, C_n(m-1)\right\} + A_n(m-1)\left\{A_n(m) - 2m\, C_n(m)\right\} = (-1)^n\,.$$

Finally employing (4.2) we get

$$f_n(2m+1)\left\{f_n(2m-1) - 2m\, g_n(2m-1)\right\} +$$
$$+ f_n(2m-1)\left\{f_n(2m+1) - 2m\, g_n(2m+1)\right\} = (-1)^n (n!)^2\,. \tag{4.5}$$

Since (4.5) holds for all integral $m \geqslant 0$ it follows immediately on replacing $2m+1$ by x that

$$(4.6) \qquad 2\, f_n(x+1)\, f_n(x-1) - x\, f_n(x+1)\, g_n(x-1) - \\ - x\, f_n(x-1)\, g_n(x+1) = (-1)^n\, (n!)^2 .$$

In particular for n even, (4.6) is identical with (1.1).

It is clear from (3.5) and (4.2) that

$$(4.7) \qquad f_{n+1}(x) = x\, f_n(x) + n^2\, f_{n-1}(x);$$

also $g_n(x)$ satisfies a like recurrence. Thus the coefficients in $f_n(x)$ and $g_n(x)$ are positive integers. If we put

$$f_n(ix) = i^n\, F_n(x),$$

then clearly the coefficients of $F_n(x)$ are real and $F_n(x)$ satisfies

$$F_{n+1}(x) = x\, F_n(x) - n^2\, F_{n-1}(x) .$$

It follows (compare [**10**, p. 44]) that the roots of $F_n(x)$ are all real and consequently the roots of $f_n(x)$ are pure imaginary.

By means of (4.7) we find easily that

$$f_1(x) = x$$

$$f_2(x) = x^2 + 1$$

$$f_3(x) = x^3 + 5\, x$$

$$f_4(x) = x^4 + 14\, x^2 + 9$$

$$f_5(x) = x^5 + 30\, x^3 + 89\, x$$

$$f_6(x) = x^6 + 55\, x^4 + 439\, x^2 + 225$$

$$f_7(x) = x^7 + 91\, x^5 + 1519\, x^3 + 3429\, x$$

$$f_8(x) = x^8 + 140\, x^6 + 4214\, x^4 + 24940\, x^2 + 11025.$$

Similarly we find that

$$g_1(x) = 1$$

$$g_2(x) = x$$

$$g_3(x) = x^2 + 4$$

$$g_4(x) = x^3 + 13\,x$$

$$g_5(x) = x^4 + 29\,x \; + 64$$

$$g_6(x) = x^5 + 54\,x^3 + 389\,x$$

$$g_7(x) = x^6 + 90\,x^4 + 1433\,x^2 + 2304$$

$$g_8(x) = x^7 + 139\,x^5 + 4079\,x^3 + 21365\,x.$$

We have also, using (3.22),

$$f_n(1) = n!\,, \qquad g_n(1) = n!\left(1 - \frac{1}{2} + \ldots + \frac{(-1)^{n-1}}{n}\right),$$

while (3.20) and (3.21) yield

$$f_{2n}(0) = \left(\frac{(2n)!}{2^n n!}\right)^2 = 1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-1)^2,$$

$$g_{2n+1}(0) = 2^2 \cdot 4^2 \ldots (2n)^2 = 2^{2n}\,(n!)^2\,.$$

Explicit expressions for $f_n(2r+1)$ are readily obtained for small integral r by means of

$$f_n(2r+1) = n!\,A_n(r) = n!\,A_r(n) = \frac{n!}{r!}\,f_r(2n+1)\,.$$

Thus for example

$$f_n(3) = \frac{n!}{1!}\,f_1(2n+1) = n!\,(2n+1),$$

$$f_n(5) = \frac{n!}{2!}\,f_2(2n+1) = \frac{n!}{2}\left((2n+1)^2 + 1\right).$$

These values can also obtained easily by means of (2.3). It is then easy to compute $g_n(2r+1)$ by making use of (3.17), that is

$$(-1)^r g_n(2r+1) - (-1)^n \frac{n!}{r!} g_r(2n+1) = (\tau_n - \tau_r) f_n(2r+1) .$$

For example

$$g_n(3) = (-1)^{n-1} n! + \left(\frac{1}{2} - \dots + \frac{(-1)^n}{n}\right) n! \, (2n+1) .$$

The value of $f_n(2r)$ can also be obtained. Indeed if we take $u = r - \frac{1}{2}$ in (3.1), we get

$$A(t) = \frac{(1+t)^{r-\frac{1}{2}}}{(1-t)^{r+\frac{1}{2}}} = \frac{(1+t)^{2r}}{(1-t^2)^{r+\frac{1}{2}}}$$

which yields

$$A_n\left(r - \frac{1}{2}\right) = \sum_{2s \leq n} \binom{2r}{n-2s} \binom{r+s-(1/2)}{s}, \tag{4.8}$$

valid for arbitrary r. In particular for integral $r \geqslant 0$, we get

$$f_n(2r) = n! \sum_{n-2r \leq 2s \leq n} \binom{2r}{n-2s} \binom{r+s-(1/2)}{s} .$$

For example

$$f_{2n}(2) = (2n)! \left\{ \binom{n+(1/2)}{n} + \binom{n-(1/2)}{n-1} \right\} = (4n+1) \{ (2n-1)(2n-3) \dots 1 \}^2,$$

$$f_{2n+1}(2) = 2(2n+1)! \binom{n+(1/2)}{n} = 2 \{ (2n+1)(2n-1) \dots 1 \}^2.$$

In the next place if we put

$$f_n(x) = \sum_{2r \leq n} a_{nr} x^{n-2r}, \qquad g_n(x) = \sum_{2r < n} b_{nr} x^{n-1-2r}, \qquad (a_{n0} = b_{n0} = 1),$$

then it follows from (4.7) that

$$a_{n+1,\,r} = a_{n,\,r} + n^2\, a_{n-1,\,r-1}\,,$$

$$b_{n+1,\,r} = b_{n,\,r} + n^2\, b_{n-1,\,r-1}\,.$$

Thus $a_{n+1,1} - a_{n1} = n^2$, which yields

$$a_{n1} = (1/6)n(n-1)\,(2n-1)\,.$$

Similarly we find that

$$a_{n2} = (1/360)n(n-1)\,(n-2)\,(n-3)\,(20n^2 - 48n + 7)\,.$$

In general a_{nr} is a polynomial in n of degree $3r$. Since

$$a_{2n+1,\,n} = a_{2n,\,n} + 4n^2\, a_{2n-1,\,r-1}\,,$$

we find that

$$\text{(4.9)} \qquad a_{2n+1,\,n} = 2^2\cdot 4^2 \ldots (2n)^2 \left\{1 + \frac{1^2}{2^2} + \frac{1^2\, 3^2}{2^2\, 4^2} + \ldots + \frac{1^2\, 3^2 \ldots (2n-1)^2}{2^2\, 4^2 \ldots (2n)^2}\right\}.$$

We have also

$$b_{n1} = a_{n1} - 1 = (1/6)\, n(n-1)\,(2n-1) - 1 \qquad (n \geqslant 2),$$

$$b_{n2} = a_{n2} - a_{n1} + 5 \qquad (n \geqslant 3)\,.$$

Indeed it follows from (5.8)′ below that

$$b_{nr} = a_{nr} - \sum_{s=1}^{n} \frac{s}{(n-s)\,(n-s+1)}\, \frac{n!}{(n-2s)!}\, a_{n-2s,\,r-s}\,.$$

We remark also that

$$\text{(4.10)} \qquad b_{2n+2,\,n} = 3^2\cdot 5^2 \ldots (2n+1)^2 \left\{1 + \frac{2^2}{3^2} + \frac{2^2\, 4^2}{3^2\, 5^2} + \ldots + \frac{2^2\, 4^2 \ldots (2n)^2}{3^2\, 5^2 \ldots (2n+1)^2}\right\},$$

corresponding to (4.9).

We note that $f_n(x)$ and $g_n(x)$ can be expressed as determinants of the following type:

$$(4.11)\qquad f_n(x) = \begin{vmatrix} x & -1 & . & . & . & . \\ 1 & x & -2 & . & . & . \\ . & 2 & x & -3 & . & . \\ . & . & 3 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & x & -(n-1) \\ . & . & . & . & n-1 & x \end{vmatrix},$$

$$(4.12)\qquad g_n(x) = \begin{vmatrix} x & -2 & . & . & . & . \\ 2 & x & -3 & . & . & . \\ . & 3 & x & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & x & -(n-1) \\ . & . & . & . & n-1 & x \end{vmatrix}.$$

5. – We now derive some additional properties of $A_n(u)$ and $C_n(u)$, or, what is the same thing, of $f_n(x)$ and $g_n(x)$. By (3.1) we have

$$(5.1)\qquad \sum_{r=0}^{\infty} A_n(u)\, t^r \cdot \sum A_n(u)\, z^n = \frac{(1+t)^u (1+z)^u}{(1-t)^{u+1}(1-z)^{u+1}}.$$

Now the right member

$$= \frac{(1+tz+t+z)^u}{(1+tz-t-z)^{u+1}} = \frac{1}{1+tz} \frac{\left(1+\dfrac{t+z}{1+tz}\right)^u}{\left(1-\dfrac{t+z}{1+tz}\right)^{u+1}} =$$

$$= \sum_{n=0}^{\infty} A_n(u) \frac{(t+z)^n}{(1+tz)^{n+1}} =$$

$$= \sum_{r,s=0}^{\infty} \binom{r+s}{s} A_{r+s}(u)\, t^r z^s \sum_{k=0}^{\infty} (-1)^k \binom{r+s+k}{k} (tz)^k.$$

Since the coefficient of $t^r z^n$ in the left member of (5.1) is $A_r(u)\,A_n(u)$, we accordingly get

$$A_r(u)\,A_n(u) = \sum_{k=0}^{\min(r,n)} (-1)^k \frac{(r+n-k)!}{(r-k)!\,(n-k)!\,k!} A_{r+n-2k}(u)\,. \tag{5.2}$$

For the proof of (5.2) compare KALUZA [7, p. 691].

Using the first of (4.2), (5.2) becomes

$$f_r(x)\,f_n(x) = \sum_k (-1)^k \binom{r}{k}\binom{n}{k}\binom{r+n-k}{k} (k!)^2 f_{r+n-2k}(x), \tag{5.3}$$

and in particular, when $r = n$,

$$f_n^2(x) = \sum_k (-1)^k \binom{n}{k}^2 \binom{2n-k}{k} (k!)^2 f_{2n-2k}(x)\,. \tag{5.4}$$

If we apply (3.7) to (5.2) (with $u = m$), we may write

$$A_m(r)\,A_m(n) = \sum_h (-1)^k \frac{(r+n-k)!}{(r-k)!\,(n-k)!\,k!} A_m(r+n-2k) =$$

$$= \sum_h (-1)^k \binom{n}{k}\binom{r+n-k}{n} A_m(r+n-2k)\,.$$

Since this is true for $r = 0, 1, 2, \ldots$, it follows that

$$A_m(x)\,A_m(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+n-k}{n} A_m(x+n-2k) \tag{5.5}$$

for arbitrary x. In particular for $n = 1$, (5.5) reduces to

$$(2m+1)\,A_m(x) = (x+1)\,A_m(x+1) - x\,A_m(x-1),$$

which is equivalent to (2.3).

It may be of interest to mention the formula

$$(1-t-z-tz)^{-1} = \sum_{m,n=0}^{\infty} A_n(m)\,t^n z^m, \tag{5.6}$$

which is a direct consequence of (3.1).

Turning now to (3.12) we apply (5.2) to get

$$(5.7) \qquad C_n(u) = \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \sum_s (-1)^s \frac{(n-1-s)!}{(r-1-s)!\,(n-r-s)!\,s!} A_{n-1-2s}(u) =$$

$$= \sum_{2s<n} \frac{(n-1-s)!}{s!} A_{n-1-2s}(u) \sum_{r=s+1}^{n-s} \frac{(-1)^{r+s-1}}{r} \frac{1}{(r-1-s)!\,(n-r-s)!}.$$

Now the inner sum on the extreme right

$$= \frac{1}{(n-1-2s)!} \sum_{r=s+1}^{n-s} \frac{(-1)^{r+s-1}}{r} \binom{n-1-2s}{r-1-s} =$$

$$= \frac{1}{(n-1-2s)!} \sum_{k=0}^{n-1-2s} \frac{(-1)^k}{k+s+1} \binom{n-1-2s}{k}.$$

Using the familiar identity

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \frac{1}{r+x} = \frac{m!}{x(x+1)\ldots(x+m)},$$

this becomes

$$\frac{1}{(s+1)(s+2)\ldots(n-s)} = \frac{s!}{(n-s)!}.$$

Substituting in (5.7), we see that

$$(5.8) \qquad C_n(u) = \sum_{2s<n} \frac{1}{n-s} A_{n-1-2s}(u).$$

It follows from (3.5) that

$$(2u+1)\, C_n(u) = A_n(u) - \sum_{0<2s<n} \frac{s\, A_{n-2s}(u)}{(n-s)(n-s+1)},$$

or what is the same thing

$$(5.8)' \qquad x\, g_n(x) = f_n(x) - \sum_{0<2s<n} \frac{s}{(n-s)(n-s+1)} \frac{n!}{(n-2s)!} f_{n-2s}(x).$$

If we differentiate (3.1) with respect to u we get

$$\sum_{1}^{\infty} A_n'(u)\, t^n = \frac{(1+t)^u}{(1-t)^{u+1}} \cdot \log \frac{1+t}{1-t},$$

so that

$$A_n'(u) = 2 \sum_{2s<n} \frac{1}{2s+1} A_{n-1-2s}(u),$$

or

$$f_n'(x) = \sum_{2s<n} \frac{1}{2s+1} \frac{n!}{(n-1-2s)!} f_{n-1-2s}(x). \tag{5.9}$$

If we take $u = -1/2$ in (5.8) and use (3.18) and (3.19), we get the following curious result

$$\sum_{s=0}^{n} \frac{1}{n+s+1} \frac{(2s)!}{2^{2s}(s!)^2} = \frac{2n}{2n+1} \frac{2n-2}{2n-1} \cdots \frac{2}{3},$$

which is included in a formula of Bailey [1, p. 93].

Substituting from (3.6) in (5.8) we get

$$C_n(u) = \sum_{r=0}^{n-1} 2^r \binom{u}{r} \sum_{2s<n-r} \frac{1}{n-s} \binom{n-1-2s}{r}. \tag{5.10}$$

A formula like (5.2) for $A_r(u)\, C_n(u)$ can be obtained. Indeed more generally, if U_z satisfies

$$(z+1)\, U_{z+1} = (2u+1)\, U_z + z\, U_{z-1}, \tag{5.11}$$

where z is not necessarily integral, we have

$$A_r(u)\, U_z = \frac{1}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} (z-k+1)_r\, U_{z+r-2k}, \tag{5.12}$$

where $(z)_r = z(z+1) \dots (z+r-1)$. For $r=0$, (5.12) is obvious, while for $r=1$ we get

$$(2u+1)\, U_z = (z+1) U_{z+1} - z\, U_{z-1},$$

in agreement with (5.11). Then assuming the truth of (5.12) we get

$$(r+1)!\,A_{r+1}U_z = r!\,\big((2u+1)A_r + rA_{r-1}\big)\,U_z =$$

$$= (2u+1)\sum_k (-1)^k \binom{r}{k}(z-k+1)_r\,U_{z+r-2k} +$$

$$+ r^2 \sum_k (-1)^k \binom{r-1}{k}(z-k+1)_{r-1}\,U_{z+r-1-2k} =$$

$$= \sum_k (-1)^k \binom{r}{k}(z-k+1)_r \{(z+r+1-2k)\,U_{z+r+1-2k} - (z+r-2k)\,U_{z+r-1-2k}\} +$$

$$+ r^2 \sum_k (-1)^k \binom{r-1}{k}(z-k+1)_{r-1}\,U_{z+r-1-2k} =$$

$$= \sum_k (-1)^k \left\{\binom{r}{k}(z-k+1)_r\,((z-k+r+1)-k) + \right.$$

$$+ \binom{r}{k-1}(z-k+2)_r\,((z-k+1)+r-k+1) -$$

$$\left. - r^2\binom{r-1}{k-1}(z-k+2)_{r-1}\right\} U_{z+r+1-2k}.$$

Since

$$-k\binom{r}{k}(z-k+1)_r + (r-k+1)\binom{r}{k-1}(z-k+2)_r -$$

$$- r^2\binom{r-1}{k-1}(z-k+2)_{r-1} = 0,$$

the right member reduces to

$$\sum_{k=0}^{r+1} (-1)^k \binom{r+1}{k}(z-k+1)_{r+1}\,U_{z+r+1-2k},$$

thus completing the induction.

In particular, (5.12) implies

$$A_r(u)\,C_n(u) = \sum_{k=0}^{r} \frac{(r+n-k)!}{(r-k)!\,(n-k)!\,k!}\,C_{r+n-2k}(u) \tag{5.13}$$

for $r < n$. The condition $r < n$ is necessary since $C_n(u)$ satisfies

$$(n+1)\, C_{n+1}(u) = (2u+1)\, C_n(u) + n\, C_{n-1}(u)$$

for $n \geqslant 1$ only and negative values of the subscripts must be avoided.

For a formula like (5.12) in the case of the HERMITE polynomials see NIELSEN [**9**, pp. 31-33].

6. – It follows immediately from (3.5) that

$$(n+1)\left\{ A_{n+1}(u)\, A_n(v) - A_n(u)\, A_{n+1}(v) \right\} =$$

$$= \left\{ (2u+1)\, A_n(x) + n\, A_{n-1}(x) \right\} A_n(v) - A_n(u) \cdot \left\{ (2v+1)\, A_n(v) + n\, A_{n-1}(v) \right\} =$$

$$= 2(u-v)\, A_n(u)\, A_n(v) - n \cdot \left\{ A_n(u)\, A_{n-1}(v) - A_{n-1}(u)\, A_n(v) \right\}$$

and threfore

$$(6.1) \qquad 2(u-v) \sum_{r=0}^{n} (-1)^r A_r(u)\, A_r(v) = (-1)^n (n+1) \left\{ A_{n+1}(u)\, A_n(v) - A_n(u)\, A_{n+1}(v) \right\}.$$

In a similar manner we may prove the companion formulas

$$(6.2) \qquad 2(u-v) \sum_{r=1}^{n} (-1)^r C_r(u)\, C_r(v) = (-1)^n (n+1) \left\{ C_{n+1}(u)\, C_n(v) - C_n(u)\, C_{n+1}(v) \right\},$$

$$(6.3) \qquad 2(u-v) = \sum_{r=1}^{n} (-1)^r A_r(u)\, C_r(v) = 1 + (-1)^n (n+1) \left\{ A_{n+1}(u)\, C_n(v) - A_n(u)\, C_{n+1}(v) \right\}.$$

In particular for $u = v$ these formulas become

$$(6.4) \qquad 2 \sum_{r=0}^{n} (-1)^r A_r^2(u) = (-1)^n (n+1) \left\{ A'_{n+1}(u)\, A_n(u) - A'_n(u)\, A_{n+1}(u) \right\},$$

$$(6.5)\qquad 2\sum_{r=1}^{n}(-1)^r C_r^2(u) = (-1)^n (n+1)\{C'_{n+1}(u)\, C_n(u) - C'_n(u)\, C_{n+1}(u)\},$$

$$(6.6)\qquad 2\sum_{r=1}^{n}(-1)^r A_r(u)\, C_r(u) = (-1)^n (n+1)\{A'_{n-1}(u)\, C_n(u) - A'_n(u)\, C_{n+1}(u)\} =$$

$$= (-1)^n (n+1)\{C'_{n+1}(u)\, A_n(u) - C'_n(u)\, A_{n+1}(u)\}.$$

The formulas (6.1), (6.2), (6.3) are special cases of the following more general identity (6.7). Let $U_z(u)$, $V_z(u)$ denote any solutions of (5.11). Then

$$(z+n+1)\{U_{z+n+1}(u)\, V_{z+n}(v) - U_{z+n}(u)\, V_{z+n+1}(v)\} =$$

$$= 2(u-v)\, U_{z+n}(u)\, V_{z+n}(v) - (z+n)\{U_{z+n}(u)\, V_{z+n-1}(v) - U_{z+n-1}(u)\, V_{z+n}(v)\},$$

which gives

$$(6.7)\qquad 2(u-v)\sum_{r=1}^{n}(-1)^r U_{z+r}(u)\, U_{z+r}(v) =$$

$$= (-1)^n (z+n+1)\{U_{z+n+1}(u)\, V_{z+n}(v) - U_{z+n}(u)\, V_{z+n+1}(v)\} -$$

$$- (z+1)\{U_{z+1}(u)\, V_z(v) - U_z(u)\, V_{z+1}(v)\}.$$

In connection with (6.1), we note that it follows from (3.1) that for $|a| < 1$,

$$\sum_{r=0}^{\infty}(-1)^r a^r \sum_{m=0}^{\infty} A_m(r)\, t^m \cdot \sum_{n=0}^{\infty} A_n(r)\, z^n =$$

$$\sum_{r=0}^{\infty}(-1)^r a^r \frac{(1+t)^r}{(1-t)^{r+1}}\,\frac{(1+z)^r}{(1-z)^{r+1}} = \{(1+a)(1+tz) + (1-a)(t+z)\}^{-1}.$$

Consequently

$$(6.8)\qquad \lim_{a\to 1-0}\sum_{r=0}^{\infty}(-1)^r a^r A_m(r)\, A_n(r) = \frac{(-1)^m}{2}\,\delta_{mn}\ ;$$

in other words the series

$$(6.9)\qquad \sum_{r=0}^{\infty}(-1)^r A_m(r)\, A_n(r)$$

is summable (A). Moreover, since by (3.6)

$$A_m(r) \sim \frac{2^m r^m}{m!} \qquad (r \to \infty)$$

for fixed m, it is clear that (6.9) is not convergent. However we can assert more than (6.8), namely that (6.9) is summable $(C, m+n+1)$. Indeed this is a consequence of the fact that $A_m(r) \cdot A_n(r)$ is a polynomial in r of degree $m+n$ [**8**, p. 496].

In view of (6.8) it may be of interest to compare $A_n(u)$ with the KRAWTCHOUK polynomials (see [**10**, pp. 34-36]). Thus (6.8) corresponds to the orthogonality relation for the latter. However $A_n(x)$ is not a special case of the KRAWTCHOUK polynomials.

7. – Arithmetic properties.

It is proved in [**4**, Theorem 1] that if $a(n)$, $b(n)$ are polynomials in n with integral coefficients and u_n is defined by means of

$$u_{n+1} = a(n)\, u_n + b(n)\, u_{n-1}, \tag{7.1}$$

$$u_0 = 1, \qquad u_1 = a(0), \qquad b(0) = 0, \tag{7.2}$$

then we have

$$\Delta^{2r} u_n \equiv \Delta^{2r-1} u_n \equiv 0 \qquad (\text{mod } m^r), \tag{7.3}$$

for all $n \geqslant 0$, $r \geqslant 1$, where

$$\Delta^r u_n = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} u_{n+sm}\, u_{(r-s)m} \tag{7.4}$$

and m is an arbitrary integer. In particular (7.3) contains

$$u_{n+m} \equiv u_m\, u_n \qquad (\text{mod } m). \tag{7.5}$$

Note also that the polynomials $a(n)$, $b(n)$ are allowed to contain additional indeterminates.

Now since the polynomial $f_n(x)$ satisfies

$$f_{n+1}(x) = x\, f_n(x) + n^2 f_{n-1}(x),$$

it is evident that the conditions of the quoted theorem are satisfied. Thus to begin with, (7.5) implies

$$f_{n+m}(x) \equiv f_n(x)\, f_m(x) \pmod{m}. \tag{7.6}$$

Since by (4.2) and (3.6)

$$f_m(x) = m!\sum_{r=0}^{m}\binom{m}{r}\binom{u}{r} = \sum_{r=0}^{m} 2^r\binom{m}{r}\frac{m!}{r!}u(u-1)\dots(u-r+1),$$

where $x = 2u+1$, we get

$$f_m(x) \equiv (x-1)(x-3)\dots(x-2m+1) \pmod{m}. \tag{7.7}$$

Thus (7.6) becomes

$$f_{n+m}(x) \equiv (x-1)(x-3)\dots(x-2m+1) f_m(x) \pmod{m}. \tag{7.8}$$

If we replace u by an odd integer c, then (7.7) reduces to $f_m(2c+1) \equiv 0 \pmod{m}$. Note also that for m equal to an odd prime p, we have

$$f_p(x) \equiv x^p - x \pmod{p}. \tag{7.9}$$

Indeed (7.9) is a special case of

$$f_{n+p}(x) \equiv (x_p - x)\, f_n(x) \pmod{p}, \tag{7.10}$$

which is implied by (7.8); the following special case of (7.10) may be noted:

$$f_{mp}(x) \equiv (x^p - x)^m \pmod{p}. \tag{7.11}$$

We also remark that (7.7) implies

$$f_{p^r}(x) \equiv (x^p - x)^{p^{r-1}} \pmod{p^r}. \tag{7.12}$$

In the next place the general result (7.3) becomes

$$\Delta^{2r} f_n(x) \equiv \Delta^{2r-1} f_n(x) \equiv 0 \pmod{m^r}, \tag{7.11}$$

where now

$$\Delta^r f_n(x) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f_{n+sm}(x)\, f_{(r-s)m}(x)\,. \tag{7.12}$$

However a stronger result than (7.11) can be asserted. Indeed by (5.2) and (4.2)

$$\Delta^r f_n(x) =$$

$$= \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \sum_k (-1)^k \frac{(n+rm-k)!\,(n+sm)!\,\big((r-s)m\big)!}{(n+sm-k)!\,\big((r-s)m-k\big)!\,k!\,(n+rm-2k)!} f_{n+rm-2k}(x) =$$

$$= \sum_k (-1)^k \binom{n+rm-k}{k} f_{n+rm-2k}(x) \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \psi(s),$$

where

$$\psi(s) = (n+sm-k+1)_k \big((r-s)\,m-k+1\big)_k\,.$$

Clearly

$$\psi(s) = a_0 + a_1\, sm + \ldots + a_{2k}\,(sm)^{2k},$$

where the a_i are integers. Then

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \psi(s) = \Delta^r \psi(0) = \sum_{i=r}^{2k} a_i\, m^2\, \Delta^r 0^i = 0 \qquad (\text{mod } r!\cdot m^r)$$

since $\Delta^r 0^i \equiv 0 \pmod{r!}$. It therefore follows that

$$\Delta^r f_n(x) \equiv 0 \qquad (\text{mod } r!\cdot m^r) \tag{7.13}$$

for all $r \geqslant 1$. For $r \geqslant 2$ this result is more precise than (7.11); for $r = 1$ it reduces to (7.6). In connection with (7.13) it may be of interest to mention that (5.2) implies

$$f_{rm}(x)\, f_{nm}(x) \equiv f_{rm+nm}(x) \qquad (\text{mod } m^2)\,. \tag{7.14}$$

The theorem stated at the beginning of this section does not apply to $g_n(x)$ since (7.2) is not satisfied. However exactly as in the proof of (7.13) we may prove that

$$\Delta^r g_n(x) \equiv 0 \pmod{r!\cdot m^r}, \tag{7.15}$$

where now

$$\Delta^r g_n(x) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} g_{n+sm}(x)\, f_{(r-s)m}(x) . \tag{7.16}$$

In particular, for $r = 1$, (7.15) becomes

$$g_{n+m}(x) \equiv f_m(x)\, g_n(x) \equiv (x-1)(x-3)\ldots(x-2m+1)\, g_n(x) \pmod{m}. \tag{7.17}$$

Also corresponding to (7.14) we have

$$f_{rm}(x)\, g_{nm}(x) \equiv g_{rm+nm}(x) \equiv g_{rm}(x)\, f_{rm}(x) \pmod{m^2}. \tag{7.18}$$

We remark that (5.8) implies

$$g_m(x) \equiv f_{m-1}(x) \pmod{m} . \tag{7.19}$$

References.

1. W. N. Bailey, *Generalized hypergeometric series*, Cambridge Univ. Press., 1935.
2. H. Bateman, *The polynomial of Mittag-Leffler*, Proc. Nat. Acad. Sci. U.S.A. **26** (1940), 491-496.
3. J. W. Bradshaw, *An infinite product for* (1/4)π *derived form Gregory's series*, Quart. J. Math. **43** (1912), 378-379.
4. L. Carlitz, *Congruence properties of the polynomials of Hermite, Laguerre and Legendre*, Math. Z. **59** (1954), 474-483.
5. J. W. L. Glaisher, *On a method of increasing the convergence of certain series for* π, π^2, *etc.*, Quart. J. Math. **34** (1903), 87-98.
6. J. W. L. Glaisher, *Methods of increasing the convergence of certain series of reciprocals*, Quart. J. Math. **34** (1903), 252-347.
7. Th. Kaluza, *Elementarer Beweis einer Vermutung von K. Friedrichs und H. Lewy*, Math. Z. **37** (1933), 689-697.

8. K. KNOPP, *Theorie und Anwendungen der unendlichen Reihen*, J. Springer, Berlin 1931 (dritte Auflage).

9. N. NIELSEN, *Recherches sur les polynomes d'Hermite*, Danske Vid. Selsk. Mat. Fys. Medd. I, **6** (1918), 1-78.

10. G. SZEGÖ, *Orthogonal polynomials*, Amer. Math. Soc. Col. Pub., Vol. **23**, New York 1939.

11. L. TOSCANO, *Polinomi associati a polinomi classici*, Riv. Mat. Univ. Parma **4** (1953), 387-402.