Riv. Mat. Univ. Parma 7 (1956), 149-185

LAMBERTO CESARI (*)

Fine-Cyclic Elements of Surfaces of the Type ν. (**)

In my book **Surface Area** [**3** (A)] I have discussed surfaces $S = (T, A)$ defined as single-valued continuous (not necessarily one-one) mappings from any admissible plane set $A \subset E_2$ into E_3, or E_N. By an admissble set A is meant either (1) any closed finitely connected JORDAN region J, or any finite sum of disjoint such regions; or (2) any plane open set G; or (3) any set $A \subset A_1$, open in A_1, where A_1 is any set as in (1). The main theorems for area, as discussed in [**3** (A)], all hold for such mappings (T, A).

Two tools have been used in the book [**3** (A)] which have been developed there only for surfaces $S = (T, A)$ defined as continuous mappings from a simple closed JORDAN region A, namely the concept of contour and the one of retraction. Indeed this was sufficient for the proof of the main theorems on area for all continuous mappings from any admissible set.

In the present paper we shall define a new the concept of retraction (§ **6**) for continuous mappings $S = (T, A)$ from a finitely connected JORDAN region $A = J$, or mappings, or surfaces of the ν-type, where ν is the connectivity of J, $0 \leqslant \nu < +\infty$.

New features will be observed for $\nu \geqslant 1$ which have no analogue in the case where J is a simple JORDAN region ($\nu = 0$). It may for instance occur that a surface of type $\nu \geqslant 1$ presents a system of « leaves » linked together to form a unique cyclic element of a rather complex structure while each leaf may be of an extremely simple type, namely of the type of the disc (see numerous examples in § **9**).

(*) Address: Department of Mathematics, Purdue University, Lafayette, Indiana, U. S. A..

(**) Received September 20, 1956.

To account for these simple elements of a surface, the new concept of fine-cyclic element σ of a surface $S = (T, J)$ of the type ν will be introduced (§ **9**) as a retraction of S according to the new concept of retraction.

The concept and the properties of fine-cyclic elements will be discussed in great details. The fine-cyclic elements may be actually « finer » than the usual cyclic elements of the same surface when $\nu \geqslant 1$, while they coincide with the usual cyclic elements when $\nu = 0$, i.e., when J is a simple closed JORDAN region. The theory is developed here independently of the usual cyclic element theory, and is self-contained, in the sense that it is based on known properties of Elementary Topology of the Euclidean plane. A fine-cyclic additivity theorem for LEBESGUE area is given (§ **18**), which is analogous to the well known MORREY's and RADÓ's cyclic additivity theorem, and states that the LEBESGUE area of a surface is equal to the countable sum of the LEBESGUE areas of its fine-cyclic elements. Finally, the fine-cyclic elements are characterized as retractions having particular maximal-minimal properties (§ **9**, § **17**).

The technique used in the present paper can be traced back in my book [**3** (A)] and in my previous papers [**3** (a)], [**3** (b)], where I discussed only questions of retraction of mappings from simple JORDAN regions.

In papers in course of publications, C. J. NEUGEBAUER has extended the present theory of fine-cyclic elements to general continuous mappings from a PEANO space, by combining the present approach with methods of Analytic Topology.

1. – Generalities.

Let $J = J_0 - (J_1 + ... + J_\nu)^0$, $J_i \subset J_0^0$, $J_i J_j = 0$, $(i \neq j;\ i, j = 1, ..., \nu)$, be any closed JORDAN region of order of connectivity ν $(0 \leqslant \nu < +\infty)$, of the oriented Euclidean w-plane π, $w = (u, v)$, let E be the Euclidean p-space, $p = (x_1, ..., x_n)$, and $|w - w'|$, $|p - p'|$ the Euclidean distances of two points $w, w' \in \pi$, $p, p' \in E$. As usual we denote by $\overline{I}$, I^*, I^0 the closure, the boundary, and the set of the interior points of any given set $I \subset \pi$, or $I \subset E$. Finally we shall use the usual notations $\operatorname{diam} A$, $\{p, A\}$, $\{A, B\}$ for the diameter of a set A, the distance of a point p from a set A, and for the distance of two sets A, B, respectively.

Let (T, J): $p = p(w)$, $w \in J$, be any single-valued continuous mapping (not necessarily one-one) from J into E, i.e., a surface $S = (T, J)$ in E. If no identifications are made on the boundary $J^* = J_0^* + ... + J_\nu^*$ of J we say that S, or (T, J), are of the ν-type. We shall denote by S also the FRÉCHET surface defined by (T, J); i.e., the class of all mappings (T', J') which are FRÉCHET equivalent to (T, J) [**3** (A)].

Let us assume as positive the counterclockwise orientation on J_0^* and the clockwise orientations on $J_1^*, \ldots, J_\nu^*$. Then (T, J) defines on J^* certain oriented closed curves $C_0, \ldots, C_\nu$ which may be not simple and even reduced to single points. The ordered system $C = \partial S = [C_0, \ldots, C_\nu]$ is said to be the boundary of the surfaces S.

Given any two ordered systems $C = [C_0, \ldots, C_\nu]$, $C' = [C_0', \ldots, C_\nu']$ of oriented closed continuous curves in E, we denote by $\| C, C' \|$, or FRÉCHET distance of C and C', the number

$$\| C, C' \| = \sum_{i=0}^{\nu} \| C_i, C_i' \| ,$$

where $\| C_i, C_i' \|$ denote the usual FRÉCHET distance [**3** (A), p. 15] of any two curves C_i, C_i'.

We shall denote as usual by $[S]$ the graph of S, that is, the set $S = T(J) \subset E$ of the points covered by S, by $[C]$ the graph of $\partial S = C$, that is, the set $[C] = T(J^*)$ covered by the curves $C_0, C_1, \ldots, C_\nu$, by $L(S) = L(T, J)$ the LEBESGUE area of the surface S. We shall denote by $\Gamma = \Gamma(T, J)$ the decomposition of J into disjoint maximal continua g of constancy for T in J.

2. – Properties of the components γ of the complement of a continuum K in J.

Let $K \subset J$ be a (nonvacuous) continuum. From the elements of topology we know that $J - K$ is a set open in J, and that the collection $\{ \gamma \}_K = \{ \gamma \}_{KJ}$ of the components γ of $J - K$ is at most countable.

(**2.** i) For every $\gamma \in \{ \gamma \}_K$ the set $\gamma\gamma^*$ is either empty, or a subset of J^*. In this second case we have

$$\gamma\gamma^* = \sum_{i=0}^{\nu} \gamma\gamma^* J_i^* ,$$

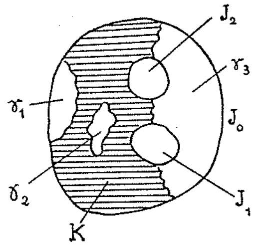
and each $\gamma\gamma^* J_i^*$ which is not empty, either coincides with J_i^*, or is an open arc l_i of J_i^* whose end points (not necessarily distinct) are both in K.

Proof. For every $w \in \gamma\gamma^*$ there is a neighborhood U of w with $UJ \subset \gamma$, $\gamma U \neq 0$, $(\pi - \gamma) U \neq 0$. By second relation and $\gamma \subset J$ we deduce $JU \neq 0$; by first and third relation we deduce $(\pi - J) U \neq 0$. Thus $w \in J^*$ and finally $\gamma\gamma^* \subset J^*$. Then $\gamma\gamma^*$ is a subset of J^* open in J^*, and hence each $\gamma\gamma^* J_i$, if not empty, is a collection of open disjoints arcs of J^*. Suppose that such a collection contain two of these arcs say $(\alpha\beta)$, $(\alpha'\beta')$ (α, β, α', β' ordered on J_i^*). Let w, w' any two ponts interior to $(\alpha\beta)$, $(\alpha'\beta')$ respectively. Then $w, w' \in \gamma$ and hence there is an arc $l \subset \gamma$ joning w, w'. Also, $w, w' \in J_i$ and hence there is an arc l' in $J_i^0 + (w) + (w')$ [or $\pi - J_0 + (w) + (w')$ if $i = 0$] joining w, w'. Now the closed curve $l + l'$ separates π into two parts with β, α' in one, and α, β' in the other, with $(l + l')K = 0$, $\alpha, \beta, \alpha', \beta' \in K$, a contradiction. Thus $\gamma\gamma^* J_i^*$ is either empty, or a simple open arc of J_i^* (with end points not necessarily distinct).

For every $\gamma \in \{\gamma\}_K$ we shall denote by $F(\gamma)$ the set $F(\gamma) = \gamma^* - \gamma\gamma^* = \bar{\gamma} - \gamma$, or boundary of γ in J.

(**2.** ii) For every $\gamma \in \{\gamma\}_K$ the set $F(\gamma)$ is a non-empty compact subset of K.

We have only to prove that $F(\gamma) \neq 0$. Indeed, if $F(\gamma) = 0$, the sets $J - \gamma$, γ would constitute a separation of J which is impossible.



We shall divide the collection $\{\gamma\}_K$ into two subclasses $\{\gamma\}'_K$, $\{\gamma\}''_K$ by putting γ in $\{\gamma\}'_K$ if either $\gamma\gamma^* = 0$, or $\gamma\gamma^* \neq 0$, and $\gamma\gamma^* J_i^* \neq 0$ for one and only one $i = 0, 1, \dots, \nu$; by putting γ in $\{\gamma\}''_K$ if $\gamma\gamma^* \neq 0$ and $\gamma\gamma^* J_i^* \neq 0$ for at least two $i = 0, 1, \dots, \nu$. We say all $\gamma \in \{\gamma\}'_K$ are of the *first class*, and all $\gamma \in \{\gamma\}''_K$ are of the *second class*.

For instance in the illustration J is a region of order $\nu = 2$, and $K \subset J$ is a continuum whose collection $\{\gamma\}_K$ has two elements γ_1, γ_2 of the first class, and one element γ_3 of the second class.

(**2.** iii) For every $\gamma \in \{\gamma\}'_K$ the set $F(\gamma)$ is a continuum. More precisely we shall prove:

(**2.** iv) For every γ with $\gamma\gamma^* = 0$ we have $\gamma^* = F(\gamma) \subset K$ and $F(\gamma)$ is a continuum. For every γ with $\gamma\gamma^* \neq 0$, $\gamma\gamma^* J_i \neq 0$, and $\gamma\gamma^* \neq J_i^*$ for one $i = 0, 1, \dots, \nu$, the set $F(\gamma)$ is a continuum containing both end points α, β of the open arc $l = l_i = \gamma\gamma^* J_i^*$ of J_i^*, and $\gamma^* = \gamma\gamma^* + F(\gamma) = l + F(\gamma)$. There may be countably many components $\gamma \in \{\gamma\}'_K$, of one or the other of these two types.

Proof. Case 1. $\gamma\gamma^* = 0$. Then $F(\gamma) = \bar{\gamma} - \gamma = \gamma^* - \gamma\gamma^* = \gamma^*$ and now we shall suppose that γ^* is not a continuum. Then γ is not simply connected and there is at least a simple polygonal region q with $q^* \subset \gamma$, such that both in q^0 and $\pi - q$ there are points of γ^*. Now $\gamma^* \subset K + J^*$ and K cannot have points both inside and outside q^*. Thus in one of the two parts the points of

γ^* are all in J^*, and also in γ. Thus $\gamma\gamma^*J^* \neq 0$, a contradiction. This assures that $F(\gamma)$ is a continuum if $\gamma\gamma^* = 0$.

Case 2. $\gamma\gamma^* \neq 0$, $\gamma\gamma^*J_i^* \neq 0$ for only one i and $\gamma\gamma^* \neq J_i$. Then $l = \gamma\gamma^*J_i^*$ is an arc of J_i^* of end points α, β (not necessarily distinct). Obviously $\alpha, \beta \in K$, α, $\beta \in F(\gamma)$. Suppose $F(\gamma)$ is not a continuum. Then $F(\gamma)$ is compact, its components are continua, and thus α, β belong to the same or different components of $F(\gamma)$.

If $\alpha, \beta \in k$ where k is a component of $F(\gamma)$, then there should be another component k' of $F(\gamma)$ and $[(\alpha) + (\beta) + l + k]k' = 0$. Let q^* be any simple closed polygonal line, $q^* \subset \pi - F(\gamma)$ separating k and k' in π. We can suppose all points of q^* so close to k', that l is completely outside q^*. If we take any two points w, w' of γ close enough to k and k' respectively then they are separated by q^* and thus a line $l \subset \gamma$ joining w, w' in γ must encounter q^* in some point w_0, $w_0 \in q^*$, $w_0 \in \gamma$. If we move along q^* from w_0 in any direction, we cannot encounter points not in γ since then there should be also points of $F(\gamma)$ on q^*, what has been excluded. Thus $q^* \subset \gamma$, $q^*K = 0$. Now $\alpha, \beta \in K$; on the other hand $k'J^* = 0$, and thus $k' \subset K$. Therefore, there are points of K both inside and outside q^*, a contradiction.

Finally suppose that α, β belong to different components, say k, k' of $F(\gamma)$, $\alpha \in k$, $\beta \in k'$. This implies that $\alpha \neq \beta$ and that α, β divide J_i^* into two parts $l = (\alpha\beta) \subset \gamma$, $l' = J_i^* - (l + \alpha + \beta)$ non containing points of γ. Now let q be a simple poligonal region with $q^* \subset \pi - F(\gamma)$ separating k and k' in π. Suppose for instance that $K' \subset q$. Then $\beta \subset q^o$, and there are points of q both inside and outside J_i^*. Now q^* must have at least one point w in l and at least one point w' in l'. Finally $w \in \gamma$ as well as any other point $w \in lq^*$. There must be also a subarc λ of q^* joining w to w' and having no other point on l but w. Now if w'' is the first point of λ on J^*, then w'' is not on l. If w'' is on l' then w'' is on γ, a contradiction; if w'' is on some J_u^*, $u \neq i$, then $w'' \in \gamma$, $w'' \in \gamma^*$, $w'' \in J_u^*$, thus $w'' \in \gamma\gamma^*J_u^*$, $u \neq i$, a contradiction. All this proves (iv).

Let us consider now any $\gamma \in \{\gamma\}_K''$. For the sake of simplicity let $J_1'^*, J_2'^*, \dots, J_a'^*$ denote all those boundary curves J_i^* of J completely contained in $\gamma\gamma^*$ (if any), let $J_1''^*, \dots, J_b''^*$ denote all those boundary curves J_i^* of J with $\gamma\gamma^*J_i^* \neq 0$, $J_i^* - \gamma\gamma^* \neq 0$, and $l_1, \dots, l_b$ the b arcs $\gamma\gamma^*J_i^*$, that is $l_j = \gamma\gamma^*J_j''^*$ $(j = 1, 2, \dots, b)$. Finally let us denote by $k_1, k_2, \dots, k_c$ the components of $F(\gamma)$.

(**2.** v) For every $\gamma \in \{\gamma\}_K''$ the components $k_1, \dots, k_c$ of $F(\gamma)$ are finite in number and

(v₁) $\quad F(\gamma) = k_1 + k_2 + \dots + k_c, \qquad 1 \leqslant c \leqslant \nu + 1,$

(v₂) $\quad \gamma\gamma^* = J_1'^* + \dots + J_a'^* + l_1 + \dots + l_b, \qquad 2 \leqslant a + b \leqslant \nu + 1,$

(v₃) $\quad \gamma^* = J_1'^* + \dots + J_a'^* + (l_1 + \dots + l_b + k_1 + \dots + k_c).$

where $J_1'^*, \dots, J_a'^*$, $(l_1 + \dots + l_b + k_1 + \dots + k_c)$ are the $a + 1$ components of γ^*, and γ has order of connectivity a. The collection $\{\gamma\}_K''$ has at most ν elements.

More precisely we prove

(2. vi) If a, b are the numbers a, b above for each of the components $\gamma \in \{\gamma\}_K''$, then $1 \leqslant a + b - 1 \leqslant \nu$, and $\sum (a + b - 1) \leqslant \nu$.

Proof. The second formula of (v) is obvious and $\gamma^* = J_1'^* + \dots + J_a'^* + l_1 + \dots + l_b + F(\gamma)$. Let γ' be the set $\gamma' = J_1'^0 + \dots + J_a'^0 + \gamma$, where we substitute $\pi - J_0$ for J_0^0 if $J_i' = J_0$ for some i. Since $\{J_i', K\} > 0$, $\{J_i'^*, l_j\} > 0$, we have $\gamma'\gamma'^* = \gamma\gamma^* - \sum J_a'^* = l_1 + \dots + l_\nu$, $\gamma'^* = \gamma^* - \sum J_a'^* = l_1 + \dots + l_b + F(\gamma)$. Let us prove that γ' is simply connected. Let q be any simple polygonal region with $q^* \subset \gamma'$. Assume there is a point $w \in q^0$ such that $w \notin \gamma'$. Since K is connected we have $w \notin K$. Moreover, $w \notin J_i''$. Since $w \in J_0$, $w \in J_i'$ for some i. Since $J_i' \subset \gamma'$ we have a contradiction. We conclude that γ' is simply connected and that $\gamma'^* = L$ is a continuum, and hence $\gamma^* = J_1'^* + \dots + J_a'^* + L$, where L is the continuum $L = \gamma'^* = l_1 + \dots + l_b + F(\gamma)$.

The proof that $F(\gamma)$ is a sum of finitely many continua will be given below.

We shall now prove (vi). For each $\gamma \in \{\gamma\}_K''$ we may join any one of the arcs $J_1'^*, \dots, J_a'^*, l_1, \dots, l_b$, say l_1 for instance, to each one of the remaining ones by means of exactly $a + b - 1$ simple arcs (cuts) $\lambda \subset \gamma$. We prove now that this operation, which involves $\sum (a + b - 1)$ cuts, does not destroy the connection of J. Let us observe first that every point $w \in \gamma$ with $\gamma \in \{\gamma\}'$, $\gamma\gamma^* = 0$ can be joined to $F(\gamma)$, and thus to K, by means of the segment s joining w to the closest point w_0 of $F(\gamma)$, and $s \subset \gamma + (w_0)$, $w_0 \in K$. Every point $w \in \gamma$ with $\gamma \in \{\gamma\}'$, $\gamma\gamma^* \neq 0$, $\gamma\gamma^* J_i^* \neq 0$ for only one i, can be joined to the first end point $\alpha \in K$ of the arc $l = (\alpha\beta) = \gamma\gamma^* J_i^*$ by means of the arc $s = s' + s''$ defined as follows: s' is an arc joining w to a point w_1 interior to l, $s' \subset \gamma$; s'' is the subarc (w_1, α) of l. Every point $w \in \gamma$ with $\gamma \in \{\gamma\}''$ can be joined to K as follows: First let s' be any arc, $s \subset \gamma$, joining w to a joint w_1 of, say, the arc l_1, let s'' be the subarc (w_1, α_1) of the arc $l_1 = (\alpha_1, \beta_1)$. Thus $s = s' + s''$ joins w to $\alpha_1 \in K$. Now s may intersect the system $\sum$ of $a + b - 1$ arcs λ of γ. Then let w', w'' be the first and last points of s from w which are in $\sum$. First let us replace the arc $(w' w'')$ of s by means of an arc s'' sum of arcs of λ, $J_i'^*$, l_1. Then, by a slight modification of the arc $(w w') + s'' + (w'', \alpha_1)$ we can obtain, as usual, a line s_0 joining w to α_1 free of points of $\sum$. We conclude that each point $w \in J - \sum\lambda$, either belongs to the continuum K, or can be joined to K by means of a line s free of points of $\sum\lambda$, $s \subset J - \sum\lambda$. Thus $J - \sum\lambda$ is connected and has order of connection

$$\nu - \sum(a + b - 1) \geqslant 0.$$

This proves (vi) and since $a + b - 1 \geqslant 1$ for every $\gamma \in \{\gamma\}''$, we conclude that there are at most ν elements $\gamma \in \{\gamma\}''$.

We shall now prove (v_1) (this proof is due to C. J. NEUGEBAUER). We shall prove in § **3** the more stringent inequality $c \leqslant \max[1, b]$. The proof is by induction on ν. If $\nu = 0$ all components of $J - K$ are of the first kind, and hence $F(\gamma)$ is connected.

Assume now that (v_1) is true for every JORDAN region of connectivity $\nu - 1 \geqslant 0$ and for every continuum $K \subset J$. We shall now prove (v_1) in case J has connectivity ν. Let K be a continuum in J.

Case 1. $K \supset J_i^*$ for some i, $1 \leqslant i \leqslant \nu$. Then consider $J' = J + J_i$, $K' = K + J_i$. Then every component γ' of $J' - K'$ is a component γ of $J - K$ and conversely. Hence $F_\nu(\gamma) = F'_\nu(\gamma')$, and since J' has connectivity $\nu - 1$, the formula (v_1) follows in this case.

Case 2. $KJ_i^* = 0$ for some i, $1 \leqslant i \leqslant \nu$. Consider $J' = J + J_i$. Then every component γ of $J - K$ is contained in a component γ' of $J' - K$ and $F'(\gamma') = F(\gamma)$. Since J' has connectivity $\nu - 1$, (v_1) follows.

Case 3. For each i, $1 \leqslant i \leqslant \nu$, $KJ_i^* \neq 0$, $J_i^* - K \neq 0$. Let γ be a component of $J - K$ such that, say, $\gamma\gamma^* J_1^* \neq 0$. We may also assume that the number of components of $F(\gamma)$ is greater than ν.

Let $l_1 = \gamma\gamma^* J_1^*$. Then l_1 is an open arc with end points α, β in K. Let k_1, k_2 be the components of $F(\gamma)$ containing α, β respectively. Consider $J' = J + J_1$, $K' = K + J_1$. Then there is a component γ' of $J' - K'$ such that $\gamma' + l_i^0 = \gamma$. If we denote by $F'(\gamma')$ the boundary of γ' in J', we have $F'(\gamma') = k_1' + k_2' + \dots + k_c'$, $1 \leqslant c \leqslant \nu$. Now, say, $k_1' = k_1 + l_1 + k_2$, and $F(\gamma) = k_1 + k_2 + k_2' + \dots + k_c'$.

Thus formula (v_1) is proved and (v) is completely proved.

(**2.** vii) If $\nu = 0$ then all $\gamma \in \{\gamma\}_K$ are of the first class and $F(\gamma)$ is a continuum.

A corollary of (iv). For a direct proof see [**3** (A), p. 505].

(**2.** viii) If $[\gamma]$ is any subcollection of $\{\gamma\}_{KJ}$ then the set $K' = K + \sum \gamma$ where $\sum$ is extended over all $\gamma \in [\gamma]$ is also a continuum $K' \subset J$, and the collection $\{\gamma\}_{K'J}$ is made up by the elements $\gamma \in \{\gamma\}_{KJ}$ which are not in $[\gamma]$, i.e., $\{\gamma\}_{K'J} = \{\gamma\}_{KJ} - [\gamma]$.

The proof is the same as the one given for $\nu = 0$ in [**3** (A), 36.2, (ii), p. 507].

Remark. The previous considerations concerning the structure of the boundary of the components γ of sets $J - K$ where K is a continuum $K \subset J$, are immediately extended to the components γ of sets $J - S$, where $S = \sum K$ is the finite sum of disjoint continua $K \subset J$. We leave the corresponding formulations to the reader.

3. – Application of Carathéodory prime-end theory.

We can now apply CARATHÉODORY's theory of ends and prime ends to each component of the boundary γ^* of γ, $\gamma \in \{\gamma\}_K$ [see **1**, or **4**, or **3**(A)].

For every $\gamma \in \{\gamma\}'$ with $\gamma\gamma^* = 0$, the families $\{\eta\}_\gamma$, $\{\omega\}_\gamma$ of all ends and prime-ends of γ [that is, of $\gamma^* = F(\gamma)$ in γ] are cyclically ordered. Let us mention also that here and in the following cases we have always $\{\eta\} \subset \{\omega\}$, as usual.

For every $\gamma \in \{\gamma\}'$ with $\gamma\gamma^* J^* \neq 0$, $\gamma\gamma^* J_i^* \neq 0$ for only one i, (**2.** iv), we have $\gamma^* = k + l$, where $l = l_i = \gamma\gamma^* J_i^*$, k a continuum, $k = F(\gamma) \subset K$. The families $\{\eta\}_\gamma$, $\{\omega\}_\gamma$ of all ends and prime ends of γ [that is of γ^* in γ] are cyclically ordered and can be divided into two consecutive intervals, say $[\eta_1, \eta_2]$, $[\eta_2, \eta_1]$ by means of the two ends η_1, η_2 corresponding to the endings of the open arc $l = (\alpha\beta)$, $w_{\eta_1} = \alpha$, $w_{\eta_2} = \beta$. Then one of the intervals say $[\eta_1, \eta_2]$ corresponds to the single points of l; the other one to the ends and prime ends of K in γ.

For every $\gamma \in \{\gamma\}''$ the collections, say $\{\eta\}_{\gamma i}$, $\{\omega\}_{\gamma i}$ of ends and prime ends relative to the components J_i' of γ coincide with the single points of J_i^* and thus are trivial, $i = 1, 2, \ldots, a$. Let us consider now the component L of γ^*, $L = l_1 + \ldots + l_b + F(\gamma)$. The collections, say $\{\eta\}_{\gamma 0}$, $\{\omega\}_{\gamma 0}$ of all ends and prime ends of L in γ are also cyclically ordered and can be divided into $2b$ consecutive intervals

$$[\eta_0, \eta_1], \quad [\eta_1, \eta_2], \quad \ldots, \quad [\eta_{2b-2}, \eta_{2b-1}], \quad [\eta_{2b-1}, \eta_{2b}]$$

by means of the $2b$ ordered ends $\eta_0 < \eta_1 < \ldots < \eta_{2b-1} < \eta_{2b} = \eta_0$, where each pair η_{2i-2}, η_{2i-1} is defined, as above, by the two endings of the arcs l_i $(i = 1, 2, \ldots, b)$. Eeach odd interval $[\eta_{2i-2}, \eta_{2i-1}]$ corresponds to the single points of the open arc $l_i = \gamma\gamma^* J_i''^*$ $(i = 1, 2, \ldots, b)$. The remaining b intervals $[\eta_{2i-1}, \eta_{2i}]$ correspond to ends η and prime ends ω of $F(\gamma)$ in γ, and we shall consider here the b sets

$$m_i = E''_{\omega_{2i-1}} + \sum E_\omega + E'_{\omega_{2i}} \qquad (i = 1, 2, \ldots, b),$$

where the sum is extended over all prime ends $\omega \in (\eta_{2i-1}, \eta_{2i})$, where $E''_{\omega_{2i-1}}$ [$E'_{\omega_{2i}}$] is the right [left] wing of the prime end ω_{2i-1} [ω_{2i}] corresponding to the end η_{2i-1} [η_{2i}]. Let us observe that if we denote by (α_i, β_i) the arc l_i $(i = 1, 2, \ldots, b)$, then

$$w_{\eta_0} = \alpha_1, \quad w_{\eta_1} = \beta_1, \quad w_{\eta_2} = \alpha_2, \quad w_{\eta_3} = \beta_2, \quad \ldots, \quad w_{\eta_{2b-2}} = \alpha_b, \quad w_{\eta_{2b-1}} = \beta_b.$$

By [**3** (d)] we know that each set m_i $(i = 1, 2, \ldots, b)$, is a continuum and

$$l_1 + l_2 + \ldots + l_b + m_1 + m_2 + \ldots + m_b = L, \qquad m_1 + m_2 + \ldots + m_b = F(\gamma).$$

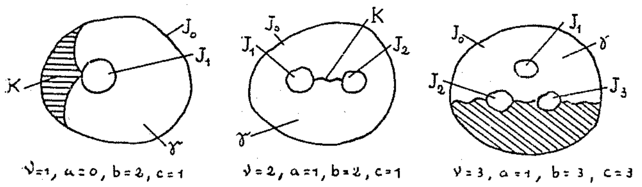
Let us mention that L is also the boundary of the simply connected open set γ' defined in the proof of (**2.** vi) and thus the application above of the results proved in [**3** (d)] for simply connected open sets [either bounded, or unbounded with bounded boundary] is correct.

The continua $m_1, \ldots, m_b$ are not necessarily disjoint, and thus $\sum m_i = F(\gamma)$ is the sum of a number c of components, say $k_1, k_2, \ldots, k_c$, with $1 \leqslant c \leqslant b$. For the sake of simplicity we have supposed above implicitely that $b \geqslant 1$. Suppose now $b = 0$. Then $L = F(\gamma) = \gamma'^*$ is the (bounded) boundary of the simply connected open set γ'. Thus $\{\eta\}_{\gamma_0}$, $\{\omega\}_{\gamma_0}$ are cyclically ordered and the unique set

$$m = \sum E_\omega = F(\gamma) \subset K,$$

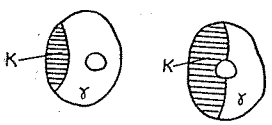
where $\sum$ is extended to all $\omega \in \{\omega\}_{\gamma_0}$, coincides with $F(\gamma)$, and is a continuum [**3** (d)]. Thus $c = 1$ if $b = 0$. We may conclude that $1 \leqslant c \leqslant \max[1, b]$ and since $b \leqslant \nu + 1$, $\nu \geqslant 0$, we have $1 \leqslant c \leqslant \nu + 1$. Thus the proof of (v) is completed.

The simple illustrations below show that the case $c < b$, $2 \leqslant b$ may actually occur.



4. – Properties P_1 and P_2 of a continuum K in J.

Let (T, J) be any continuous mapping from any closed JORDAN region J of order ν, $0 \leqslant \nu < +\infty$, $J \subset \pi$, into E. Let $K \subset J$ be a continuum, and $\{\gamma\}_{KJ}$ be the collection of all components of $J - K$.



We say that K has *property* P_1 with respect to (T, J) if for every $\gamma \in \{\gamma\}_{KJ}$ the mapping T is constant on each component k of $F(\gamma)$.

We say that K has *property* P_2 with respect to (T, J) if every $\gamma \in \{\gamma\}_{KJ}$ is of the first class.

Property P_2 concerns only the region J and not the mapping T. Obviously the two properties P_1, P_2 are independent as the illustrations show. In the first one the property P_2 holds since the only element γ belongs to the first class, and the property P_2 may hold if T is constant on $F(\gamma)$. In the second example the only element γ belongs to the second class and $F(\gamma)$ has two components. The property P_2 does not hold, and the property P_1 may hold if T is constant on each of the two components of $F(\gamma)$.

Let us observe that, if $\nu = 0$, then by (**2.** vii) every γ is of the first class and property P_2 is necessarily satisfied.

We say that a continuum $K \subset J$ has property P' with respect to (T, J) if it occurs that for every $g \in \Gamma$ with $gK \neq 0$ we have $g \subset K$. In other words K has property P' if and only if K is the sum of continua $g \in \Gamma$, $K = \sum g$. Properties P_1, P_2 and P' are independent as it can be seen by examples. All continua $g \in \Gamma$ have both properties P_1 and P'.

(**4.** i) If $K \subset J$ has property P_1 and γ is any component of $J - K$, then $F(\gamma)$ is covered by s continua $g \in \Gamma$, $1 \leqslant s \leqslant \nu + 1$.

A consequence of (v) of § **2** and the definition. Indeed each component $k = k_j$ of $F(\gamma)$ $(j = 1, \ldots, c)$, is contained in a continuum $g \in \Gamma$, and $c \leqslant \nu + 1$. Thus $1 \leqslant s \leqslant c \leqslant \nu + 1$.

Let J be any closed JORDAN region of order ν, $0 \leqslant \nu < +\infty$, let (T, J) be any continuous mapping from J into E, let $K \subset J$ be any continuum satisfying condition P_1, and $\{\gamma\} = \{\gamma\}_{KJ}$ the countable collection of all components of $J - K$. The following statement holds:

(**4.** ii) Given $\varepsilon > 0$, the relation diam $T(\gamma) > \varepsilon$ is satisfied by at most finitely many elements $\gamma \in \{\gamma\}$.

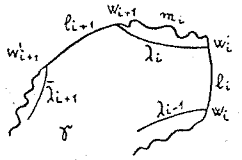
Proof. By virtue of (v) we may prove (ii) only for the components $\gamma \in \{\gamma\}'$. The proof is now the same as the one given for $\nu = 0$ in [**3** (A), 36.2, i, p. 506].

5. – Further properties of the collection $\{\gamma\}_{KJ}$.

(**5.** i) If K has property P_1 with respect to (T, J), then there is a JORDAN region J_0 of connectivity μ, $0 \leqslant \mu \leqslant \nu$, with $K \subset J_0 \subset J$, such that K has both properties P_1 and P_2 with respect to (T, J_0).

Proof. If all elements $\gamma \in \{\gamma\}_{KJ}$ are of the first class, we can take $J_0 = J$. Otherwise we shall consider the elements $\gamma \in \{\gamma\}''$. For these we have $1 \leqslant c \leqslant b \leqslant \nu + 1$, $2 \leqslant b$ and $F(\gamma) = l_1 + m_1 + l_2 + m_2 + \ldots + l_b + m_b$, where the continua m_i are not necessarily distinct. Let us consider on each l_i, a pair of ordered points $w_1, w_1' \in l_1$, $w_2, w_2' \in l_2$, ..., $w_b, w_b' \in l_b$. Then there are b disjoint cross-cuts $\lambda_1 = (w_1' w_2)$, $\lambda_2 = (w_2' w_3)$, ..., $\lambda_b = (w_b' w_1)$. Let us denote by l_i' the subarc $w_i w_i'$ of l_i. We can take λ_i made up of points all so close to

m_i that the part γ_{0i} separated by λ_i in γ (and containing arcs defining all ends of m_i) does not contain any of the region J_j' (with $J_j^* \subset \gamma$), $j = 1, 2, \ldots, a$ (if any). We can for instance require that all points of γ_{0i} are at a distance $< \varepsilon$ from m_i where 2ε is the minimum distance $\{F(\gamma), J_1^* + \ldots + J_a^*\}$, or any even smaller given number. Let us observe that the sets γ_{0i} are simply connected, open, and γ_{0i}^* is the sum of the continuum m_i and of the arc $(\beta_i w_i') + \lambda_i + (w_{i+1} \alpha_{i+1})$. In addition $F(\gamma_{0i}) = m_i$ is a continuum. Now the continuous line $\lambda^* = l_1' + \lambda_1 + l_2' + \lambda_2 + \ldots + l_b' + \lambda_b$ is simple and closed and defines a JORDAN region λ containing all the regions $J_1', \ldots, J_a'$. Thus $\lambda' = \lambda - (J_1' + \ldots + J_a')^0$ is a JORDAN region contained in J.



The previous process can be applied to every element $\gamma \in \{\gamma\}''$ (at most ν elements γ have this property). We obtain a finite collection of region $\lambda' \subset J$. Now we shall remove the lines l' separating the regions λ' from regions J_i'' corresponding to elements γ. We shall obtain new regions J_{0u} ($u = 1, 2, \ldots, \mu$) and we put $J_0 = J - \sum J_{0u}$. This procedure reduces the order ν of connectivity of J of at most $\sum(a + b - 1)$ units, but does not destroy the connection of J as it can be proved by a reasoning analogous to the one in § **2**. Obviously $K \subset J_0 \subset J$.

If we consider now the collection $\{\gamma\}_{KJ_0}$ of all components γ of $J_0 - K$, then $\{\gamma\}_{KJ_0}$ contains: all elements $\gamma \in \{\gamma\}_{KJ}'$; all new elements γ_{0i} for which $F(\gamma_{0i}) = m_i$. Thus K satisfies property P_2 with respect to (T, J_0). Finally since no continuum m_i [**4**] may be made of more than one continuum $k_1, k_2, \ldots, k_c$, and T is constant on each k_i, we conclude that T is constant on each m_i. Thus K satisfies condition P_2 with respect to (T, J_0).

(**5.** ii) If K has property P_1 with respect to (T, J) and $\varepsilon > 0$ is a given number then there is a JORDAN region J_0 of some connectivity $\mu \geqslant 0$, with $K \subset J_0 \subset J$, such that K has properties P_1, P_2 with respect to (T, J_0), and, for each component $\gamma \in \{\gamma\}_{KJ_0}$; $T(\gamma)$ belongs to the ε-neighborhood of the point $T[F(\gamma)]$ in E.

Proof. By (**4.** ii) only a finite collection $[\gamma]_\varepsilon$ of elements $\gamma \in \{\gamma\}$ may verify the relation $\operatorname{diam} T(\gamma) \geqslant \varepsilon$. For each element $\gamma \in [\gamma]_\varepsilon$ we can now define some region $J_\gamma \subset \gamma$ such that T has an oscillation $< \varepsilon$ in each component of $\gamma - J_\gamma$, by proceeding as in the proof of (i). Then $J_0 = J - \sum J_\gamma$ where $\sum$ is extended over all $\gamma \in [\gamma]_\varepsilon$ has the required properties.

Remark 1. In (ii) we may require as well that for every $\gamma \in \{\gamma\}_{KJ_0}$ and every point $w \in \gamma$ we have $\{w, F(\gamma)\} < \varepsilon$.

Remark 2. As in § **2** we may observe that the previous considerations hold even if we are concerned with finite sums $S = \sum K$ of disjoint continua

$K \subset J$. Properties P_1, P_2 remain unchanged. Theorems (i), (ii) hold provided we replace J_0 by a finite sum $\sum J_0$ of disjoint subregions J_0 of J.

The following statements concerning mappings (T, J) defined on polygonal regions $J \subset \pi$ will be used in the sequel and give a few more details. They can be proved, as usual in elementary topology, by considering convenient finite subdivisions of J into triangles of diameter sufficiently small.

(**5.** iii) Let J be a polygonal region, (T, J) a continuous mapping, $K = [K_1, ..., K_N]$ a finite collection of disjoint continua $K_i \subset J$, each satisfying condition P_1 with respect to (T, J), ε a positive number, then there are disjoint polygonal regions J_i, $K_i \subset J_i \subset J$ $(i = 1, ..., N)$ such that K_i has both properties P_1 and P_2 with respect to (T, J_i) $(i = 1, ..., N)$ and for every component γ of $J_i - K_i$ and point $w \in \gamma$ we have $\{w, F(\gamma)\} < \varepsilon$.

(**5.** iv) Let J be a polygonal region, (T, J) a continuous mapping, $K = [K_1, ..., K_N]$ a finite collection of continua $K_i \subset J$, each satisfying conditions P' and P_1 with respect to (T, J); $S = [g_1, ..., g_M]$ a finite collection of distinct continua $g_h \in \Gamma$; ε a positive number. Suppose that for each $i \neq j$ $(i, j = 1, ..., N)$, we have (a) $s_{ij0} = K_i K_j$ is either empty, or a finite sum $s_{ij0} = \sum' g_h$ of continua $g_h \in S$; (b) there is a finite sum $s_{ij} = \sum'' g_h$ of continua $g_h \in S$ such that s_{ij} separates $K_i - s_{ij}$ and $K_j - s_{ij}$ in J. Then there are N polygonal regions J_i, $K_i \subset J_i \subset J$ $(i = 1, ..., N)$ such that (1) K_i has properties P', P_1 and P_2 with respect to (T, J_i); (2) for every $i \neq j$ $(i, j = 1, ..., N)$, we have $s_{ij0} \subset J_i J_j$, where $J_i J_j$ is either empty or a figure according as s_{ij0} is empty or not; (3) for every component γ of $J_i - K_i$ and point $w \in \gamma$ we have $\{w, F(\gamma)\} < \varepsilon$.

(**5.** v) In (iv) and for every $\varepsilon > 0$, there are disjoint polygonal regions f_h with $g_h \subset f_h \subset J$ and $\operatorname{diam} T(f_h) < \varepsilon$ $(h = 1, ..., M)$ such that (4) for every $i \neq j$ $(i, j = 1, 2, ..., N)$ with $s_{ij0} = \sum' g_h \neq 0$ we have $J_i J_j = \sum' f_h$.

6. – Retraction.

Let J be any closed JORDAN region of order ν, $0 \leqslant \nu < +\infty$, let (T, J) be any continuous mapping from J into E, and let $K \subset J$ be any continuum.

Let J_0 be any closed JORDAN region of some order μ, $0 \leqslant \mu < +\infty$, with $K \subset J_0 \subset J$, where we do not exclude $J_0 = J$, or $K = J_0$. Let $\{\gamma\}_{KJ_0}$ be the countable collection of all components γ of $J_0 - K$. Suppose that K satisfies both conditions P_1 and P_2 with respect to (T, J_0). Then for each element

$\gamma \in \{\gamma\}_{KJ_0}$ $F(\gamma)$ is a continuum and T is constant on $F(\gamma)$. Let (T_0, J_0) be the new mapping defined by $T_0 = T$ on K, $T_0(w) = T[F(\gamma)]$ for all $w \in \gamma$, $\gamma \in \{\gamma\}_{KJ_0}$. The mapping (T_0, J_0) is said to be the *retraction of* (T, J) *with respect to* K *in* J_0.

(**6.** i) The mapping (T_0, J_0) is continuous.
The proof is the same as the one given for $\nu = 0$, $J = J_0$ in [**3** (A), p. 508].

Remark. The process of retraction itself could be defined even under somewhat weaker conditions, namely: P_1 and P_2': $c = 1$ for every $\gamma \in \{\gamma\}_{KJ_0}$; or P'': T constant on $F(\gamma)$ for each $\gamma \in \{\gamma\}_{KJ_0}$.

7. – Some metric lemmas.

(**7.** i) Lemma. Given N distinct points $P_i \in E$ $(i = 1, ..., N)$ and a number $\varepsilon > 0$, there is a $\sigma > 0$, $0 < \sigma \leqslant \varepsilon$, and a quasi-linear single-valued continuous transformation τ from E onto itself (not necessarily one-one) such that

(a) $|\tau(p) - \tau(p')| \leqslant |p - p'|$ for all $p, p' \in E$;

(b) $|\tau(p) - p| < \varepsilon$ for all $p \in E$;

(c) τ is constant on each sphere F_i of center P_i and radius σ $(i = 1, 2, ..., N)$;

(d) each triangle Δ of E is mapped into a polyhedral surface Δ' with $a(\Delta') \leqslant a(\Delta)$.

A proof is given in [**3** (A), 6.5.i, p. 67]. Here $a(\Delta)$, $a(\Delta')$ denote elementary areas.

(**7.** ii) Lemma. Let J be any polygonal region of order ν, $0 \leqslant \nu < +\infty$, of the w-plane π, (T, J) a continuous mapping, $K \subset J$ any continuum satisfying conditions P' and P_1 with respect to (T, J); J_0 a polygonal region of order μ, $0 \leqslant \mu < +\infty$, such that $K \subset J_0 \subset J$ and K satisfy conditions P_1 and P_2 with respect to (T, J_0); (T_0, J_0) the retraction of (T, J) with respect to K in J_0. Then, for every $n = 1, 2, ...$, there is a quasi linear mapping (P_n, J) and a polygonal region R_j, $K \subset R_n \subset J_0 \subset J$, satisfying conditions P_1 and P_2 with respect to (P_n, J_0) and such that if (P_{n0}, J_0) is the retraction of (P_n, J) with respect to R_n in J_0, then $P_n(w) \rightrightarrows T(w)$ in J; $P_{n0}(w) \rightrightarrows T_0(w)$ in J_0, $a(P_n, J) \to L(T, J)$, $a(P_{n0}, J_0) \to L(T_0, J_0)$ as $n \to \infty$ and $a(P_{n0}, J_0) \leqslant a(P_n, J)$.

The proof is the same as the one given in [**3** (A), 36.3, i, p. 511] for $\nu = 0$, $J_0 = J$.

The statement (iii) below is only slightly more general than (ii), and the proof is analogous. By figure we mean the finite sum of disjoint polygonal regions. Use is made of § **2**, (iv) an (v).

(**7**. iii) Lemma. Let J be any polygonal region of order ν, $0 \leqslant \nu < +\infty$, (T, J) be any continuous mapping; $K = [K_1, \ldots, K_N]$ any finite collection of continua K_i each satisfying conditions P' and P_1 with respect to (T, J); $S = [g_1, \ldots, g_M]$ any finite collection of distinct continua $g_h \in \Gamma$; J_i any polygonal region, $K_i \subset J_i \subset J$, such that K_i has both properties P_1 and P_2 with respect to (T, J_i) $(i = 1, \ldots, N)$. Suppose that, for each $i \neq j$ $(i, j = 1, \ldots, N)$ (a) $K_i K_j = s_{ij0}$ is either empty, or the sum $s_{ij0} = \sum' g_h$ of continua $g_h \in S$; (b) there is a sum $s_{ij} = \sum'' g_h$ of continua $g_h \in S$ such that $s_{ij0} \subset s_{ij}$ and s_{ij} separates $K_i - s_{ij}$ and $K_j - s_{ij}$ in J; (c) $s_{ij0} \subset J_i J_j$ where $J_i J_j$ is either empty, or a figure according as s_{ij0} is empty or not. Let (T_i, J_i) denote the retraction of (T, J) with respect to K_i in J_i $(i = 1, \ldots, N)$. Then, for each $n = 1, 2, \ldots$, there is a quasi linear mapping (P_n, J) and certain figures R_{ni} such that (1) $K_i \subset R_{ni} \subset J_i$, $R_{ni} \supset R_{n+1,i}$, $R_{ni} \downarrow K_i$ as $n \to \infty$; (2) R_{ni} satisfies both conditions P_1 and P_2 with respect to (P_n, J_i); (3) $P_n(w) \rightrightarrows T(w)$ in J; $P_{ni}(w) \rightrightarrows T_i(w)$ in J_i $(i = 1, \ldots, N)$, where (P_{ni}, J_i) denotes the retraction of (P_n, J) with respect to R_{ni} in J_i; (4) for every $h = 1, \ldots, M$ there exists a polygonal region f_{nh} such that $g_h \subset f_{nh}$, and P_n is constant on f_{nh}; (5) if s_{ij0} is empty then $R_{ni} \cdot R_{nj}$ is empty; if $s_{ij0} = \sum' g_h$ then $R_{ni} \cdot R_{nj} = \sum' f_{nh}$, $i \neq j$ $(i, j = 1, 2, \ldots, N)$; (6) for every $h = 1, \ldots, M$ and $i = 1, \ldots, N$, with $g_h \subset K_i$ we have $f_{nh} \subset R_{ni}$ and $P_n(w) = P_{ni}(w) =$ constant on f_{nh} $(n = 1, 2, \ldots)$; (7) $a(P_n, J) \to L(T, J)$, $a(P_{ni}, J_i) \to L(T_i, J_i)$ $(i = 1, 2, \ldots, N)$ as $n \to \infty$.

8. – Rigid equivalence.

We say that two mappings (T_1, J_1), (T_2, J_2) are rigidly equivalent if we can pass from T_1 to T_2 by means of finitly many operations R of the following type, or of their inverse:

(R) For some finitely connected closed Jordan region J' we have $J_1 + J' = J_2$, T_2 is constant on J', and $T_2 = T_1$ on J_1.

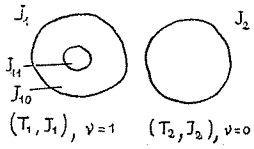
A particular case of this operation has been considered in [**3** (e), p. 26]. We proved there the invariance of Lebesgue area. This is true in general:

(**8**. i) If (T_1, J_1), (T_2, J_2) are *rigidly equivalent* then $L(J_1, T_1) = L(J_2, T_2)$.

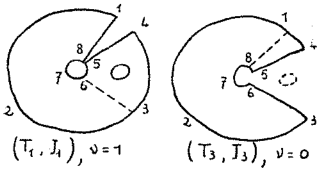
The proof is the same as in [**3** (e)], or as in [**3** (A), 6.5. ii, p. 69] and based on Lemma (i) of § **7**. Also it is an immediate consequence of [**3** (A), 21.4, i, p. 337].

The following examples show cases of rigid equivalence where J_1, J_2 need not have the same order of connectivity.

Example 1. Let (T_1, J_1), $J_1 = J_{10} - J_{11}^0$, be a given mapping, which is constant on J_{11}^*. Let $J' = J_{11}$ and $T_2 = T_1$ on J_1, and constant and equal to $T_1(J_{11}^*)$ on $J' = J_{11}$. Then (T_2, J_2) is rigidly equivalent to (T_1, J_1), $J_2 = J_{10}$.



Example 2. Let (T_1, J_1), $J_1 = (1\,2\,3\,4\,5\,6\,7\,8)$, be a given mapping, which is constant on $(3\,6\,4\,5)$ and $(1\,8)$ with two different values $p_1 \neq p_2$ there. Let $J' = (3\,6\,5\,4)$ and « suppress » J' from J_1, that is let $J_2 = (1\,2\,3\,6\,7\,8\,1)$ (operation R^{-1}), and let $T_2 = T_1$ in J_2. Then let $J' = (1\,4\,5\,8)$, $J_3 = J_2 + J'$, and let $T_3 = T_2$ in J_2 and $T_3 = p_2$ in $(1\,4\,5\,8)$. Then (T_3, J_3) is rigidly equivalent to (T_1, J_1).



9. – Fine-cyclic elements.

Let us consider the collection $\Gamma = \Gamma(T, J)$ of the maximal continua of constancy for T in J. Then each continuum $g \in \Gamma$ has obviously property P_1 with respect to (T, J), but not necessarily property P_2.

A (non vacuous) continuum $K \subset J$ is said to be a *fine-cyclic element* (set) of (T, J), if

(1) K has both properties P_1 and P';

(2) T is not constant on K, i.e., K is not an element $g \in \Gamma$;

(3) K is minimal with respect to properties (1) and (2); that is, every proper subcontinuum of K having property (1) is a continuum $g \in \Gamma$.

We shall denote by $\{K\}$ the collection of all distinct fine-cyclic elements of (T, J).

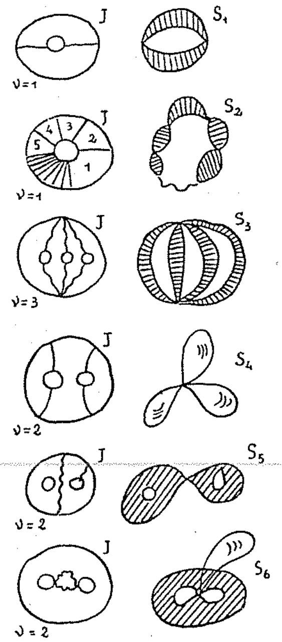
Let K be a fine-cyclic element of (T, J). Then K has both properties P_1 and P' with respect to (T, J). Let $\{J_0\}$ be the class of all regions J_0 with $K \subset J_0 \subset J$ such that K has all properties P_1, P_2, P' with respect to (T_0, J_0). By (**5.** ii) we know that the class $\{J_0\}$ is not empty.

For every $J_0 \in \{J_0\}$ let (T_0, J_0) be the retraction of (T, J) with respect to K in J_0.

(**9.** i) For every two regions $J_1, J_2 \in \{J_0\}$ the retractions (T_1, J_1), (T_2, J_2) of (T, J) with respect to K in J_1, J_2 respectively are rigidly equivalent.

The proof does not offer difficulties.

As a consequence any two continuous mappings (T_1, J_1), (T_2, J_2), $J_1, J_2 \in \{J_0\}$, obtained by retractions of (T, J) with respect to K in J_1, J_2 are rigidly equivalent, and thus the class



$$(1) \qquad \{(T_0, J_0)\}, \qquad J_0 \in \{J_0\},$$

is a single entity up to a rigid equivalence. We denote this element as a *fine-cyclic element* (surface) of (T, J). By LEBESGUE area $L(T_0, J_0)$ of a fine-cyclic element we denote the LEBESGUE area of any element (T_0, J_0) of the class (1).

The following illustrations show various types of fine-cyclic elements for surfaces of the ν-type.

The surface S_1 presents two fine-cyclic elements each of the type of the disc. The surface S_2 presents five fine-cyclic elements each of the type of the disc, joined in chain and by a thread. The surface S_3 presents four fine-cyclic elements each of the type of a disc, joined at two distinct points. The surface S_4 presents three fine-cyclic elements each of the type of the sphere, joined at a single point. The surface S_5 presents two fine-cyclic elements. The surface S_6 also presents two fine-cyclic elements, one of which is of the type of the sphere.

10. – Properties of separations of fine-cyclic elements.

(**10.** i) If K, K' are any two distinct fine-cyclic elements of (T, J) then (a) $KK' = S$ where S is a finite sum of continua $g \in \Gamma$, which may be empty; (b) there is a component $\gamma \in \{\gamma\}_{KJ}$ such that $K' \subset \gamma + S$; (c) there is a component $\gamma' \in \{\gamma\}_{K'J}$ such that $K \subset \gamma' + S$.

P r o o f. If $KK' = 0$ then necessarily K' is completely contained in one and only one component $\gamma \in \{\gamma\}_{KJ}$ and K is in one and only one component $\gamma' \in \{\gamma'\}_{K'J}$. Thus (i) is proved if $KK' = 0$.

Suppose $KK' \neq 0$. Since K, K' are distinct there must be a point w which is in one of them and not in the other one; say, $w \in K'$, $w \in J - K$. Then $w \in \gamma$ for some $\gamma \in \{\gamma\}_{KJ}$ and certainly K' cannot be contained in γ, otherwise we would have $KK' = 0$. Thus K' goes beyond γ and hence must have at least one point in common with $F(\gamma)$ and thus with some of the components k_1,

$k_2, \ldots, k_c$ of $F(\gamma)$. Since K has property P_1, T is constant on each k_i and thus $k_i \subset g_i$ for some $g_i \in \Gamma$, and since K has property P' and $k_i \subset K$, $g_i K \neq 0$, we conclude $g_i \subset K$ $(i = 1, 2, \ldots, c)$. As a consequence, if we denote by $k_1, k_2, \ldots, k_d$, $1 \leqslant d \leqslant c$, those components of $F(\gamma)$ with $K' k_i \neq 0$ $(i = 1, 2, \ldots, d)$, we have $K' g_i \neq 0$, $g_i \subset K$, and, since also K' has property P' we have $g_i \subset K'$. Thus, if $S = g_1 + \ldots + g_d$, we have $S \subset K$, $S \subset K'$ while $K' \cdot (g_{d+1} + \ldots + g_c) = 0$.

Let us consider now the set $K'' = K' \cdot (\gamma + S)$, which is obviously a continuum $K'' \subset J$. Since each set K', γ, S is a sum of continua $g \in \Gamma$, also K'' has the same property, i.e., K'' has property P'. Now let us consider any component γ'' of $J - K''$ and observe that $F(\gamma'')$ is the finite sum of its components, say $k_1'', \ldots, k_c''$, and that $F(\gamma) \subset K'' \subset \gamma + S$. If $w \in F(\gamma'')$, say $w \in k_1''$ for instance, and $w \in \gamma$, then $w \in K'$, and there is a neighborhood U of w such that $UK'' = UK'$, $U\gamma'' = U\gamma$. Therefore every component γ_0 of $U\gamma''$ is contained in a component γ' of $J - K'$ and $U\gamma_0 = U\gamma'$. Thus w is on some component k' of $T(\gamma')$ for some $\gamma' \in \{\gamma'\}_{K'J}$. All this proves that T has on each k_i'' at most the values taken by T on the component k_i' of the sets $\gamma' \in \{\gamma\}_{K'J}$ plus the values taken by T on the continua $g \in S$. Since K' has property P_1 with respect to (T, J), T has constant values on each k_i' above, and thus T takes at most countably many values on each k_i''. This implies that T is constant on each k_i'', and thus K'' has property P_1 with respect to (T, J). Finally T cannot be constant on the whole of K'' since we would have then $S = g = K''$, $g \in \Gamma$, and this is impossibile since K'' has some points in γ, and thus is larger than S. We conclude that K'' is a fine-cyclic element for (T, J), with $K'' \subset K'$. This implies that $K'' = K'$ and (a), (b) of (i) are proved.

By exchanging K and K' we may deduce (c). Thus (i) is completely proved.

(**10.** ii) If $g \in \Gamma$ and $K \in \{K\}$ then $K - g \subset \gamma$ for some component γ of $J - g$.

Proof. Suppose if possible, that $(K - g)\gamma' \neq 0$, $(K - g)\gamma'' \neq 0$ for two distinct components γ', γ'' of $J - \gamma$. Then it is immediate that both sets

$$K' = g + (K - g)\gamma, \qquad K'' = g + (K - g)\gamma''$$

are uncountable sums of continua $g \in \Gamma$, are proper subcontinua of K, and have properties P_1 and P', a contradiction.

11. – Further properties of separation.

(**11.** i) If $g \in \Gamma$, if γ, γ' are two distinct components of $J - g$, and $g \subset KK'$, $K - g \subset \gamma$, $K' - g \subset \gamma'$, where K, K' are distinct fine-cyclic elements, then $KK' = g$.

Proof. By $g \subset KK'$, $K - g \subset \gamma$, $K' - g \subset \gamma'$, and $\gamma\gamma' = 0$, it follows that $K = g + (K - g)\gamma$, $K' = g + (K' - g)\gamma'$, $KK' = g$.

(**11.** ii) If $g \in \Gamma$, if γ is any component of $J - g$ and $g \subset KK'$, $K - g \subset \gamma$, $K' - g \subset \gamma$ where K, K' are distinct fine-cyclic elements, then $gJ^* \neq 0$, and, if G is the component of $J - K$ containing $K' - KK'$, then $F(G) = m_1 + ... + m_b$, $GG^* = J_1'^* + ... + J_a'^* + l_1 + ... + l_b$, $G^* = GG^* + F(G)$, $2 \leqslant b \leqslant \nu + 1$ (§ **2**, § **3**) and for at least one of the continua m_i of $F(G)$, say m_1, we have $m_1 \subset g \subset KK'$.

Proof. By (§ **10**) we have $KK' = S$, $K' \subset G + S$, where S is a finite sum of continua $g_1 = g, g_2, ..., g_\mu \in \Gamma$. By (§ **2**, § **3**) we have either (a) $F(G) = k_1 + ... + k_c = m_1 + ... + m_b$, $GG^* = J_1'^* + ... + J_a'^* + l_1 + ... + l_b$, $b \geqslant 1$, $G^* = GG^* + F(G)$, or (b) $F(G) = m_1$, $GG^* = J_1' + ... + J_a'$, $G^* = GG^* + F(G)$. Suppose, if possible, that the case (b) is true. We have $K - g \subset \gamma$, $K' - g \subset \gamma$, $g \subset KK'$ and, since K, K' are distinct, there must be either a point $w \in (K - K')\gamma$, or a point $w' \in (K' - K)\gamma$, or both. Since neither K is a proper part of K', nor K' is a proper part of K (§ **9**, (3)), both points $w \in (K - K')\gamma$, $w' \in (K' - K)\gamma$ exist. Thus every point $w' \in G$, and, therefore, $w' \in G\gamma$ and $G\gamma \neq 0$. Since $g \subset K$ and γ is a component of $J - g$ we have $gG = 0$ and, finally, $G \subset \gamma$. On the other hand, by (b), $F(G) = m_1 \subset K$, and since K has properties P' and P_1, also $m_1 \subset g' \subset K$ for some $g' \in \Gamma$. If $g' \neq g = g_1$, then G is a component of $J - g'$, say γ', and γ' contains $w' \in K'$. Thus, by the reasoning of (**10**, ii), $K' \subset \gamma' + g'$, where $\gamma' g = 0$, $g'g = 0$, and hence $K'g = 0$, a contradiction. If $g' = g = g_1$, then $F(G) = m_1 \subset g$, and G is a component of $J - g$, say γ', and since $w' \in G\gamma$, we have $\gamma' = \gamma$, and $G = \gamma$. By $g \subset K$, $K - g \subset \gamma$, $KG = 0$, we have also $Kg = g$, $K\gamma = KG = 0$, $K \cdot (J - g - \gamma) = K - g - \gamma = 0$, and $g \subset K = Kg + K\gamma + K \cdot (J - g - \gamma) = g$, that is, $K = g$, a contradiction. Thus (b) is contradictory, and (a) holds. Now suppose, if possible, that (a) holds with $b = 1$. The same reasoning abowe shows that also this assumption leads to a contradiction. Thus (a) holds with $2 \leqslant b \leqslant \nu + 1$. Here $l_i = GG^* J_i''^*$ $(i = 1, 2, ..., b)$, where $J_1''^*, ..., J_b''^*$ are b distinct of the $\nu + 1$ boundary curves $J_0^*, ..., J_\nu^*$, and the end points α_i, β_i of the arcs $l_i = (\alpha_i \beta_i)$ [α_i, β_i not necessarily distinct] belong to two consecutive continua m_i, with $m_i \subset g_i \subset K$ $(i = 1, ..., b)$. The b continua $g_i \in \Gamma$ are not necessarily distinct, and the continua $g_i = g, ..., g_\mu$, whose sum is S, are μ of these b continua g_i (thus $1 \leqslant \mu \leqslant b$). Since the second end point β_1 of l_1 and the first end point α_2 of l_2 certainly belong to m_1 and hence to $g_1 = g$, we conclude that $g\, J_i''^* \neq 0$ for some $i = 1, 2, ..., b$, and thus $gJ^* \neq 0$.

Note. In the considerations above we have $G \subset \gamma$. Indeed, since G is a component of $J - K$, $g \subset K$, and γ is a component of $J - \gamma$, we conclude that $G \subset \gamma$.

If we denote by $m_i = m_1, \ldots, m_{i_u}$ those continua m_i contained in $g_1 = g$, and by $m_{j_1}, \ldots, m_{j_v}$ the remaining ones, we have $u + v = b$, and the same reasoning above proves that $u \geqslant 1$, $v \geqslant 1$. Thus $1 \leqslant u \leqslant b - 1$, $1 \leqslant v \leqslant b - 1$.

(**11.** iii) If $g \in \Gamma$ belongs to the non-zero intersection of μ distinct fine-cyclic elements K_i $(i = 1, 2, \ldots, \mu)$, $g \subset K_1 K_2 \ldots K_\mu$, $\mu \geqslant 2$, if $K_i - g \subset \gamma$ $(i = 1, 2, \ldots, \mu)$, where γ is a given component of $J - g$, then $F(\gamma) = m_1 + \ldots + m_b$, $\gamma\gamma^* = J_1' + \ldots + J_a'^* + l_1 + \ldots + l_b$, $\gamma^* = \gamma\gamma^* + F(\gamma)$ (§ **2**, § **3**), and $2 \leqslant b \leqslant \nu + 1$, $\mu \leqslant b$.

Proof. Let us apply (i) to $K = K_i$ and $K' = K_1$ $(i = 2, 3, \ldots, \mu)$. Then for every $2 \leqslant i \leqslant \mu$ have $K_i K_1 = S_i$, $K_1 \subset G_i + S_i$, where $g \subset S_i$, $G_i \subset \gamma$, and G_i is a component of $J - K_i$. For every $i = 2, \ldots, \mu$, we have

$$F(G_i) = m_{i1} + \ldots + m_{iu_i} + m'_{i1} + \ldots + m'_{iv_i}, \qquad u_i + v_i = b_i, \qquad u_i, v_i \geqslant 1,$$

$$G_i G_i^* = J_{i1}'^* + \ldots + J_{ia_i}'^* + l_{i1} + \ldots + l_{ib_i}, \qquad G_i^* = G_i G_i^* + F(G_i),$$

where $m_{i1} + \ldots + m_{iu_i} \subset g$, $(m_{i1} + \ldots + m_{iu_i})\, g = 0$. Actually γ is larger than each G_i $(i = 2, 3, \ldots, \mu)$ but the continua $m_{i1}, \ldots, m_{iu_i}$ belong to $F(\gamma)$ as well as to $F(G_i)$ $(i = 2, \ldots, \mu)$. Thus also for γ the case (a) occurs with b [proof of (ii)] and we have $F(\gamma) = m_1 + \ldots + m_b$, $\gamma\gamma^* = J_1'^* + \ldots + J_a'^* + l_1 + \ldots + l_b$, $\gamma^* = \gamma\gamma^* + F(\gamma)$, where each collection $[m_{i1}, \ldots, m_{iu_i}]$ is a proper part of $[m_1, \ldots, m_b]$, each collection $[J_{i1}'^*, \ldots, J_{iu_i}'^*]$ is a part of $[J_1'^*, \ldots, J_a'^*]$, each arc l_{is} is a part of (proper or not) of an arc l_i of a curve $J_i''^*$, where $m_1 + \ldots + m_b \subset g$, and $2 \leqslant b \leqslant \nu + 1$, $0 \leqslant a \leqslant \nu + 1$, $a + b \leqslant \nu + 1$, $a_i \leqslant a$.

Let us observe that T cannot be constant on the arcs $l_i = (\alpha_i \beta_i)$ nor on any subarc $l_i' = (\alpha_i' \beta_i)$, $l_i'' = (\alpha_i \beta_i')$ of l_i. Indeed suppose T constant on an arc l_i', then T should be constant on the closed arc $(\alpha_i') + l_i' + (\beta_i) = \bar{l}_i'$, and then, by $\beta_i \in \bar{l}_i$, $\beta_i \in m_i$, $\bar{l}_i \subset g'$, $m_i \subset g$, $g, g' \in \Gamma$, it follows $g' = g$, $l_i' \subset g$, $l_i'\gamma = 0$, a contradiction, since $l_i' \subset \gamma\gamma^* \subset \gamma$.

It may occur that for a subarc as above, say $l_1' = (\alpha_1' \beta_1)$ we have $l_1 K_i = 0$ for some $i = 1, 2, \ldots, \mu$. Then the whole arc l_1' belongs to G_i and since $\beta_1 \subset m_1 \subset g$, the point β_1 cannot be in G_i. Thus m_1 is a continuum m_{is} and (*) there are two subarcs $l_1' = (\alpha_1' \beta_1)$ of $l_1 = (\alpha_1 \beta_1)$, and $l_2'' = (\alpha_2 \beta_2')$ of l_2 with $l_1' K_i = 0$, $l_2'' K_i = 0$. The same result holds for $i = 1$, by exchanging K_1 and K_i.

Suppose that for a subarc $l_1' = (\alpha_1' \beta_1)$ of l_1 we have $l_1' \subset K_i$ for some $i = 2, \ldots, \mu$. Then we can show that for a subarc l_1' sufficiently small we have also

$l_1' K_j = 0$ for every $j \neq i$ ($j = 1, 2, \ldots, \mu$). Suppose this is not true. Then there would be points $w_n = l_1' K_i K_j$ ($n = 1, 2, \ldots$) as close as we want to β_1 and thus we can suppose $w_n \to \beta_1$ as $n \to \infty$. By an extraction we can suppose $w_n \in l_1' K_i K_j$ ($n = 1, 2, \ldots$) for the same j independently of n. These points cannot belong to the same continuum $g' \in \Gamma$, since then we would have $\beta_1' \in g'$, $g'g \neq 0$, $g' = g$, $gl_1 \neq 0$, a contradiction, since $l_1 \subset \gamma$, $\gamma g = 0$. Analogously the points w_n cannot belong to finitely many continua $g' \in \Gamma$. Thus the points w_n ($n = 1, 2, \ldots$) belong to a denumerable family $[g]$ of distinct continua $g \in \Gamma$, and we have $g' \subset K_i K_j$ for all $g \in [g]$, a contradiction, since, by (**10.** i), $K_i K_j = S_{ij}$ is a finite collection of continua $g' \in \Gamma$. Thus we have proved that if $l_1' = (\alpha_1' \beta_1) \subset K_i$ then we have also $l_1' K_j = 0$, for all $j \neq i$, $j = 1, 2, \ldots, \mu$, and a subarc l_1' sufficiently small.

Suppose that for every subarc $l_1' = (\alpha_1' \beta_1)$ of l_1 we have both $l_1' K_i \neq 0$, $l_1' - K_i \neq 0$ (l_i open) for some $i = 1, 2, \ldots, \mu$. Then these relations occur also for a well determined $i = 1, 2, \ldots, \mu$, and every subarc l_1'. Since the set $l_1 - K_i$ is open and thus the countable sum of disjoint open arcs of l_1, we deduce that there certainly exists a sequence $\omega_n = (u_n v_n)$ of open arcs, with $u_n \to \beta_1$, $v_n \to \beta_1$, $u_n, v_n \subset K_i$, $\omega_n K_i = 0$, ($n = 1, 2, \ldots$). Each ω_n must belong to a component $\tilde{\gamma}_n$ of $J - K_i$, $\tilde{\gamma}_n \neq G_i$, $\tilde{\gamma}_n G_i = 0$ (at least for all n large enough), and $\tilde{\gamma}_n$ must be of the first class, at least for all n large enough (§ **2**). For n large we have then $F(\tilde{\gamma}_n) = \tilde{m}_n$, $\tilde{\gamma}_n \tilde{\gamma}_n^* = \omega_n$, $\tilde{\gamma}_n^* = \tilde{\omega}_n + \tilde{m}_n$. Since K_i has both properties P_1 and P' we have $\tilde{m}_n \subset \tilde{g}_n$ for some $\tilde{g}_n \in \Gamma$ and now we can prove that $\tilde{\gamma}_n K_j = 0$ for all $j = 1, 2, \ldots, \mu$. This relation for $j = i$ is trivial. For $j \neq i$ we must consider the set $K' = K_j \tilde{\gamma}_n + \tilde{g}_n$. This set K' is obviously a proper subcontinuum of K_j, not reduced to a single $g' \in \Gamma$, and having both properties P_1 and P', a contradiction (§ **9**, (3)). Thus $\tilde{\gamma}_n K_j = 0$ ($j = 1, 2, \ldots, \mu$). Now we can show that $l_1' K_i \neq 0$, $l_1' - K_i \neq 0$ for every subarc $l_1' = (\alpha_1' \beta_1)$, implies that $l_1' K_j = 0$ ($j = 1, 2, \ldots, \mu$; $j \neq i$) for some subarc l_i'. Indeed suppose that the contrary is true. Then we have $l_1' K_i K_j \neq 0$ for every subarc l_1' and some $j \neq i$, $j = 1, 2, \ldots, \mu$. Then the same relation occur for a fixed j and thus there exists a sequence w_n ($n = 1, 2, \ldots$) of points $w_n \in l_1' K_i K_j$ for some l_1', with $w_n \to \beta_1$. As before these points must belong to a denumerable collection $[g']$ of continua $g' \in \Gamma$, and $g' \subset K_i K_j$ for all $g' \in [g']$, a contradiction.

The last three paragraphs show that if $l_1' K_i \neq 0$ for some i and for every subarc $l_1' = (\alpha_1' \beta_1)$ of l_1 sufficiently small, then there exists some subarc l_1' with $l_1' K_j \neq 0$ for all $j \neq i$, $j = 1, 2, \ldots, \mu$.

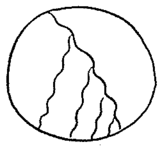
Let observe that the following situation cannot occur: (a) $K_j l_i' = 0$ for all arcs $l_i' = (\alpha_i' \beta_i)$, for every $i = 1, \ldots, b$, and some $j = 1, \ldots, \mu$. Indeed (a) would imply $K_j l_i'' = 0$ for all arcs $l_i'' = (\alpha_i \beta_i')$, and m_i would be one of the continua $m_{j1}, \ldots, m_{j\mu_i}$ for $i = 1, \ldots, b$, a contradiction, since the last collection is a proper part of $[m_1, \ldots, m_b]$.

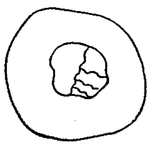
By combining this statement with the previous one we conclude as follows: For each $i = 1, 2, \ldots, b$, there is a pair of arcs $l'_i = (\alpha'_i \beta_i)$, $l''_i = (\alpha_{i+1} \beta'_{i+1})$ with the following property: either (m) $l'_i K_s = 0$, $l''_i K_s = 0$, for all $s = 1, 2, \ldots, \mu$; or (m′) $l'_i K_t \neq 0$ for every two subarcs l'_i, l''_i as above and some $t = 1, 2, \ldots, \mu$, and $l'_i K_s = 0$, $l''_i K_s = 0$ ($s \neq t$; $s = 1, 2, \ldots, \mu$). For every $s = 1, \ldots, \mu$, case (m′) occurs for at least one $i = 1, \ldots, b$.

Since we have exactly b arcs l_i and μ continua K_s, we must have $\mu \leqslant b$. Thus (iii) is proved.

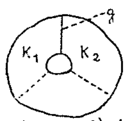
12. – Examples concerning § 11.

We may consider now a finite collection $[K]$ of given distinct fine-cyclic elements K with non-zero intersection, $S_0 = \Pi K \neq 0$. Then, by (§ **10**), S_0 is the finite sum of, say n, distinct continua $g \in \Gamma$, $S_0 = g_1 + \ldots + g_n$, $1 \leqslant n < +\infty$. Let m be the number of the elements $K \in [K]$. Finally let $\{\gamma\}_{S_0}$ be the collection of all components γ of $J - S_0$.

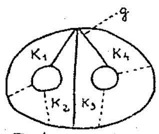

$1 \leqslant m \leqslant +\infty$, $n = 1$


$1 \leqslant m \leqslant +\infty$, $n = 1$

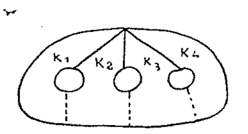
If $n = 1$ and thus $S_0 = g = g_1$ and if either $gJ^* = 0$, or $gJ^*_i \neq 0$ for (at most) one value $i = 0, 1, \ldots, \nu$, then by (**11**.ii), the sets $K - g$, $K \in [K]$, belong to distinct components $\gamma \in \{\gamma\}_g$, we have $g = \Pi K$, and the family $[K]$ is finite or countable. The situation we have described here occurs necessarily if $\nu = 0$. The illustrations show that the various cases are all possibile.



$m = 2$, $n = 1$, $\nu = 1$



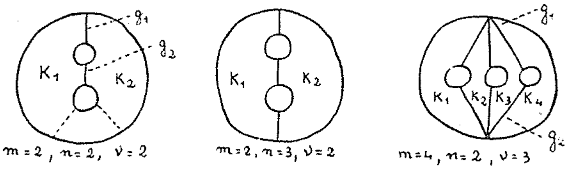
$m = 4$, $n = 1$, $\nu = 2$



$m = 4$, $n = 1$, $\nu = 1$

If $n = 1$, and $gJ^*_i \neq 0$ for exactly N values of $i = 0, 1, \ldots, \nu$, $2 \leqslant N \leqslant \nu + 1$, then we may consider first all the components $\gamma \in \{\gamma\}_g$ of the first class, or of the second class with $b \leqslant 1$. Then there is at most an element K with $K - g \subset \gamma$ for each γ of these types. We shall then consider the components γ of the second class with $b \geqslant 2$, $a \geqslant 0$. Then $\sum (a + b - 1) \leqslant \nu$, and thus there are at most ν of these components. Since for each γ of this second type there are exactly b pairs of arcs like l'_i, l''_i, we may have at most b elements $K \in [K]$ with $K - g \subset \gamma$, and thus a total number $\sum b \leqslant 2\nu$ of elements K of this type. Let us observe we whave in any case $b \leqslant N$ for every γ. The illustrations below give examples of the situation just now discussed.

If $n \geqslant 2$ then by § **10** there is exactly one component γ_0 of $J - g_1$ with $g_2 + \ldots + g_n \subset \gamma_0$. For every component $\gamma' \neq \gamma_0$ of $J - g_1$, we cannot have $K - S_0 \subset \gamma'$, $S_0 = g_1 + \ldots + g_n$. Let $F(\gamma_0) = m_1 + \ldots + m_b$, $\gamma_0 \gamma_0^* = J_1'^* + \ldots + J_a'^* + l_1 + \ldots + l_b$, $\gamma_0^* = \gamma_0 \gamma_0^* + F(\gamma_0)$, and $m_1 + \ldots + m_b \subset g_1$. Now it may occur that $g_2 \, l_i \neq 0$ for some $i = 1, 2, \ldots, b$, and then there will be at most as many components $\gamma \in \{\gamma\}_{S_0}$, $S_0 = g_1 + \ldots + g_n$, with $F(\gamma)\, g_1 \neq 0$, $F(\gamma)\, g_2 \neq 0$, as arcs $l_i' = (\alpha_i' \beta_i)$, $l_i'' = (\alpha_i \beta_i')$ with $\alpha_i' \in g_2$, $\beta_i \in g_1$, or $\alpha_i \in g_1$, $\beta_i' \in g_2$. Since there are at most $2b$ of these arcs we have $m \leqslant 2b \leqslant 2(\nu + 1)$. The illustrations below show examples of this situation.



We shall not discuss whether m may be > 2 with $n \geqslant 3$.

13. – Denumerability of fine-cyclic elements.

(**13.** i) Given $\varepsilon > 0$ the collection of all distinct fine-cyclic elements K of (T, J) with $\operatorname{diam} T(K) \geqslant \varepsilon$ is finite.

(**13.** ii) The collection $\{K\}$ of all distinct fine-cyclic elements K of (T, J) is either empty, or finite, or denumerable.

P r o o f. Since (ii) is a corollary of (i) we have only to prove (i). Suppose that (i) is not true. Then there exists a sequence $[K_n]$ of distinct elements $K_n \in \{K\}$ with $\operatorname{diam} T(K_n) \geqslant \varepsilon$. Let $K_{10} = K$, and let us prove that we have $K_{10} K_n = 0$ for all but finitely many n. Suppose, if possibile that $K_{10} K_n \neq 0$ for infinitely many n. Since $K_{10} K_n$ is the finite sum of continua $g \in \Gamma$ and $K_n - K_{10} K_n = K_n - K_{10}$ is completely contained in one of the components G of $J - K_{10}$, we have $\varepsilon \leqslant \operatorname{diam} T(K_n) = \operatorname{diam} T(K_n - K_{10}) \leqslant \operatorname{diam} T(G)$ for infinitely many n. Since only finitely many components G of $J - K_{10}$ satisfy the relation $\operatorname{diam} T(G) \geqslant \varepsilon$, we conclude that infinitely many sets $K_n - K_{10}$ should be in the same component G and satisfy the relation $K_n K_{10} \neq 0$. But $F(G)$ is contained in finitely many continua $g \in \Gamma$, $g \subset K_{10}$; thus infinitely many K_n should have in common with K_{10} a well determined continuum $g_0 \in \Gamma$, $g_0 \subset K_{10}$, a contradiction, because of (**11.** iii). We have proved that there is an index $n_1 \geqslant 2$ with $K_n K_{10} = 0$ for all $n \geqslant n_1$. Let $K_{20} = K_{n_1}$, hence $K_{10} K_{20} = 0$ for all $n \geqslant n_1$, and let us repeat the reasoning above on the new sequence K_n, $n \geqslant n_1$. We will obtain an index n_2 and an element $K_{30} = K_{n_2}$, such that $K_{10} K_{20} = K_{10} K_{30} = K_{20} K_{30} = 0$, $(K_{10} + K_{20}) K_n = 0$ for all $n \geqslant n_2$. By indefinite repetition of this procedure we will obtain a sequence $[K_{n0}]$ of elements K with $K_{r0} K_{s0} = 0$, $\operatorname{diam} T(K_{r0}) \geqslant \varepsilon$ $(r \neq s;\ r, s = 1, 2, \dots)$. We shall denote this sequence by $[K_n]$ for the sake of simplicity. Thus $\operatorname{diam} T(K_n) \geqslant \varepsilon$, $K_m K_n = 0$, $(m \neq n;\ m, n = 1, 2, \dots)$.

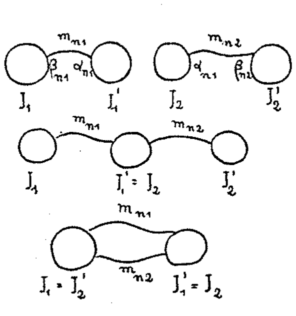
For every n there are two points $w_{n1}, w_{n2} \in K_n$ with $| T(w_{n1}) - T(w_{n2}) | \geqslant \varepsilon$. Then, by two successive extractions, we can pick a subsequence, say $[K_n]$, such that $w_{n1} \to w_1$, $w_{n2} \to w_2$ as $n \to \infty$ for some $w_1, w_2 \in J$. Then we have $| T(w_1) - T(w_2) | \geqslant \varepsilon$, and therefore $4d \equiv | w_1 - w_2 | > 0$.

Let d_1, $0 < d_1 \leqslant d$, be a number such that $w, w' \in J$, $| w - w' | \leqslant d_1$, implies $| T(w) - T(w') | \leqslant \varepsilon/4$.

Let δ, $0 < \delta \leqslant d_1$, be a number such that any two points $w_1, w' \in J$ with $| w - w' | \leqslant \delta$ belong to some arc $\lambda \subset J$ of diameter $\leqslant d_1$ (uniform local connectivity of J). Since $| w_{n1} - w_1 | \to 0$, $| w_{n2} - w_2 | \to 0$, we may discard finitely many K_n in such a way that for the new sequence, say $[K_n]$, we have $| w_{n1} - w_1 |$, $| w_{n2} - w_2 | < \delta$.

Then the two closed circles c_1, c_2 of centers w_1, w_2 and radii d_1 are disjoint. In addition, for every n, there are two paths $\lambda_{n1} = (w_{n1} w_1)$, $\lambda_{n2} = (w_{n2} w_2)$, of diameters $\leqslant d_1$ and thus $[\lambda_{n1}] \subset c_1$, $[\lambda_{n2}] \subset c_2$.

Since $K_n K_{n+1} = 0$, by (§ **10**. i) there is a component γ_n of $J - K_n$ with $K_{n+1} \subset \gamma_n$. We have $F(\gamma_n) = m_1 + \dots + m_b$, $\gamma_n \gamma_n^* = J_1'^* + \dots + J_a'^* + l_1 + \dots + l_b$, $\gamma_n^* = \gamma_n \gamma_n^* + F(\gamma_n)$, where m_i, l_i, $J_i'^*$, a, b may depend on n. Certainly $F(\gamma_n)$ separates both $w_{n1} + w_{n2}$ from $w_{n+1,1}$, $w_{n+1,2}$ in J. Thus we have $(\lambda_{n1} + \lambda_{n+1,1}) \cdot F(\gamma_n) \neq 0$, $(\lambda_{n2} + \lambda_{n+1,2}) \cdot F(\gamma_n) \neq 0$, since $\lambda_{ns} + \lambda_{n+1,s}$ joins w_{ns} to $w_{n+1,s}$ $(s = 1, 2)$ in J. Let w_{n1}', w_{n2}' be the last points on $\lambda_{n1} + \lambda_{n+1,1}$, $\lambda_{n2} + \lambda_{n+1,2}$ which are on $F(\gamma_n)$. We may say that $w_{n1}' \in m_{n1}$, $w_{n2}' \in m_{n2}$ where m_{n1}, m_{n2} are two of the continua m_i relative to $F(\gamma_n)$. Let us prove that $m_{n1} m_{n2} = 0$, for every n. Indeed we have $| T(w_{n1}') - T(w_{n2}') | \geqslant | T(w_1) - T(w_2) | - 2\varepsilon/4 \geqslant \varepsilon/2$, and thus $T(w_{n1}') \neq T(w_{n2}')$. Since K_n has property P' we have $m_{n1} \subset g_{n1}$, $m_{n2} \subset g_{n2}$, $g_{n1}, g_{n2} \in \Gamma$, and T is constant on g_{n1} $[g_{n2}]$ where it has the value $T(w_{n1}')$ $[T(w_{n2}')]$, and $T(w_{n1}') \neq T(w_{n2}')$. Thus $g_{n1} g_{n2} = 0$, and γ_n is certainly of the second class (§ **2**).



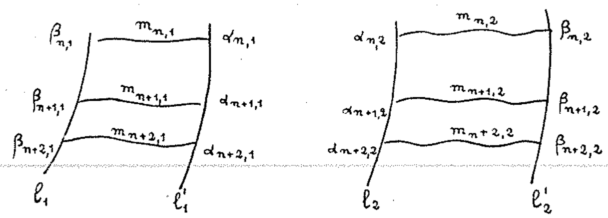
By (§ **3**) we know that m_{n1} joins two points β_{n1}, α_{n1} of two arcs, say l_1, l_1', of two curves J_s^* $(s = 0, 1, \dots, \nu)$, say $l_1 \subset J_1^*$, $l_1' \subset J_1'^*$, $J_1 \neq J_1'$. Analogously m_{n2} joins two points α_{n2}, β_{n2} of two arcs, say l_2, l_2', of two curves J_s^*, say $l_2 \subset J_2^*$, $l_2' \in J_2'^*$, $J_2 \neq J_2'$. We cannot exclude that J_2, or J_2' coincide with J_1, or J_1', as the illustrations show.

Since the curves J_s^* $(s = 0, 1, \dots, \nu)$ are disjoint and compact, we can estract a new subsequence, say $[K_n]$, such that $\beta_{n1} \to \beta_1$, $\alpha_{n1} \to \alpha_1$, $\alpha_{n2} \to \alpha_2$, $\beta_{n2} \to \beta_2$,

$w'_{n1} \to w_1$, $w'_{n2} \to w_2$, as $n \to \infty$. As a consequence we can also suppose that the points $\beta_{n1}, \alpha_{n1}, \alpha_{n2}, \beta_{n2}$ belong to the same curves J_1, J'_1, J_2, J'_2 for all n, and then the same occurs for $\beta_1, \alpha_1, \alpha_2, \beta_2$:

$$\beta_{n1}, \beta_1 \in J_1, \qquad \alpha_{n1}, \alpha_1 \in J'_1, \qquad \alpha_{n2}, \alpha_2 \in J_2, \qquad \beta_{n2}, \beta_2 \in J'_2 .$$

We can even suppose that the sequences $[\beta_{n1}]$, $[\alpha_{n1}]$, $[\alpha_{n2}]$, $[\beta_{n2}]$ are ordered on J_1, J'_1, J_2, J'_2 respectively. Finally let us observe that $\alpha_{ns} \beta_{ns} \in m_{ns}$, $m_{ns} \subset g_{ns}$, $\beta_{ns} \to \beta_s$, $\alpha_{ns} \to a_s$, $(s = 1, 2)$, and then, by the upper semicontinuity of the collection Γ, there are two continua g_1, g_2 with $w_1, \beta_1, \alpha_1 \in g_1$, $w_2, \beta_2, \alpha_2 \in g_2$.



Since $|T(g_1) - T(g_2)| = |T(w_1) - T(w_2)| \geq \varepsilon$, we have $\{g_1, g_2\} \geq \delta$ and thus $|\alpha_s - \beta_t| \geq \delta$, $|\alpha_t - \beta_t| \geq \delta$ $(s \neq t;\ s, t = 1, 2)$. From here it follows that we may suppose that the points $[\beta_{n1}]$, $[\alpha_{n1}]$, $[\alpha_{n2}]$, $[\beta_{n2}]$ belong to four arcs $l_1 = (\beta_{n1} \beta_1)$, $l'_1 = (\alpha_1 \alpha_{n1})$, $l_2 = (\beta_{n2} \beta_2)$, $l'_2 = (\alpha_2 \alpha_{n2})$ completely disjoint.

The consequence of all this is that the two couples

$$M_1 = g_{n1} + g_{n+2,1}, \qquad M_2 = g_{n2} + g_{n+2,2},$$

separate J. In particular $M_1 + M_2$ separates $h = g_{n+1,1}$ and $h' = g_{n+1,2}$ in J, a contradiction, since $h, h' \subset K_{n+1}$, K_{n+1} is a continuum, and $(M_1 + M_2) K_{n+1} = 0$.

Thus we have proved that $\operatorname{diam} T(K_n) \geq \varepsilon$ for infinitely many distinct $K_n \in \{K\}$ is impossible, and (i) is proved.

(**13.** iii) If w_0 is any point $w_0 \in J$ and $[K]$ any collection of fine-cyclic elements K of (T, J) with $w_0 \in K$, then the sum $\omega = \sum K$ of all elements $K \in [K]$ is a continuum of J having both properties P_1 and P'.

Proof. Property P' is trivial. Denote by g_0 the continuum $g_0 \in \Gamma$ with $w_0 \in g_0$. Let us prove that ω is closed. This is trivial if $[K]$ is finite. Let us suppose $[K]$ denumerable. Let $\overline{w}$ be a point of accumulation of ω, and $[w_n]$ any sequence of points $w_n \in \omega$, $w_n \to \overline{w}$. If infinitely many w_n belong to the same $K \in [K]$, then also $\overline{w} \in K$, and $\overline{w} \in \omega$. Otherwise we can extract from $[w_n]$ a subsequence, say still $[w_n]$, such that each w_n belongs to the element $K_n \in [K]$, and the elements $K_1, K_2, \ldots$, are all distinct. Since $w_0 \in K_n$, and $w_n \to \overline{w}$, $w_n \in K_n$, we conclude that both points w_0 and $\overline{w}$ belong to the set $\liminf K_n$ which, therefore, is not empty. Thus $s = \limsup K_n$ is a continuum containing both w_0 and $\bar{w}$. On the other hand, by (i) above, diam $T(K_n) \to 0$ as $n \to \infty$ and, hence diam $T(s) = 0$. Thus $s \subset \bar{g}$ for some $\bar{g} \in \Gamma$. Since $w_0 \in s \subset \bar{g}$, $w_0 \in g_0$, we deduce $g_0 \, \bar{g} \neq 0$, and, finally, $\bar{g} = g_0$, $\bar{w} \in g_0$ where $g_0 \subset \omega$. Thus $\bar{w} \in \omega$, and ω is closed.

Let us prove property P_1. Let γ be any component of $J - \omega = J - \sum K$. Then $F(\gamma)$ is characterized by (v) of § **2**. If w_1 is any point of one of the continua k_i, say $w_1 \in k_1$, then w_1 belongs to some element $K \in [K]$, say $w_1 \in K_1$. On the other hand γ must be completely contained in one of the components, say G_1, of $J - K_1$ and hence $w_1 \in F(G_1)$. Therefore w_1 belongs to one of the continua $g \in \Gamma$, finite in number, covering $F(G_1)$ [(i) of § **4**]. Thus k_1 should be covered by countable many continua $g \in \Gamma$; hence K_1 should also be covered by countably many distinct (and thus disjoint) continua $g \in \Gamma$. This is possible only if k_1 is covered by just one of these continua, say $k_1 \subset g_1$, $g_1 \in \Gamma$. This implies that T is constant on k_1. Thus T is constant on each component of $F(\gamma)$ and ω has property P_1.

14. – A topological lemma.

(**14**. i) Given any two points $w_1, w_2 \in J$ and a finite system $S = g_1 + \dots + g_n$ of distinct continua $g_i \in \Gamma$ $(i = 1, 2, \dots, n)$, such that S separates w_1 and w_2 in J, then there is a subsystem $S' \subset S$, say $S' = g_1 + \dots + g_\mu$, separating w_1 and w_2 in J, with $\mu \leqslant \nu + 1$, $1 \leqslant \mu \leqslant n$.

P r o o f. The points w_1, w_2 are in two distinct components G_1, G_2 of $J - S$, $S = g_1 + \dots + g_n$, say $w_1 \in G_1$, $w_2 \in G_2$. Let $K = G_1 + F(G_1)$. Then K is a continuum of J. Let G_2' be the component of $J - K$ containing G_2. Then by virtue of (**2**. v) $F(G_2')$ reduces to a finite number of components $k_1 + \dots + k_c$, $1 \leqslant c \leqslant \nu + 1$ and each k_i is contained in a $g_j \in \Gamma$. Let then $g_1, \dots, g_d$, $d \leqslant c$, be the continua in Γ contained in S for which $k_1 + \dots + k_c \subset g_1 + \dots + g_d$. We assert that $S' = g_1 + \dots + g_d$ separates w_1, w_2 in J. If this were not the case we would have a simple arc α in $J - S'$ connecting w_1, w_2. But then the intersection $\alpha \cdot F(G_2')$ would be not empty and hence $\alpha S' \neq 0$, a contradiction. Since $1 \leqslant d \leqslant \nu + 1$ the proof is complete.

15. – A covering theorem.

As usual we shall denote by PEANO space M the continuous image of a closed interval.

(**15**. i) Given any PEANO space M and for every point $p \in M$ a neighborhood $U = U(p)$ of p in M then there are two finite chains of points of M, not necessarily distinct p_i, p'_i $(i = 1, 2, ..., N;\ p_1 = p'_1)$, such that, if $U'_i = U(p'_i)$ $(i = 1, 2, ..., N)$, we have

$$\sum_{i=1}^{N} U'_i \supset M, \qquad p_i,\ p_{i+1} \in U'_i \quad (i = 1, ..., N-1), \qquad p_N \in U'_N .$$

Proof. Let $p = p(t)$, $0 \leqslant t \leqslant 1$, be any continuous mapping from $0 \leqslant t \leqslant 1$ onto M. For every $t \in [0, 1]$ let us denote by $u(t)$ a subinterval of $[0, 1]$, open in $[0, 1]$, containg t and such that

$$(1) \qquad p[u(t)] \subset U[p(t)] .$$

The existence of such a subinterval $u(t)$ of $[0, 1]$ follows from the continuity of $p(t)$ at t. Now, by BOREL covering theorem, there is a finite chain of points $0 \leqslant t''_1 < t''_2 < ... < t''_M \leqslant 1$ such that $\sum u(t''_j)$ is a covering of $[0, 1]$. Let t'_1 denote that point t''_s for which $u(t'_1) = u(t''_s) = [0, \beta_1)$ covers $t = 0$ and has maximum β_1. Let t'_2 denote that point t''_s for which $u(t'_2) = u(t''_s) = (\alpha_2, \beta_2)$ covers $t = \beta_1$ and has maximum β_2. Let t'_3 denote that point t''_s for which $u(t'_3) = u(t''_s) = (\alpha_3, \beta_3)$ covers $t = \beta_2$ and has maximum β_3. By finitely many repetitions of this procedure we will determine a finite chain $t'_1, t'_2, ..., t'_N$ of points $t \in [0, 1]$ such that $\sum u(t'_i)$ is a covering of $[0, 1]$ and $u(t'_1) = [0, \beta_1)$, $u(t'_2) = (\alpha_2, \beta_2)$, ..., $u(t'_{N-1}) = (\alpha_{N-1}, \beta_{N-1})$, $u(t'_N) = [\alpha_N, 1]$, $0 < \beta_1 < ... < \beta_N = 1$, $\alpha_2 < \beta_1$, $\alpha_3 < \beta_2$, ..., $\alpha_N > \beta_{N-1}$.

A consequence of method above for the choice of the intervals $u(t'_i)$ is that we will have also $0 = \alpha_1 < \alpha_2 < \beta_1 < \alpha_3 < \beta_2 < \alpha_4 < \beta_3 < ... < \alpha_N < \beta_{N-1} < \beta_N = 1$. Now take arbitrary points t_i as follows $t_1 = t'_1$, $\alpha_2 < t_2 < \beta_1$, $\alpha_3 < t_3 < \beta_2$, ..., $\alpha_N < t_N < \beta_{N-1}$, and put $p_i = p(t_i)$, $p'_i = p(t'_i)$, $(i = 1, 2, ..., N)$. We have $p_1 = p'_1$, and, since $t_1, t_2 \in u(t'_1)$, $t_2, t_3 \in u(t'_2)$, ..., $t_{N-1}, t_N \in u(t'_{N-1})$, $t_N \in u(t'_N)$, we conclude that $p_1, p_2 \in U'_1$, $p_2, p_3 \in U'_2$, ..., $p_{N-1}, p_N \in U'_{N-1}$, $p_N \in U'_N$, where $U'_i = U(p'_i)$ $(i = 1, 2, ..., N)$. Statement (i) is completely proved.

16. – Properties of separation in J.

(**16.** i) Given any point $w_0 \in J$ and any simple arc $l \subset J$ such that each point $w \in l$ is separated from w_0 in J by some finite system S of continua $g \in \Gamma(T, J)$, then there is a finite system $S_0 = g_1 + ... + g_\mu$, $1 \leqslant \mu \leqslant \nu + 1$, of continua $g \in \Gamma$ separating l from w_0 in J.

Proof. Let us consider l as the continuous image l: $w = w(t)$, $0 \leqslant t \leqslant 1$, of $[0, 1]$. Then for every $t \in [0, 1]$ there is a subinterval $u = u(t)$ of $[0, 1]$, containing t and open in $[0, 1]$ such that the correponding arc α of l, containing the point $w = w(t)$, is completely contained in the component $\gamma = \gamma(w)$ of $J - S$, where $S = g_1 + \dots + g_n = S(w)$, in the finite system of continua $g \in \Gamma$ separating $w(t)$ from w_0 in J. We can now apply (**15.** i) and we have a pair of chains

$$0 = t_1 < t_2 < \dots < t_N \leqslant 1, \qquad 0 = t_1' < t_2' < \dots < t_N' = 1,$$

such that

$$t_1, t_2 \in u(t_1'), \quad \dots, \quad t_{N-1}, t_N \in u(t_{N-1}'), \quad t_N \in u(t_N'), \quad u(t_1') + \dots + u(t_{N-1}') + u(t_N') = [0, 1].$$

As a consequence, if we put $w_i = w(t_i)$, $w_i' = w(t_i')$, $(i = 1, 2, \dots, N)$, we have

$$w_1, w_2 \in \gamma(w_1'), \qquad \dots, \qquad w_{N-1}, w_N \in \gamma(w_{N-1}'), \qquad w_N \in \gamma(w_N'),$$

and the whole arc $(w_1 w_2)$ of l is in $\gamma(w_1')$, the whole arc $(w_2 w_3)$ of l is in $\gamma(w_2')$, ..., $(w_{N-1} w_N)$ of l is in $\gamma(w_{N-1}')$. We may observe that each set $\gamma_i = \gamma(w_i')$ is a component of $J - S_i$, $S_i = g_{i1} + \dots + g_{i\mu_i}$, where S_i is a system of $1 \leqslant \mu_i \leqslant \nu + 1$ (**14**, i) continua $g_{is} \in \Gamma$ $(s = 1, \dots, \mu_i)$, and thus, in general (§ **2**, § **3**)

$$F(\gamma_i) = m_{i1} + \dots + m_{ib}, \qquad \gamma_i \gamma_i^* = J_1'^* + \dots + J_a'^* + l_1 + \dots + l_b,$$

$$\gamma_i^* = \gamma_i \gamma_i^* + F(\gamma_i),$$

where a and b may depend upon i, and

$$m_{i1} + \dots + m_{ib} \subset g_{i1} + \dots + g_{i\mu_i}.$$

Let $S_1'' = S_1' = S_1$. Then let us denote by S_2' the sum of all distinct continua $g \in S_1'' = S_1$ and all distinct continua $g \in S_2$ which are not contained in $\gamma_1' = \gamma_1$. Then $\gamma_1' = \gamma_1$ is still a component of $J - S_2'$ and, by (**14.** i), there is a subsystem $S_2'' \subset S_2'$ of $\leqslant \nu + 1$ continua $g \in S_2'$, $g \in \Gamma$, separating w_2 from w_0 in J. Thus w_2 belongs to a component of γ_2' of $J - S_2''$ with $\gamma_2' \supset \gamma_1'$, $\gamma_2' \supset \gamma_2$, and we have

$$(w_1 w_2) \subset \gamma_1' \subset \gamma_2', \qquad (w_2 w_3) \subset \gamma_2'.$$

Now let S_3' be the sum of all distinct continua $g \in S_2''$ and of all distinct continua $g \in S_3$ which are not contained in γ_2'. Then γ_2' is still a component of $J - S_3''$ and, by (**14.** i), there is a subsystem $S_3'' \subset S_3'$ of $\leqslant \nu + 1$ continua

$g \in S_3'$, $g \in \Gamma$, separating w_3 from w_0 in J. Thus w_3 belongs to a component γ_3' of $J - S_3''$ with $\gamma_3' \supset \gamma_2' \supset \gamma_1'$, $\gamma_3' \supset \gamma_3$, and we have

$$(w_1\, w_2) \subset \gamma_1' \subset \gamma_2' \subset \gamma_3', \qquad (w_2 w_3) \subset \gamma_2' \subset \gamma_3', \qquad (w_3\, w_4) \subset \gamma_3',$$

and so on. After N repeated applications of this process, we obtain a finite system $S = S_N'' = g_1 + \ldots + g_\mu$ of $\mu \leqslant \nu + 1$ continua g, separating w_N from w_0 in J, and such that, if $\gamma' = \gamma_N$, is the component of $J - S = J - S_N''$ containing w_N, we have

$$(w_1\, w_2) \subset \gamma_1' \subset \ldots \subset \gamma_N', \qquad (w_2\, w_3) \subset \gamma_2' \subset \ldots \subset \gamma_N', \qquad \ldots, \qquad (w_{N-1}\, w_N) \subset \gamma_N',$$

and thus $l \subset \gamma_N'$, and the entire arc l is separated from w_0 in J by the system S of $\leqslant \nu + 1$ continua $g \in \Gamma$. Statement (i) is proved.

Remark. In the reasoning above we have not used explicitely the hypothesis that l is a simple arc. Thus (iii) holds even if l is any continuous curve, since we can always consider any representation l: $w = w(t)$, $0 \leqslant t \leqslant 1$, of l on the unit interval $[0, 1]$.

(**16.** ii) Given any point $w_0 \in J$ and any PEANO space $M \subset J$ such that each point $w \in M$ is separated from w_0 in J by some finite system S of continua $g \in \Gamma$, then there is a finite system $S_0 = g_1 + \ldots + g_\mu$, $1 \leqslant \mu \leqslant \nu + 1$, of continua $g \in \Gamma$ separating M from w_0 in J.

This statement is a consequence of (i) and of the remark above, since M can always be considered as the set of points covered by a continuous curve (not necessarily simple) l: $w = w(t)$, $0 \leqslant t \leqslant 1$, $[l] = M \subset J$.

(**16.** iii) If $\overline{R} \subset J$ is a JORDAN region, if $R \subset J$ is open in J, and

$$RR^* = J_1' + \ldots + J_a' + l_1 + \ldots + l_b, \qquad l_j = J_j''^*\, R^*,$$

$$F(R) = \lambda_1 + \ldots + \lambda_b, \qquad R^* = RR^* + F(R),$$

and thus

$$R^* = J_1'^* + \ldots + J_a'^* + (l_1 + \lambda_1 + l_2 + \lambda_2 + \ldots + l_b + \lambda_b),$$

where the last line gives the boundary curves of R^*, if for each line λ_i there is a finite system $S_i = g_{i1} + \ldots + g_{i\mu_i}$, of $0 \leqslant \mu_i \leqslant \nu + 1$ continua $g_{is} \in \Gamma$ separating λ_i from a point $w_0 \in J - \overline{R}$, then there is also a finite system $S = g_1 + \ldots + g_\mu$, $1 \leqslant \mu \leqslant \nu + 1$, of μ continua $g \in \Gamma$, $g \in S_1 + \ldots + S_b$, separating the whole region $\overline{R}$ from w_0 in J.

Proof. We consider the system $\overline{S} = S_1 + \ldots + S_b$ and the component γ_0 of $J - \overline{S}$ containing w_0. Then $F(\gamma_0) = m_1 + \ldots + m_b \subset g_1 + \ldots + g_\mu$, where $g_1 + \ldots + g_\mu$, $1 \leqslant \mu \leqslant \nu + 1$, is a subsystem S of $\overline{S}$. It is easy to prove that S separates the whole region R from w_0 in J.

17. – A characterization of fine-cyclic elements.

Given any point $w_0 \in J$ let g_0 be the continuum $g_0 \in \Gamma$ containing w_0, and let $\Omega = \Omega(w_0)$ be the set of all points $w \in J$ which have the following property: there is no finite system S of continua $g \in \Gamma$, $S = g_1 + \ldots + g_n$, separating w_0 from w_0 in J. Thus $g_0 \subset \Omega(w_0)$. In addition, if $w_1 \in \Omega(w_0)$, then $w_0 \in \Omega(w_1)$. Also, if w_0, w_1, w_2 are distinct points of J, and $w_1 \in \Omega(w_0)$, $w_2 \in \Omega(w_1)$, then $w_2 \in \Omega(w_0)$. Indeed, in the contrary case, there would be a system S separating w_0 and w_2 in J, then $w_0 \in \gamma$, $w_2 \in \gamma'$, where γ, γ' are distinct components of $J - S$, and $\gamma\gamma' = 0$. On the other hand, $w_1 \in \Omega(w_0)$, $w_1 \in \Omega(w_2)$, and hence $b \in \gamma$, $b \in \gamma'$, $b \in \gamma\gamma'$, $\gamma\gamma' \neq 0$, a contradiction. Thereby we have proved that $w_2 \in \Omega(w_0)$.

(**17.** i) For every $w_0 \in J$ and $\Omega = \Omega(w_0)$ we have either $\Omega = \sum K$, or $\Omega = g_0$, according as w_0 belongs, or does not belong to some fine-cyclic element K of (T, J), and then, in the first case $\sum$ is extended over all fine-cyclic elements K with $w_0 \in K$.

Proof. By definition we have $w_0 \in g_0$, $g_0 \subset \Omega$, and thus Ω is not empty. Suppose if possibile that a point of accumulation $\overline{w}$ of Ω does not belong to Ω. Then $\overline{w}$ is separated from w_0 by some finite system $S = g_1 + \ldots + g_n$ of continua $g \in \Gamma$, and thus $\overline{w}$, w_0 belong to different components, say γ, γ_0, of $J - S$. Then $F(\gamma) = m_1 + \ldots + m_k \subset g_1 + \ldots + g_n = S$, $\gamma S = 0$, and $\{\overline{w}, S\} \equiv \delta > 0$. No point w of the neighborhood of $\overline{w}$ of radius δ may belong to Ω. Thus $\overline{w}$ is not a point of accumulation of Ω, a contradiction. This proves that all points of accumulation of Ω belong to Ω and thus Ω is closed.

Let us now prove that Ω is a continuum. Indeed, in the contrary case, there would be a component Ω_1 of Ω, distinct from the component Ω_0 of Ω containing w_0, and there would be also a JORDAN region $\overline{R} \subset J$ with $R \subset J$, open in J, and

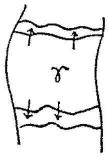
$$\Omega_1 \subset R \subset J, \qquad F(R) = \lambda_1 + \ldots + \lambda_b, \qquad R^* = RR^* + F(R),$$

$$RR^* = J_1'^* + \ldots + J_a'^* + l_1 + \ldots + l_b, \qquad l_j = J_j''^* R^*,$$

$$R^* = J_1'^* + \ldots + J_a'^* + (l_1 + \lambda_1 + \ldots + l_b + \lambda_b), \qquad F(R) \cdot \Omega = 0.$$

Thus each point $w \in \lambda_i$ is separated from w_0 in J by some finite system $S = S(w)$ of continua $g \in \Gamma$. Then, by (**16.** i) there is a system $S_i = S_{i1} + \dots + g_{i\mu_i}$ of $1 \leqslant \mu_i \leqslant \nu + 1$, continua $g \in \Gamma$ separating λ_i from w_0 in R, and also, by (**16.** iii), there is a finite system $S = g_1 + \dots + g_\mu$, $1 \leqslant \mu \leqslant \nu + 1$, of μ continua $g \in \Gamma$ separating the whole of R from w_0 in J. Thus it is contradictory to suppose that R contains points $w \in \Omega$. This proves that Ω is a continuum.

If w_0 does not belong to any fine-cyclic element K, then we should prove that $\Omega = g_0$. We shall prove first that Ω has properties P_1 and P'.



The property P' is trivial. In order to prove property P_1, let us consider any component γ of $J - \Omega$, let $\bar{w}$ be any point $\bar{w} \in \gamma$, and let $Q_1, Q_2, \dots, Q_n, \dots$ be a sequence of JORDAN regions $\bar{w} \in Q_1 \subset Q_2 \subset \dots \subset Q_n \subset \dots$ invading γ. Then for each Q_n there is, by (**16.** iii), a finite system $S_n = g_{n1} + \dots + g_{n\mu_n}$ of $1 \leqslant \mu_n \leqslant \nu + 1$, of continua $g \in \Gamma$ separating Q_n from w_0 in J and thus $Q_n \subset \gamma_n$ where γ_n is a component of $J - S_n$.

By proceeding as in (**16.** i) we can now define a modified sequence $[S_n']$ of systems S_n', each made up of at most $\nu + 1$ continua $g \in \Gamma$, such that, if γ_n' is the component of $J - S_n'$ containing Q_n, we have $\gamma_1' \subset \gamma_2' \subset \gamma_3' \subset \dots$, $Q_1 \subset Q_2 \subset Q_3 \subset \dots$, $Q_n \subset \gamma_n'$ $(n = 1, 2, \dots)$. Thus $\gamma_n' \uparrow \gamma'$ where γ' is a connected subset of J. Since $Q_n \uparrow \gamma$, we have $\gamma \subset \gamma'$. On the other hand, since each point of γ' belongs to some γ_n' and hence to $J - \Omega$, we have $\gamma' \subset \gamma$, and finally $\gamma' = \gamma$. According to § **2** and § **3** we have now $F(\gamma) = k_1 + \dots + k_c = m_1 + \dots + m_b$, $\gamma\gamma^* = J_1'^* + \dots + J_a'^* + l_1 + \dots + l_b$, for some a, b, c, and $m_i = \Omega$. Then for n large enough we have $F(\gamma_n') = m_{n1} + \dots + m_{nb}$, $\gamma_n'\gamma_n'^* = J_1'^* + \dots + J_a'^* + l_{n1} + \dots + l_{nb}$ for the same a, b, and $l_{ni} \subset l_i^{(0)}$, $l_{ni} \uparrow l_i^{(0)}$, $\liminf m_{ni} = \limsup m_{ni} = m_i$, $i = 1, \dots, b$, as $n \to \infty$. Since $m_{n1} + \dots + m_{nb} \subset g_{n1} + \dots + g_{n\mu_n}$, each continuum m_{ni} is contained in some g_{nj} $(j = 1, \dots, \mu_n)$. Since Γ is an upper semicontinuous collection, each m_i must be contained in some $g \in \Gamma$, hence $k_1 + \dots + k_c = m_1 + \dots + m_b \subset g_1 + \dots + g_\mu$, $1 \leqslant \mu \leqslant \nu + 1$. Thus each continuum k_i is contained in some g_j $(j = 1, \dots, \mu)$, and T is constant on k_i $(i = 1, \dots, c)$. This proves that Ω satisfies condition P_1.

Let us prove now that $\Omega = g_0$. We know that w_0 is contained in no element K. Thus Ω itself is no such element. Suppose, if possible that Ω contains an element K as a proper subset. Then $w_0 \in J - K$ and hence $w_0 \in G$, where G is some component of $J - K$. By (v) of § **2** and (i) of § **4**, $F(G)$ is contained in a finite system $S = g_1 + \dots + g_\mu$ of continua $g \in \Gamma$ and S separates w_0 from some point $w' \in K$, with $K \subset \Omega$ and hence $w' \in \Omega$, a contradiction. Thus Ω has both properties (1) and (3) of § **9** but Ω is no element K. Therefore Ω does not satisfy property (2) of § **9**; i. e., $\Omega = g_0 \in \Gamma$.

If $w_0 \in K$ for some K, let $\omega = \sum K$, where $\sum$ ranges over all elements K

with $w_0 \in K$. Then $w_0 \in \omega$. By (iii) of § **13** we know that ω is a continum of J satisfying both properties P_1 and P'. For every K with $w_0 \in K$, no point $w \in K$, $w \neq w_0$, can be separated from w_0 in J by some finite system S of continua $g \in \Gamma$. Indeed, if this were the case, then the points w_0, w would belong to distinct components γ_0, γ of $J - S$ and then $K' = K - \gamma$ would be a proper subcontinuum K' of K satisfying both conditions P_1 and P', a contradiction. Thus $K \subset \Omega$ for every element K with $w_0 \in K$, and hence $\omega = \sum K \subset \Omega$. Let us prove that $\Omega = \omega$. Suppose $\Omega - \omega \neq 0$ and let $\bar{w}$ be any point $\bar{w} \in \Omega - \omega$. Then $\bar{w}$ belongs to one component, say $\bar{\gamma}$, of $J - \omega$. Suppose first that $\bar{w}$ belongs to an element K, say $K = \bar{K}$. Because of what we have just proved we have $\bar{K} \subset \Omega(\bar{w})$ where $\bar{w} \in \Omega = \Omega(w_0)$, hence $\bar{K} \subset \Omega$. If $g_0 \bar{K} \neq 0$, then $g_0 \subset \bar{K}$, $w_0 \in \bar{K}$, and $\bar{K} \subset \omega$, i e., $\bar{w} \in \omega$, a contradiction. If $g_0 \bar{K} = 0$, then g_0 is in one of the components, say G_0, of $J - \bar{K}$, and hence there is, also, a finite system S separating $\bar{K} - S$ from g_0, i. e., $\bar{K} - S$ from w_0 in J. If $\bar{w} \in \bar{K} - S$, then S separates $\bar{w}$ from w_0 in J, a contradiction, since $\bar{w} \in \Omega$. If $\bar{w} \in S$ then all points $\bar{\bar{w}} \in J$ with $\{\bar{\bar{w}}, \bar{w}\}_T < \delta$ [**3** (A), 10. 7, p. 159] for δ sufficiently small belong to γ, and, on the other hand, not all such points $\bar{\bar{w}}$ can belong to S (since S is a finite sum of continua $g \in \Gamma$), and thus some must belong to $\bar{K}$. Thus there is at least one point $\bar{\bar{w}} \in \gamma$, $\bar{\bar{w}} \in \bar{K} - S$, which is separated by S from w_0 in J, a contradiction since $\bar{\bar{w}} \in \Omega$. All this proves that $\bar{w}$ belongs to no element K. Then the set $\bar{\Omega} = \Omega(\bar{w})$ of all points $w \in J$ which are not separated from $\bar{w}$ by systems S is the single continuum $\bar{g} \in \Gamma$, with $\bar{w} \in \bar{g}$, and $\bar{g} \subset \bar{\gamma}$. Let R denote any finitely connected JORDAN region with $\bar{g} \subset R'$, $R \subset \gamma$, so that $F(R)$ separates $\bar{g}$ from $F(\gamma)$ in J. Thus $F(R)$ separates $\bar{g}$ from w_0 in J. Now each point $w' \in F(R)$ is separated from $\bar{g}$ in J by some finite system S' of continua $g \in \Gamma$. By (**16.** iii) we conclude that there is a finite system S' of continua $g' \in \Gamma$ separating $\bar{g}$ from the whole of $F(R)$ in J. Thus S separates $\bar{w}$ from $F(R)$ and $F(R)$ separates $\bar{w}$ from w_0 in J. We deduce that S separates $\bar{w}$ from w_0 in J, a contradiction since $\bar{w} \in \Omega$. All this shows that the hypothesis $\Omega - \omega \neq 0$ leads to a contradiction. Since $\Omega \supset \omega$, we conclude that $\Omega = \omega = \sum K$, and statement (i) is thereby completely proved.

18. – Fine-cyclic additivity theorem.

By (**13.** ii) we know that the collection $\{K\}$ of all distinct fine-cyclic elements K of (T, J) is countable. Let us denote, therefore, such a collection by

$$\{K\} = \{K_1, K_2, \ldots, K_i, \ldots\},$$

where we do not exclude that this collection is finite, or empty.

According to § **9** we may now associate to each element K_i a class $\{(T_i, J_i)\}$ of all the continuous mappings which can be obtained from (T, J) by retraction with respect to K_i relative to any subregion J_i, $K_i \subset J_i \subset J$, such that K_i has property P_2 in J_i. For each i, all the continuous mappings (T_i, J_i) are rigidly equivalent and thus define, a unique element which we have denoted as a fine-cyclic element $(T_i, J_i) = S_i$ (surface) of (T, J). We shall consider the sequence

$$\{S\} = \{S_1, S_2, \ldots, S_i, \ldots\}$$

of these surfaces and by $L(S_i)$ $(i = 1, 2, \ldots)$ their LEBESGUE area. The following statement holds

(**18.** i) (Fine-cyclic additivity Theorem.) *For every continuous mapping* $S = (T, J)$ *we have:*

$$L(S) = \sum_{i=1}^{\infty} L(S_i).$$

Proof. We can suppose J a polygonal region. The proof we give is only a modification of the one given in [**3** (A), 36.3, i, p. 511].

(a) Let us prove first that $L(S) \geqslant \sum L(S_i)$. Let $K_1, K_2, \ldots, K_\mu$ be any finite system of disjoint fine-cyclic elements. Then $S = K_1 + \ldots + K_\mu$ is a finite sum S of $1 \leqslant m \leqslant \mu$ distinct continua having properties P_1 and P'. For each i there exists a region $J_i, K_i \subset J_i \subset J$, such that K_i has properties P', P_1, P_2 with respect to (T, J_i), and we shall denote by $S_i = (T_i, J_i)$ the retraction of (T, J) with respect to K_i in J_i, $i = 1, \ldots, \mu$. By (**7.** iii) there exists a sequence of quasi linear mappings (P_n, J) and certain figures F_{ni} such that, if $F_n = F_{n1} + \ldots + F_{n\mu}$, we have

(α1) $S \subset F_n \subset J$, $K_i \subset F_{ni} \subset J_i$, $F_{ni} \supset F_{n+1,i}$;

(α2) F_{ni} satisfies conditions P_1, P_2 with respect to (P_n, J_i);

(α3) (P_{ni}, J_i) denote the retraction of (P_n, J) with respect to F_{ni} in J_i;

(α4) $P_n(w) = P_{ni}(w) =$ constant on each component of the intersection $F_{ni}F_{nj}$ $(i \neq j;\ i, j = 1, 2, \ldots, \mu)$; thus the μ mappings (P_{ni}, F_{ni}) $(i = 1, \ldots, \mu)$, define univocally a quasi linear mapping (P_{n0}, F_n) on F_n;

$$(\alpha 5) \quad (P_n,\ J) \to (T,\ J), \qquad a(P_n,\ J) \to L(T,\ J) = L(S);$$

$$(\alpha 6) \quad a(P_{n0},\ F_n) \leqslant a(P_n,\ J), \qquad a(P_{n0},\ F_{ni}) = a(P_{ni},\ J_i);$$

$$(\alpha 7) \quad (P_{ni},\ J_i) \to (T_i,\ J_i) \quad (i = 1,\ 2, \ldots,\ \mu).$$

Now we have

$$\sum_{i=1}^{\mu} L(S_i) \leqslant \sum_{i=1}^{\mu} \underline{\lim}\ a(P_{ni},\ J_i) = \sum_{i=1}^{\mu} \underline{\lim}\ a(P_{n0},\ F_{ni}) \leqslant \underline{\lim} \sum_{i=1}^{\mu} a(P_{n0},\ F_{ni}) =$$

$$= \underline{\lim}\ a(P_{n0},\ F_n) \leqslant \lim a(P_n,\ J) = L(S)$$

for every μ, and thus $\sum_{i=1}^{\infty} L(S_i) \leqslant L(S)$.

(b) Let us prove now that $L(S) \leqslant \sum L(S_i)$.

Let $\varepsilon > 0$ be any positive number and let $[K] = [K_i,\ (i = 1, \ldots, N)]$ be the finite collection of all fine-cyclic elements with $\operatorname{diam} T(K_i) \geqslant \varepsilon$ $(i = 1, \ldots, N)$. Then the set $H = K_1 + \ldots + K_N$ is a compact subset of J having properties P' and P_1 with respect to $(T,\ J)$. The set $G = J - H$ is open in J and the family of its components γ is countable. The subfamily of all components γ of G with diam $T(\gamma) \geqslant \varepsilon$ is finite, say $[\gamma_j\ (j = 1, \ldots, \mu)]$. Since each γ_j has a boundary $F(\gamma_j)$ which is contained in a finite sum b_j of continua $g \in \Gamma$, $g \subset H$, there is a finite collection, say $[g_1, \ldots, g_{M_0}]$, of continua $g \in \Gamma$, $g \subset H$, separating each γ_j $(j = 1, \ldots, \mu)$ from the remaining part of A in J. We shall enlarge this collection into a collection $B = [g_h\ (h = 1, \ldots, M)]$, $M \geqslant M_0$, with continua $g \in \Gamma$, $g \subset H$, such that for any $i \neq j$ and $i, j = 1, \ldots, N$,t here exists a finite sum $s_{ij} = \sum'' g_h$ of continua $g_h \in B$ such that $K_i - s_{ij}$ and $K_j - s_{ij}$ are separated by s_{ij} in J. Obviously, if $s_{ij0} = K_i K_j$, then either s_{ij0} is empty, or $s_{ij0} = \sum' g_h \subset s_{ij}$, for all $i \neq j$ and $i, j = 1, 2, \ldots, N$.

Now $\widetilde{\gamma}_j = \gamma_j + b_j$ $(j = 1, \ldots, \mu)$ is a continuum of J. For each point $w_0 \in \widetilde{\gamma}_j$ we shall consider the set $U(w_0)$ of all points $w \in J$ such that $\{w,\ w_0\}_T \leqslant \varepsilon$; i. e., whose distance with respect to T from w_0 is $\leqslant \varepsilon$ [**3** (A), 10.7, p. 159]. The set $U(w_0)$ is a continuum having w_0 as an interior point in J, and the set $\widetilde{\gamma}_j \cdot U(w_0)$ is a compact subset of $\widetilde{\gamma}_j$, also having w_0 as an interior point (interior with respect to $\widetilde{\gamma}_j$). We shall denote by $U_j(w_0)$ the component of $\widetilde{\gamma}_j \cdot U(w_0)$ containing w_0; hence w_0 is an interior point of $U_j(w_0)$, $U_j(w_0) \subset \widetilde{\gamma}_j$, and $U_j(w_0)$ is a subcontinuum of $\widetilde{\gamma}_j$.

Finally let $V = V_j(w_0)$ denote the set

$$V = V_j(w_0) = U_j(w_0) + \sum K,$$

where $\sum$ ranges over all fine-cyclic elements $K \in \{K\} - [K]$ with $K \cdot U_j(w_0) \neq 0$, $K \subset \tilde{\gamma}_j$. The set V is a subcontinuum of γ_j having both properties P' and P_1 with respect to (T, J). The property P' is trivial; the proofs that V is a continuum and has property P_1 are analogous to the ones given in (**13**. iii).

Let us observe that $\text{diam} T(V) \leqslant 4\varepsilon$. Indeed, if $w', w'' \in V$ and it occurs that $w', w'' \in U = U_j(w_0)$ then $| T(w') - T(w'') | \leqslant | T(w') - T(w_0) | + | T(w_0) - T(w'') | \leqslant \varepsilon + \varepsilon = 2\varepsilon$. If $w' \in K'$, $w'' \in K''$, $K', K'' \subset \sum K$, then there are points $w_1' \in K'U$, $w_1'' \in K''U$, and thus

$$| T(w') - T(w'') | \leqslant | T(w') - T(w_1') | + | T(w_1') - T(w_1'') | + \\ + | T(w_1'') - T(w'') | \leqslant \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon .$$

Analogously if $w' \in K' \subset \sum K$, $w'' \in U$, or viceversa. Hence we have $| T(w') - T(w'') | \leqslant 4\varepsilon$ for all $w', w'' \in V$; i. e., diam $T(V) \leqslant 4\varepsilon$.

Each point w_0 of the compact set $\tilde{\gamma}_j$ belongs to a set $V_j(w_0)$ and w_0 is an interior point of $V_j(w_0)$ (interior with respect to $\tilde{\gamma}_j$). By BOREL covering theorem there exists a finite collection $V_{j1}, \ldots, V_{jt_j}$ of sets V covering $\tilde{\gamma}_j$. Finally we can say that the whole finite collection $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_\mu$ is covered by a finite collection $[V_{js}]$ of sets V.

Each continuum V_{js} has property P_1 with respect to (T, J); hence the collection $\{\tau\}_{js}$ of all components τ of $J - V_{js}$ is countable, for each τ the boundary $F(\tau)$ is contained in a finite sum b' of continua $g \in \Gamma$, $g \subset A$, and the subcollection $[\tau]_{js}$ of all $\tau \in \{\tau\}_{js}$ with $\text{diam} T(\tau) > \varepsilon$ is finite. Denote by $B_1 = [g_1', \ldots, g_{M'}']$ the sum of all collections b' relative to the components $\tau \in [\tau]_{js}$, and all s and j, and put $B_2 = B + B_1$. We shall denote by B, B_1, B_2 also the sets covered by the continua $g \in B, B_1, B_2$, and by C the set $C = H + B_2$. All sets B, B_1, B_2, C are closed and $B \subset B_2$, $B_1 \subset B_2$, $B \subset H \subset C$. Finally, if $B_2 - B = [g_s'' \, (s = 1, \ldots, M'')]$, the continua g_s'' $(s = 1, \ldots, M'')$ are distinct and $H \cdot (B_2 - B) = 0$.

The set C has both properties P' and P_1 with respect to (T, J). In addition, if γ is any component of $J - C$ we have $\text{diam} T(\gamma) \leqslant 4\varepsilon$. Indeed, either γ is a component of $J - H$ distinct from $\gamma_1, \ldots, \gamma_\mu$ and then $\text{diam} T(\gamma) < \varepsilon$; or γ has at least one point in common with some γ_j and then $\gamma \subset \gamma_j \subset \tilde{\gamma}_j$ and also $\gamma V_{js} \neq 0$ for some s. Then $\gamma\tau = 0$ for every $\tau \in [\tau]_{js}$ and hence $\gamma \subset V_{js} + \sum' \tau$ where $\sum'$ ranges over all $\tau \in \{\tau\}_{js} - [\tau]_{js}$, and diam $T(\tau) \leqslant \varepsilon$, diam $T(V_{js}) \leqslant 4\varepsilon$. Thus, by the same reasoning above, we have diam $T(\gamma) \leqslant 6\varepsilon$.

We now apply (**7**. iii) to the systems $S = B_2$ and $K = H + (B_2 - B)$.

If J_i $(i = 1, \ldots, N)$ are polygonal regions with $K_i \subset J_i \subset J$ such that K_i satisfies properties P_1 and P_2 with respect to (T, J_i), such that conditions (a), (b), (c) of (**7**. iii) are satisfied [see **5**. (iv) and (v)], we shall denote by (T_i, J_i) the retraction of (T, J) with respect to K_i in J_i, $(i = 1, \ldots, N)$. Also, since $g''_s \in \Gamma$ for every $g''_s \in B_2 - B$, and $H \cdot (B_2 - B) = 0$, there are certain disjoint regions F_s with $g''_s \subset F_s \subset J$ such that g''_s satisfies conditions P_1 and P_2 with respect to (T, F_s) and the retraction (T''_s, F_s) of (T, J) with respect to g''_s in F_s is the constant mapping (T''_s, F_s) with $T''_s(w) = T(g''_s)$ for all $w \in F_s$.

By (7. iii), for all $n = 1, 2, \ldots$, there exists a quasi linear mapping (P_n, J) and certain polygonal regions R_{ni}, r_{ns} such that $K_i \subset R_{ni} \subset J_i$, $R_{ni} \supset R_{n+1,i}$, $R_{ni} \downarrow K_i$ as $n \to \infty$, $(i = 1, \ldots, N)$; $g''_s \subset r_{ns} \subset F_s$, $r_{ns} \downarrow g''_s$ as $n \to \infty$, $r_{ns} \supset r_{n+1,s}$, $(s = 1, \ldots, M'')$; (2) R_{ni} $[r_{ns}]$ satisfies both conditions P_1 and P_2 with respect to (P_n, J_i), $i = 1, \ldots, N$, $[(P_n, F_s), (s = 1, \ldots, M)]$; (3) $P_n(w) \rightrightarrows T(w)$ in J; $P_{ni}(w) \rightrightarrows T_i(w)$ in J_i, $(i = 1, \ldots, N)$; $P''_{ns}(w) \rightrightarrows T_s(w) = T(g''_s)$ in F_s $(s = 1, \ldots, M')$, where (P_{ni}, J_i), (P''_{ns}, F_s) denote the retractions of (P_{ni}, J) with respect to R_{ni} in J_i, and with respect to r_{ns} in F_s; (4) for every $h = 1, \ldots, M$, there exists a polygonal region f_{nh} such that $g_h \subset f_{nh}$ and P_n is costant on f_{nh}; P_n is constant on each r_{ns} $(s = 1, \ldots, M'')$; (5) if $s_{ij0} = K_i K_j = 0$ then $R_{ni} R_{nj} = 0$; if $s_{ij0} = \sum' g \neq 0$ then $R_{ni} R_{nj} = \sum' f_{nh}$ $(i \neq j;\ i, j = 1, \ldots, N)$; for every $h = 1, \ldots, M$, $i = 1, \ldots, N$, with $g_h \subset K_i$ we have $f_{nh} \subset R_{ni}$ and $P_n(w) = P_{ni}(w) =$ constant on f_{nh}, for every $s = 1, \ldots, M''$, we have $P_n(w) = P''_{ns}(w) =$ constant on r_{ns}; (6) $a(P_n, J) \to L(T, J)$, $a(P_{ni}, J_i) \to L(T_i, J_i)$ as $n \to \infty$, $(i = 1, \ldots, N)$.

Thus, for a convenient n, we have $| P_n(w) - T(w) | < \varepsilon$ for all $w \in J$, $| P_{ni}(w) - T_i(w) | < \varepsilon$ for all $w \in J_i$ $(i = 1, \ldots, N)$, and $a(P_{ni}, J_i) < L(T_i, J_i) + \varepsilon/N$ $(i = 1, \ldots, N)$. Now let us define a quasi linear mapping (P, J) as follows. First of all let $P(w) = P_n(w)$ for all $w \in R_{ni}$ $(i = 1, \ldots, N)$. This definition implies that $P(w) = P_{ni}(w)$ for all $w \in R_{ni}$ $(i = 1, \ldots, N)$. Indeed (P_{ni}, J_i) is a retraction of (P_n, J) with respect to R_{ni} in J_i and hence $P_n(w) = P_{ni}(w)$ for all $w \in R_{ni}$. Also, let us observe that, for each $i \neq j$, $i, j = 1, \ldots, N$, either $s_{ij0} = 0$ and $R_{ni} R_{nj} = 0$, or $s_{ij0} = \sum' g_h \neq 0$ and then $R_{ni} R_{nj} = \sum' f_{nh}$ and $P_{ni}(w) = P_{nj}(w) = P_n(w) =$ constant in each $f_{nh} \subset \sum' f_{nh}$. Now let us consider the set $J - \sum_i R_{ni}$ and its closure Q_n. Each component q of Q_n is a polygonal region whose boundary $F(q)$ is the finite sum of disjoint arcs l, each l being part of the boundary of a region R_{ni} or r_{ns}, and $P_n(w) = P_{ni}(w) =$ constant on l, or $P_n(w) = P''_{ns}(w) =$ constant on l; i. e., $\lambda = P_n(w)$, $w \in l$, is a point. We may define $P(w)$ on each q in such a way that $a(P, q) = 0$, $P(w) = P_n(w) = \lambda$ on each arc l, and $P_n(q)$ is contained in the minimum convex body (polyhedron) containing the points λ which are the images of arcs l of q^*. This can be done quite elementarily as, for instance, in [**3** (A), 23.15, p. 387]. Let us observe that the arcs l of q^* not covering a component of q^* leave uncovered on the same component an equal number of arcs, say l', which all belong to J^*.

In such a way $P(w)$ is defined for every $w \in J$; i. e., the quasi linear mapping (P, J) is completely defined.

For every $w \in R_{ni}$ we have $P(w) = P_n(w)$ and hence $|P(w) - T(w)| = |P_n(w) - T(w)| < \varepsilon$. For every $w \in q$ where q is a component of Q we certainly have $|P_n(w) - T(w)| < \varepsilon$. On the other hand q is completely contained in some component γ of $J - C$ and hence $\operatorname{diam} T(\gamma) \leqslant 6\varepsilon$, and also $|P(w') - T(w')| < \varepsilon$ for all points $w' \in l$. Thus the points $\lambda = P_n(w)$, $w \in l$, belong to the ε-neighborhood of $T(\gamma)$ and hence $\operatorname{diam} P(w) \leqslant 8\varepsilon$. Finally, for every $w \in q$, we have $|P(w) - T(w)| \leqslant |P(w) - P(w')| + |P(w') - T(w')| + |T(w') - T(w)|$ for any $w' \in l$, and also, $|P(w) - T(w)| \leqslant \operatorname{diam} P(q) + \varepsilon + \operatorname{diam} T(\gamma) \leqslant 8\varepsilon + \varepsilon + 6\varepsilon = 15\varepsilon$. This proves that $|P(w) - T(w)| \leqslant 15\varepsilon$ for all $w \in J$. On the other hand, by combining the triangles t of linearity of (P_n, J) in each R_{ni} and of (P, q) in each q, we obtain a unique subdivision S_0 of J into triangles t such that $a(\Delta) = a[P(t)] > 0$ implies that t is in one and only one of the regions R_{ni} $(i = 1, \dots, N)$ and then $(P, t) = (P_n, t) = (P_{ni}, t)$. This implies that

$$a(P, J) = \sum a(\Delta) = \sum_{i=1}^{N} a(P_{ni}, R_{ni}) = \sum_{i=1}^{N} a(P_{ni}, J_i) \leqslant$$

$$\leqslant \sum_{i=1}^{N} L(J_i, T_i) + N(\varepsilon/N).$$

Thus $|P(w) - T(w)| \leqslant 15\varepsilon$ for all $w \in J$, and

$$a(P, J \leqslant \sum_{i=1}^{\infty} L(T_i, J_i) + \varepsilon .$$

By taking $\varepsilon = 1/m$ $(m = 1, 2, \dots)$ we obtain a sequence of quasi linear mappings (P_m, J) $(m = 1, 2, \dots)$, such that $(P_m, J) \to (T, J)$ as $m \to \infty$ and

$$a(P_m, J) \leqslant \sum_{i=1}^{\infty} L(T_i, J_i) + 1/m = \sum_{i=1}^{\infty} L(S_i) + 1/m .$$

Finally

$$L(T, J) \leqslant \varliminf_{m \to \infty} a(P_m, J) \leqslant \sum_{i=1}^{\infty} L(S_i) .$$

Thus (b) is proved.

(c) By combining (a) and (b) we have

$$L(T, J) = \sum_{i=1}^{\infty} L(S_i)$$

and thereby (i) is completely proved.

Bibliography.

[1] C. CARATHÉODORY: *Über die Begrenzung einfach zusammenhängender Gebiete*, Math. Ann. **73** (1913), 323-370.

[2] J. CECCONI: *Sulla additività ciclica degli integrali sopra una superficie*, Riv. Mat. Univ. Parma **4** (1953), 43-67.

[3] L. CESARI: (A) **Surface Area**, Princeton University Press, 1956. — (a) *Su un particolare processo di retrazione per superficie*, Riv. Mat. Univ. Parma **3** (1952), 25-42. — (b) *Una uguaglianza fondamentale per l'area delle superficie*, Mem. Accad. Italia **14** (1944), 891-951. — (c) *Contours of a Fréchet surface*, Riv. Mat. Univ. Parma **4** (1953), 173-194. — (d) *Properties of contours*, Rend. Mat. Appl. Roma (5) **15** (1956), 341-365. — (e) *Sulle superficie di Fréchet*, Riv. Mat. Univ. Parma **1** (1950), 19-44. — (f) *Retraction of mappings from finitely connected Jordan regions*, Bull. Amer. Math. Soc. **61** (1955), p. 529 (abstract). — (g) *A new process of retraction and the definition of fine-cyclic elements*, An. Acad. Brasil Ci. **29** (1957), 1-6.

[4] B. v. KERÉKJÁRTÓ: **Topologie**, Springer, Berlin 1923.

[5] R. L. MOORE: **Foundations of point set theory**, Amer. Math. Soc. Col. Pub., Vol. **13**, 1932.

[6] T. RADÓ: **Length and Area**, Amer. Math. Soc. Col. Pub., Vol. **30**, 1948.

[7] G. T. WHYBURN: **Analytic Topology**, Amer. Math. Soc. Col. Pub., Vol. **27**, 1942.

Previous studies on cyclic elements of higher orders:

[8] W. L. AYRES: *On the structure of a plane continuous curve*, Proc. Nat. Acad. Sci. **13** (1927), 650-657.

[9] D. W. HALL: *On a decomposition of true cyclic elements*, Trans. Amer. Math. Soc. **47** (1940), 305-321.

[10] G. NOEBELING: *Eine Verschärfung des n-Beinsatses*, Fund. Math. **18** (1932), 23-38.

[11] H. E. RUTT: *Concerning the cut points of a continuous curve when the arc curve contains N independent arcs*, Amer. J. Math. **51** (1929), 217-246.

[12] G. T. WHYBURN: *Cyclic elements of higher orders*, Amer. J. Math. **56** (1934), 133-146.

[13] J. W. T. YOUNGS: *k-cyclic elements*, Amer. J. Math. **62** (1940), 449-456.