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On Generalised Riemann Spaces. (**)

1. – A space with coordinates x^i $(i = 1, 2, \ldots, n)$ with which is associated a non-symmetric tensor g_{ij} has been termed by EISENHART as a Generalised RIEMANN Space. If we denote the symmetric and skew-symmetric parts of the fundamental tensor g_{ij} by $g_{\underline{ij}}$ and $g_{\underset{\vee}{ij}}$ respectively, then

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}), \qquad g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}).$$

Using the notation

$$(1 \cdot 1) \qquad \Delta_{pij} = \frac{1}{2}(g_{ip,j} + g_{pj,i} - g_{ij,p}),$$

where comma followed by an index denotes partial derivative, the coefficients of the affine connection for the space are defined by the relation

$$\Delta^h_{ij} = g^{\underline{hp}} \Delta_{pij}.$$

The components of the fundamental tensor g_{ij} and $g_{\alpha\beta}$ in the two coordinate systems x^i and y^α are related by the equation

$$(1 \cdot 2) \qquad g_{ij} = g_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j},$$

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where in this and in what follows the quantities with Greek indices correspond to the coordinate system y^{α}.

Differentiating equations (1·2) partially with respect to x^k, we have

$$(1\cdot3)\qquad g_{ij,k} = g_{\alpha\beta,\gamma}\, y^{\alpha}_{,i}\, y^{\beta}_{,j}\, y^{\gamma}_{,k} + g_{\alpha\beta}\, y^{\alpha}_{,ik}\, y^{\beta}_{,j} + g_{\alpha\beta}\, y^{\alpha}_{,i}\, y^{\beta}_{,jk}\,.$$

Interchanging the covariant indices i, j, k in equations (1·3), we have

$$(1\cdot4)\qquad \begin{cases} g_{ik,j} = g_{\alpha\beta,\gamma}\, y^{\alpha}_{,i}\, y^{\beta}_{,k}\, y^{\gamma}_{,j} + g_{\alpha\beta}\, y^{\alpha}_{,ij}\, y^{\beta}_{,k} + g_{\alpha\beta}\, y^{\alpha}_{,i}\, y^{\beta}_{,kj} \\ g_{kj,i} = g_{\alpha\beta,\gamma}\, y^{\alpha}_{,k}\, y^{\beta}_{,j}\, y^{\gamma}_{,i} + g_{\alpha\beta}\, y^{\alpha}_{,ki}\, y^{\beta}_{,j} + g_{\alpha\beta}\, y^{\alpha}_{,k}\, y^{\beta}_{,ji}\,. \end{cases}$$

Calculating one half of the sum of the equations (1·4) minus one half of the equation (1·3), we get

$$(1\cdot5)\qquad \Delta_{kij} = \Delta_{\gamma\alpha\beta}\, y^{\alpha}_{,i}\, y^{\beta}_{,j}\, y^{\gamma}_{,k} + g_{\underline{\alpha\beta}}\, y^{\alpha}_{,ij}\, y^{\beta}_{,k}\,.$$

Thus:

As in Riemannian space the quantities Δ_{ijk} are not the components of a tensor in Generalised Riemann space.

Equation (1·5) can be simplified as

$$\Delta_{kij} = g_{\underline{\alpha\varepsilon}}\, y^{\varepsilon}_{,k}\cdot(y^{\alpha}_{,ij} + y^{\delta}_{,i}\, \Delta^{\alpha}_{\delta\beta}\, y^{\beta}_{,j})\,.$$

From (1·1) it follows that

$$(1\cdot6)\qquad \Delta_{ijk} + \Delta_{jik} = g_{\underline{ij},k}$$

and

$$(1\cdot7)\qquad \Delta_{jki} + \Delta_{ikj} = g_{i\underline{j},k}\,.$$

Differentiating partially with respect to x^k the identity

$$g_{\underline{ij}}\, g^{\underline{il}} = \delta^{l}_{j}\,,$$

we get

$$(1\cdot8)\qquad g_{\underline{ij},k}\, g^{\underline{il}} + g_{\underline{ji}}\, g^{\underline{il}}_{,k} = 0,$$

which in consequence of (1·6) reduces to

$$g_{\underline{ij}}\ g^{\underline{il}}_{,k} = -g^{\underline{il}}\cdot(\Delta_{ijk} + \Delta_{jik})\,.$$

Multiplying this equation by $g^{\underline{jp}}$, we get

$$\delta^i_p\ g^{\underline{il}}_{,k} = -g^{\underline{il}}\ g^{\underline{jp}}\cdot(\Delta_{ijk} + \Delta_{jik})\,,$$

or

(1·9) $$g^{\underline{pl}}_{,k} = -g^{\underline{il}}\,\Delta^p_{ik} - g^{\underline{jp}}\,\Delta^l_{jk}\,.$$

Hence

$$g^{\underline{pl}}_{;k} = g^{\underline{pl}}_{,k} + g^{\underline{il}}\,\Delta^p_{ik} + g^{\underline{jp}}\,\Delta^l_{jk} = 0,$$

where semi-colon followed by an index denotes covariant differentiation.
Substituting for $g_{\underline{ij},k}$ from (1·7) in (1·8), we get

$$g_{\underline{ij}}\ g^{\underline{il}}_{,k} = -g^{\underline{il}}\cdot(\Delta_{jki} + \Delta_{ikj}),$$

$$g_{\underline{ij}}\ g^{\underline{jp}}\ g^{\underline{il}}_{,k} = -g^{\underline{il}}\ g^{\underline{ip}}\cdot(\Delta_{jki} + \Delta_{ikj})$$

or

(1·10) $$g^{\underline{pl}}_{,k} = -g^{\underline{il}}\,\Delta^p_{ki} - g^{\underline{jp}}\,\Delta^l_{kj}\,.$$

Comparing (1·9) and (1·10), we get the identity

(1·11) $$g^{\underline{il}}\,\Delta^p_{\underset{\vee}{ki}} + g^{\underline{jp}}\,\Delta^l_{\underset{\vee}{kj}} = 0\,.$$

2. – Let v_i be the covariant components of a vector. Its covariant derivative is then given by

(2·1) $$v_{i;j} = \frac{\partial v_i}{\partial x^j} - v_p\,\Delta^p_{ij}\,.$$

Subtracting the corresponding equation obtained by interchanging i and j from this we have the relation

$$(2\cdot2)\qquad v_{i;j}-v_{j;i}=\frac{\partial v_i}{\partial x^j}-\frac{\partial v_j}{\partial x^i}+2\,v_p\,\underset{\vee}{\Delta}{}^p_{ji},$$

where $\underset{\vee}{\Delta}{}^p_{ji}$ denotes the skew-symmetric part of Δ^p_{ji} in the covariant indices.

If the vector v is a gradient then $(2\cdot2)$ reduces to

$$(2\cdot3)\qquad v_{i;j}-v_{j;i}=2\,v_p\,\underset{\vee}{\Delta}{}^p_{ji}.$$

Hence we get the result:

The curl of a gradient vanishes in a Riemannian space but it does not generally vanish in Generalised Riemann Space. The condition that the curl of grad φ *may vanish is*

$$\frac{\partial\varphi}{\partial x^p}\,\underset{\vee}{\Delta}{}^p_{ij}=0\,.$$

This condition is identically satisfied for a space with constant fundamental metric tensor. Hence:

The curl of a gradient vanishes identically for a Generalised Riemann Space with constant fundamental metric tensor.

Taking covariant derivative of $(2\cdot1)$ with respect to x^k, we have

$$v_{i;jk}=v_{i,jk}-v_{i,a}\,\Delta^a_{jk}-v_{a,j}\,\Delta^a_{ik}-v_{a,k}\,\Delta^a_{ij}-v_a\,\frac{\partial}{\partial x^k}\Delta^a_{ij}+v_a\,\Delta^a_{bj}\,\Delta^b_{ik}+v_a\,\Delta^a_{ib}\,\Delta^b_{jk}\,.$$

Subtracting the equation obtained by interchanging j and k from this, we get

$$(2\cdot4)\qquad v_{i;jk}-v_{i;kj}=2v_{i,a}\,\underset{\vee}{\Delta}{}^a_{kj}-2\,\Delta^a_{ib}\,\underset{\vee}{\Delta}{}^b_{kj}+v_a\cdot[\Delta^a_{ik,j}-\Delta^a_{ij,k}+\Delta^a_{bj}\,\Delta^b_{ik}-\Delta^a_{bk}\,\Delta^b_{ij}]=$$

$$=2v_{i;a}\,\underset{\vee}{\Delta}{}^a_{kj}-v_a\,\Gamma^a_{ijk}\,,$$

where

$$(2\cdot5)\qquad \Gamma^a_{ijk}=\Delta^a_{ij,k}-\Delta^a_{ik,j}+\Delta^b_{ij}\,\Delta^a_{bk}-\Delta^b_{ik}\,\Delta^a_{bj}\,.$$

Hence Γ^a_{ijk} is a mixed tensor of fourth order skew-symmetric in j and k.

Taking a tensor of any order $P^{i_1 i_2 \ldots i_p}_{s_1 s_2 \ldots s_q}$ say, we have by a similar process

$$P^{\cdots}_{\ldots;\, l} = \frac{\partial P^{\cdots}_{\ldots}}{\partial x^l} + \sum_{r=1}^{p} \binom{a}{i_r} P^{\cdots}_{\ldots} \Delta^{i_r}_{al} - \sum_{t=1}^{q} \binom{s_t}{a} P^{\cdots}_{\ldots} \Delta^{a}_{s_t l},$$

where $P^{\cdots}_{\ldots}$ denotes the tensor $P^{i_1 i_2 \ldots i_p}_{s_1 s_2 \ldots s_q}$, and $\binom{a}{i_r} T^{\cdots}_{\ldots}$ denotes the result of replacing i_r by a in $T^{\cdots}_{\ldots}$ with similar meanings of the other brackets. Hence:

$$P^{\cdots}_{\ldots;\, lm} = \frac{\partial}{\partial x^m} P^{\cdots}_{\ldots;\, l} + \sum_{r=1}^{p} \binom{a}{i_r} P^{\cdots}_{\ldots;\, l} \Delta^{i_r}_{am} - \sum_{t=1}^{q} \binom{s_t}{a} P^{\cdots}_{\ldots;\, l} \Delta^{a}_{s_t m} - P^{\cdots}_{\ldots;\, a} \Delta^{a}_{lm},$$

$$P^{\cdots}_{\ldots;\, lm} - P^{\cdots}_{\ldots;\, ml} =$$

$$= -2 \Delta^{a}_{\underset{\vee}{lm}} P^{\cdots}_{\ldots;\, a} - \sum_{r=1}^{p} \sum_{n=1}^{p} \binom{a}{i_r} \binom{b}{i_n} P^{\cdots}_{\ldots} \Delta^{i_r}_{al} \Delta^{i_n}_{bm} + \sum_{r=1}^{p} \sum_{d=1}^{q} \binom{a}{i_r} \binom{s_d}{b} P^{\cdots}_{\ldots} \Delta^{i_r}_{al} \Delta^{b}_{s_d m} +$$

$$+ \sum_{t=1}^{q} \sum_{n=1}^{p} \binom{b}{i_n} \binom{s_t}{a} P^{\cdots}_{\ldots} \Delta^{a}_{s_t l} \Delta^{i_n}_{bm} - \sum_{d=1}^{q} \sum_{t=1}^{q} \binom{s_t}{a} \binom{s_d}{b} P^{\cdots}_{\ldots} \Delta^{a}_{s_t l} \Delta^{b}_{s_d m} -$$

$$- \sum_{t=1}^{q} \Delta^{a}_{s_t m} \left[\sum_{n=1}^{p} \binom{b}{i_n} \binom{s_t}{a} P^{\cdots}_{\ldots} \Delta^{i_n}_{bl} - \sum_{d=1}^{q} \binom{s_d}{b} \binom{s_t}{a} P^{\cdots}_{\ldots} \Delta^{b}_{s_d l} \right] +$$

$$+ \sum_{r=1}^{p} \Delta^{i_r}_{am} \left[\sum_{n=1}^{p} \binom{b}{i_n} \binom{a}{i_r} P^{\cdots}_{\ldots} \Delta^{i_n}_{bl} - \sum_{d=1}^{q} \binom{a}{i_r} \binom{s_d}{b} P^{\cdots}_{\ldots} \Delta^{b}_{s_d l} \right] +$$

$$+ \sum_{r=1}^{p} \binom{a}{i_r} P^{\cdots}_{\ldots} \left(\frac{\partial}{\partial x^m} \Delta^{i_r}_{al} - \frac{\partial}{\partial x^l} \Delta^{i_r}_{am} \right) + \sum_{t=1}^{q} \binom{s_t}{a} P^{\cdots}_{\ldots} \left(\frac{\partial}{\partial x^l} \Delta^{a}_{s_t m} - \frac{\partial}{\partial x^m} \Delta^{a}_{s_t l} \right).$$

On further reduction this reduces to

$$(2 \cdot 6) \qquad P^{\cdots}_{\ldots;\, lm} - P^{\cdots}_{\ldots;\, lm} =$$

$$= -2 \Delta^{a}_{\underset{\vee}{lm}} P^{\cdots}_{\ldots;\, a} + \sum_{r=1}^{p} \binom{a}{i_r} P^{\cdots}_{\ldots} \Gamma^{i_r}_{alm} + \sum_{t=1}^{q} \binom{s_t}{a} P^{\cdots}_{\ldots} \Gamma^{a}_{s_t m l}.$$

Formulae (2·4) *and* (2·6) *are* *Ricci's formulae for the Generalised Riemann Space.*

From (2·5) it follows that

$$\Gamma^h_{ijk} + \Gamma^h_{jki} + \Gamma^h_{kij} = \tag{2·7}$$

$$= 2\,[\Delta^h_{\underset{\vee}{ij},k} - \Delta^h_{\underset{\vee}{ik},j} + \Delta^h_{\underset{\vee}{jk},i} + \Delta^h_{lk}\,\Delta^l_{\underset{\vee}{ij}} - \Delta^h_{lj}\,\Delta^l_{\underset{\vee}{ik}} + \Delta^h_{li}\,\Delta^l_{\underset{\vee}{jk}}]\,.$$

Let

$$P^h_{ijk} = \Gamma^h_{ijk} + \Gamma^h_{jki} + \Gamma^h_{kij}\,. \tag{2·8}$$

It is deduced from the definition of P^h_{ijk} that

$$P^h_{ijk} = P^h_{jki} = P^h_{kij}\,. \tag{2·9}$$

From (2·7) it is easily seen that P^h_{ijk} are completely skew-symmetric in all the three covariant indices, i.e. it is a non-zero tensor only when i, j, k are distinct and unequal.

From the tensor P^h_{ijk} we get on contraction

$$P_{ij} \equiv P^h_{ijh} = P^h_{ijh} + P^h_{jhi} + P^h_{hij} = \Gamma_{ij} - \Gamma_{ji} = 2\,\Gamma_{\underset{\vee}{ij}}\,.$$

But since (EISENHART [2] (1))

$$\Gamma_{\underset{\vee}{ij}} = \Delta^h_{\underset{\vee}{ij}|h}\,, \qquad \Gamma_{\underline{ij}} = -R_{ij} + \Delta^l_{\underset{\vee}{ih}}\;\Delta^h_{\underset{\vee}{jl}}\,,$$

where solidus (|) followed by an index denotes covariant differentiation with respect to the CHRISTOFFEL symbols and R_{ij} is the RICCI tensor corresponding to the fundamental tensor $g_{\underline{ij}}$. Hence

$$P_{ij} = 2\,\Delta^h_{\underset{\vee}{ij}|h} \tag{2·10}$$

and

$$\Gamma_{ij} = \frac{1}{2}\,P_{ij} - R_{ij} + \Delta^l_{\underset{\vee}{ih}}\,\Delta^h_{\underset{\vee}{jl}}\,.$$

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(1) Numbers in bracket refer to « References » at the end of the paper.

Multiplying (2·10) by $g^{\underline{ij}}$ we have

$$P_{ij}\, g^{\underline{ij}} = 0\,.$$

Hence

$$\Gamma_{ij}\, g^{\underline{ij}} = -R + \Gamma^{l}_{\underset{\vee}{ih}}\, \Gamma^{h}_{\underset{\vee}{jl}}\, g^{\underline{ij}},$$

where R is the curvature scalar for the Riemannian space.

3. – Using the notation

$$\Gamma_{pijk} = g_{\underline{hp}}\, \Gamma^{h}_{ijk}\,, \tag{3·1}$$

we have

$$\Gamma_{pijk} = g_{\underline{hp}}\cdot[\Delta^{h}_{ij,k} - \Delta^{h}_{ik,j} + \Delta^{l}_{ij}\,\Delta^{h}_{lk} - \Delta^{l}_{ik}\,\Delta^{h}_{lj}] =$$

$$= \Delta_{pij,k} - \Delta^{h}_{ij}\,\Delta_{\underline{hp},k} - \Delta_{pik,j} + \Delta^{h}_{ik}\, g_{\underline{hp},j} + \Delta^{l}_{ij}\,\Delta_{plk} - \Delta^{l}_{ik}\,\Delta_{plj}\,.$$

But since

$$g_{\underline{ij},k} = \Delta_{jik} + \Delta_{ijk},$$

therefore

$$\Gamma_{pijk} = \Delta_{pij,k} - \Delta_{pik,j} + \Delta^{h}_{ik}\,\Delta_{hpj} - \Delta^{h}_{ij}\,\Delta_{hpk}\,. \tag{3·2}$$

Further simplifying it we have the alternative formula

$$\Gamma_{pijk} = \frac{1}{2}\,(g_{pj,ik} - g_{ij,pk} - g_{pk,ij} + g_{ik,pj}) + \Delta^{h}_{ik}\,\Delta_{hpj} - \Delta^{h}_{ij}\,\Delta_{hpk}\,.$$

From this formula we deduce that

$$\Gamma_{pijk} = -\Gamma_{ipjk} \qquad \text{and} \qquad \Gamma_{pijk} = -\Gamma_{pikj}\,.$$

Thus:

The tensor Γ_{pijk} is skew-symmetric in the first two as well as in the last two indices.

From (3·2) we have

$$\Gamma_{pijk} + \Gamma_{jkpi} =$$

$$= [g_{\underline{p}j,ik} - g_{\underline{i}j,pk} - g_{\underline{p}k,ij} + g_{\underline{i}k,pj}] + \Delta^{l}_{\underset{\vee}{ki}} \cdot \left(\frac{\partial g_{l\underline{p}}}{\partial x^{j}} + 2\Delta_{\underset{\vee}{lip}}\right) + \Delta^{l}_{\underset{\vee}{ij}} \cdot \left(\frac{\partial g_{l\underline{p}}}{\partial x^{k}} - 2\Delta_{\underset{\vee}{lkp}}\right) =$$

$$= 2R_{pijk} + \Delta^{l}_{\underset{\vee}{ki}} \cdot \left(\frac{\partial g_{l\underline{p}}}{\partial x^{j}} + 2\Delta_{\underset{\vee}{ljp}}\right) + \Delta^{l}_{\underset{\vee}{ij}} \left(\frac{\partial g_{l\underline{p}}}{\partial x^{r}} - 2\Delta_{\underset{\vee}{lkp}}\right),$$

where R_{pijk} are the RIEMANN symbols of first kind.

Also

$$\Gamma_{pijk} - \Gamma_{jkpi} = g_{\underset{\vee}{pj},ik} - g_{\underset{\vee}{ij},pk} - g_{\underset{\vee}{pk},ij} + g_{\underset{\vee}{ik},p} +$$

$$+ 2\,[\Delta^{h}_{\underline{i}k}\, \Delta_{\underset{\vee}{hpj}} + \Delta^{h}_{\underset{\vee}{ik}}\, \Delta_{h\underline{p}j} - \Delta^{h}_{\underline{i}j}\, \Delta_{\underset{\vee}{hpk}} - \Delta^{h}_{\underset{\vee}{ij}}\, \Delta_{\underset{\vee}{hpk}}]\,.$$

From (3·2) we have

$$\Gamma_{pijk} + \Gamma_{pjki} + \Gamma_{pkij} = \tag{3·3}$$

$$= (g_{\underset{\vee}{ji},pk} + g_{\underset{\vee}{ik},pj} + g_{\underset{\vee}{kj},pi}) + 2\,\Delta^{h}_{\underset{\vee}{ik}}\, \Delta_{hpj} + 2\,\Delta^{h}_{\underset{\vee}{ji}}\, \Delta_{hpk} + 2\,\Delta^{h}_{\underset{\vee}{kj}}\, \Delta_{hpi}\,.$$

(2·9) *and* (3·3) *are the forms of first* *Bianchi* *identities for the Generalised* *Riemann* *Space.*

By simple calculation we get

$$\Gamma^{h}_{ijk;l} + \Gamma^{h}_{ikl;j} + \Gamma^{h}_{ilj;k} = 2\,[\Gamma^{h}_{ial}\, \Delta^{a}_{\underset{\vee}{jk}} + \Gamma^{h}_{iaj}\, \Delta^{a}_{\underset{\vee}{kl}} + \Gamma^{h}_{iak}\, \Delta^{a}_{\underset{\vee}{lj}}]\,. \tag{3·4}$$

This equation when multiplied by $g_{h\underline{b}}$ gives

$$\Gamma_{bijk;l} + \Gamma_{bikl;j} + \Gamma_{bilj;k} = 2\,[\Gamma_{bial}\, \Delta^{a}_{\underset{\vee}{jk}} + \Gamma_{biaj}\, \Delta^{a}_{\underset{\vee}{kl}} + \Gamma_{biak}\, \Delta^{a}_{\underset{\vee}{lj}}]\,. \tag{3·5}$$

Thus:

(3·4) *and* (3·5) *are the second* *Bianchi* *identities in the Generalised* *Riemann* *Space.*

4. – As in the Riemannian space, we define parallelism of vectors with respect to a gives curve C in Generalised RIEMANN Space and say that a vector u of constant magnitude is parallel with respect to V_n along the curve C, if its derived vector in the direction of the curve is zero at all points of C. Since the coordinates of any point on the curve may be expressed as functions of the arc lenght s, the condition for vector u to be parallel along C is that

$$u^i_{;k} \frac{dx^k}{dx} = 0,$$

i.e.

$$\frac{du^i}{ds} + \Delta^i_{lk} \; u^l \frac{dx^k}{ds} = 0 \,. \tag{4·1}$$

The condition corresponding to (4·1) for Riemannian space is

$$\frac{du^i}{ds} + \Delta^i_{\underline{lk}.} \, u^l \frac{dx^k}{ds} = 0 \,.$$

Hence:

The condition that a vector u undergoes parallel displacement along a curve C in Riemannian space may also undergo parallel displacement along that curve in Generalised Riemann Space is that

$$\Delta^i_{\underset{\vee}{lk}} \; u^l \frac{du^k}{ds} = 0 \,. \tag{4·2}$$

Condition (4·2) is satisfied when u is a vector tangential to the curve, and then (4·1) represents the equation of a Geodesic for the Generalised RIEMANN space. Hence:

Geodesic in a Generalised Riemann Space are indentical with those in Riemannian space.

It is easy to prove that

If two vectors of constant magnitudes undergo parallel displacement along a given curve in Generalised Riemann Space, they are inclined at a constant angle.

The divergence of a vector λ^i in Generalised RIEMANN Space is defined by

$$\operatorname{div} \lambda^i = \lambda^i_{;i} = \frac{d\lambda^i}{dx^i} + \lambda^l \Delta^i_{li} \,.$$

But since (EISENHART [2])

$$\underset{\vee}{\Delta}{}^{i}_{li} = 0 .$$

We have

$$\operatorname{div} \lambda^i = \frac{\partial \lambda^i}{\partial x^i} + \lambda^l \Delta^i_{\underline{li}} = \lambda^i_{|i} .$$

Hence:

Divergence of a vector in Generalised Riemann Space has the same value as in Riemannian space.

Let $e_{(h)}$ $(h = 1, 2, \ldots, n)$ be the unit tangents to the congruences of an orthogonal ennuple in the Generalised RIEMANN Space. Then we have the identities

$$\sum_h e^i_{(h)}\, e_{(h)j} = \delta^i_j, \qquad \sum_h e^i_{(h)}\, e^j_{(h)} = g^{\underline{ij}}, \qquad \sum_h e_{(h)i}\, e_{(h)j} = g_{\underline{ij}} .$$

The invariant defined by

$$\bar{\gamma}_{lhk} = e_{(l)i;j}\, e^i_{(h)}\, e^j_{(k)} \tag{4·3}$$

are the RICCI's coefficients of rotation for the generalised space:

$$\bar{\gamma}_{lhk} = e^i_{(h)}\, e^j_{(k)} \cdot \left[\frac{\partial e_{(l)i}}{\partial x^j} - e_{(l)p}\, \Delta^p_{ij}\right] = e^i_{(h)}\, e^j_{(h)} \cdot \left[e_{(l)i|j} - e_{(l)p}\, \underset{\vee}{\Delta}{}^p_{ij}\right] .$$

If γ_{lhk} are the RICCI's coefficients of rotation for the Riemannian space, then

$$\bar{\gamma}_{lhk} = \gamma_{lhk} + e^i_{(h)}\, e^j_{(k)}\, e_{(l)p}\, \underset{\vee}{\Delta}{}^p_{ij} \tag{4·4}$$

Interchanging h and k and changing the dummy indices we get

$$\bar{\gamma}_{lkh} = \gamma_{lkh} + e^i_{(h)}\, e^j_{(k)}\, e_{(l)p}\, \underset{\vee}{\Delta}{}^p_{ij} . \tag{4·5}$$

Adding and subtracting the equations (4·4) and (4·5) we get the results

$$\bar{\gamma}_{l\underline{hk}} = \gamma_{l\underline{hk}} \tag{4·6}$$

and

$$\bar{\gamma}_{l\underset{\vee}{hk}} = \gamma_{l\underset{\vee}{hk}} + e^j_{(k)}\, e^i_{(h)}\, e_{(l)p}\, \underset{\vee}{\Delta}{}^p_{ji} . \tag{4·7}$$

From (4·7) it is evident that $\overline{\gamma}_{hll} = \gamma_{hll}$. Hence:

If the curves of the congruence, whose unit tangent is $e_{(l)}$, be geodesics in Riemannian space, they will also be geodesics in Generalised Riemann Space.

Taking covariant derivative of the identity

$$e_{(h)i}\, e^i_{(l)} = \delta^h_l$$

with respect to x^j we have

$$e_{(h)i;j}\, e^i_{(l)} + e_{(h)i}\, e^i_{(l);j} = 0 .$$

Multiplying by $e^j_{(k)}$ and suming with respect to j we obtain

$$e_{(h)i;j}\, e^i_{(l)}\, e^j_{(k)} + e^i_{(l);j}\, e_{(h)i}\, e^j_{(k)} = 0$$

wich is equivalent to

$$(4\cdot 8) \qquad \overline{\gamma}_{hlk} + \overline{\gamma}_{lhk} = 0 .$$

But we have (WEATHERBURN [**4**], p. 99)

$$(4\cdot 9) \qquad \gamma_{hlk} + \gamma_{lhk} = 0 .$$

Substituting for $\overline{\gamma}_{hlk}$ and $\overline{\gamma}_{lhk}$ from (4·5) and using (4·9), we get

$$e^j_{(k)}\, \underset{\vee}{\Delta^p_{ij}} \cdot (e^i_{(h)}\, e_{(l)p} + e^i_{(l)}\, e_{(h)p}) = 0 .$$

5. – We consider an infinitesimal point transformation

$$(5\cdot 1) \qquad \overline{x}^i = x^i + \in \varepsilon^i(x) .$$

LIE derivative $L\Omega$ of a given field Ω of a geometric object with respect to ε^i is defined by the equation

$$(5\cdot 2) \qquad D\Omega = \in(L\Omega) = \Omega(\overline{x}) - \overline{\Omega}(\overline{x}),$$

where $\overline{\Omega}(\overline{x})$ denotes the components of this object in the coordinate system $(\overline{x}^i)$, regarding (5·1) as a coordinate transformation from x^i to $\overline{x}^i$.

Let T^i_{jk} be the components of a tensor, then

$$\overline{T}^i_{jk}(\overline{x}) = T^l_{mn}(x)\ \frac{\mathrm{d}\overline{x}^i}{\mathrm{d}x^l}\ \frac{\partial x^m}{\partial \overline{x}^j}\ \frac{\partial x^n}{\partial \overline{x}^k}$$

and

$$T^i_{jk}(\overline{x}) = T^i_{jk}(x) + \frac{\partial}{\partial x^p}\, T^i_{jk} \in \varepsilon^p .$$

Hence we get

$$LT^i_{jk} = T^i_{jk,\,l}\ \varepsilon^l - T^a_{jk}\ \varepsilon^i_{,a} + T^i_{ak}\ \varepsilon^a_{,j} + T^i_{ja}\ \varepsilon^a_{,k} .$$

Now, taking a tensor of any order, $P^{i_1 i_2 \ldots i_p}_{s_1 s_2 \ldots s_q}$ say, we have by a similar process

$$(5\cdot 3)\quad L\, P^{i_1 i_2 \ldots i_p}_{s_1 s_2 \ldots s_q} = P^{i_1 i_2 \ldots i_p}_{s_1 s_2 \ldots s_q,\, l}\ \varepsilon^l - \sum_{r=1}^{p} P^{i_1 \ldots i_{r-1} a i_{r+1} \ldots i_p}_{s_1 s_2 \ldots s_q}\ \varepsilon^{i_r}_{,a} + \sum_{t=1}^{q} P^{i_1 i_2 \ldots i_p}_{s_1 \ldots s_{t-1} a s_{t+1} \ldots s_q}\ \varepsilon^a_{,s_t} .$$

From (5·3) we get

$$L(g^{\underline{ij}}) = g^{\underline{ij}}_{,k}\ \varepsilon^k - g^{\underline{aj}}\ \varepsilon^i_{,a} - g^{\underline{ia}}\ \varepsilon^j_{,a} = (- g^{\underline{il}}\ \Delta^j_{lk} - g^{\underline{lj}}\ \Delta^i_{lk})\varepsilon^k - g^{\underline{aj}}\ \varepsilon^i_{,a} - g^{\underline{ia}}\ \varepsilon^j_{,a} =$$

$$= - g^{\underline{lj}}\ \varepsilon^i_{;l} - g^{\underline{il}}\ \varepsilon^j_{;l} - 2\,\varepsilon^k \cdot (g^{\underline{lj}}\ \Delta^i_{\underset{\vee}{lk}} + g^{\underline{il}}\ \Delta^j_{\underset{\vee}{lk}}) .$$

Using (1·11) this reduces to

$$L(g^{\underline{ij}}) = - \varepsilon^{i;j} - \varepsilon^{i;j},$$

where

$$\varepsilon^{i;j} = g^{\underline{jl}}\ \varepsilon^i_{;l} .$$

Also

$$(5\cdot 4)\qquad L(g_{\underline{ij}}) = g_{\underline{ij},k}\ \varepsilon^k + g_{\underline{aj}}\ \varepsilon^a_{,i} + g_{\underline{ia}}\ \varepsilon^a_{,j} =$$

$$= (\Delta_{jki} + \Delta_{ikj})\varepsilon^k + g_{\underline{aj}}\ \varepsilon^a_{,i} + g_{\underline{ia}}\ \varepsilon^a_{,j} = g_{\underline{aj}}\ \varepsilon^a_{;i} + g_{\underline{ia}}\ \varepsilon^a_{;j} = \varepsilon_{j;i} + \varepsilon_{i;j} .$$

It can be easily seen that

The LIE derivative of the sum (difference) of two tensors is nothing but the sum (difference) of the LIE derivatives of the two tensors.

Since

$$\underline{\Gamma}^i_{jk}(\bar{x}) = \underline{\Gamma}^i_{jk}(x) + \frac{\partial}{\partial x^l}\,\underline{\Gamma}^i_{jk} \in \varepsilon^l$$

and

$$\overline{\Gamma}^i_{jk}(\bar{x}) = \Gamma^l_{mn}(x)\;\frac{\partial \bar{x}^i}{\partial x^l}\;\frac{\partial x^m}{\partial \bar{x}^j}\;\frac{\partial x^n}{\partial \bar{x}^k} + \frac{\partial \bar{x}^i}{\partial x^a}\;\frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k}\,,$$

we have ,

$$(5\cdot5) \qquad L\underline{\Gamma}^i_{jk} = \frac{\partial^2 \varepsilon^i}{\partial \bar{x}^j \partial \bar{x}^k} + \underline{\Gamma}^i_{jk,l}\,\varepsilon^l - \underline{\Gamma}^a_{jk}\,\varepsilon^i_{,a} + \underline{\Gamma}^i_{ak}\,\varepsilon^a_{,j} + \underline{\Gamma}^i_{ja}\,\varepsilon^a_{,k}\,.$$

Also, $\underset{\vee}{\Gamma}{}^i_{jk}$ being components of a tensor

$$L\underset{\vee}{\Gamma}{}^i_{jk} = \underset{\vee}{\Gamma}{}^j_{jk,l}\,\varepsilon^l - \underset{\vee}{\Gamma}{}^a_{jk}\,\varepsilon^i_{,a} + \underset{\vee}{\Gamma}{}^i_{ak}\,\varepsilon^a_{,j} + \underset{\vee}{\Gamma}{}^i_{ja}\,\varepsilon^a_{,k}\,.$$

Hence

$$L\Gamma^i_{jk} = \frac{\partial^2 \varepsilon^i}{\partial x^j \partial x^k} + \Gamma^i_{jk,l}\,\varepsilon^l - \Gamma^a_{jk}\,\varepsilon^i_{,a} + \Gamma^i_{ak}\,\varepsilon^a_{,j} + \Gamma^i_{ja}\,\varepsilon^a_{,k}\,.$$

From (5·4) and (5·5) we get

Any infinitesimal point transformation which is a motion (infinitesimal affine collineation) in Riemannian space is also a motion (infinitesimal affine collineation) in Generalised Riemann Space.

I take this opportunity to thank Dr. R. S. MISHRA for his kind guidance and help in the preparation of this paper.

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VLADETA VUČKOVIĆ (*)

Ein Satz über reelle Folgen. (**)

In seiner Note [1] unternahm Herr L. TANZI CATTABIANCHI eine Untersuchung über den Satz von MERCER und verwandte Sätze auf eine elementare und methodisch interessante Weise, insbesondere in Bezug auf nichtmonotone Folgen. Ich möchte hier auf einen Satz Acht legen, der sich aus den Darlegungen des Herrn TANZI CATTABIANCHI herleiten lässt, und der, ohne Acht der MERCERschen Sätze, für sich selbst ziemlich Interesse haben soll. Das ist der

Satz 1. *Seien* $\{ a_n \}$ *und* $\{ b_n \}$ *zwei reelle Folgen und die Folge* $\{ a_n \}$ *nicht monoton. Wenn für jedes n für welche* $a_n \gtreqless a_{n-1}$ *auch* $b_n \gtreqless a_n$ *ist (wobei in jeder Inklusion der gleiche Zeichen zu nehmen ist), so gilt*

$$\underline{\lim}\, b_n \leqslant \overline{\lim}\, a_n \leqslant \overline{\lim}\, b_n \, .$$

Den Beweis dieses Satzes kann man den Paragraphen 7, 8 und 10 der genannten Note des Herrn TANZI CATTABIANCHI fast wörtlich entnehmen. Dabei soll man von den dort betrachteten Folgen $\{ x_n \}, \{ y_n \}$ und $\{ X_n \}$ die Folge $\{ x_n \}$ ausser Acht lassen, die Folge $\{ y_n \}$ als die Folge $\{ b_n \}$ und die Folge $\{ X_n \}$ als die Folge $\{ a_n \}$ deuten, und die Erklärungen des § 7, die mit der Folge $\{ x_n \}$ angeführt sind, sinngemäss mit der Folge $\{ b_n \}$ ersetzen. Alles andere bleibt wörtlich den Ausführungen des Herrn TANZI CATTABIANCHI treu, so dass es mir nicht notwendig erscheint sie hier wiederzugeben.

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(**) Eigegangen am 17-I-1956.

Ich möchte noch eine leichte Verallgemeinerung des obigen Satzes angeben. Das ist der

Satz 2. *Seien* $\{a_n\}$ *und* $\{b_n\}$ *zwei reelle Folgen, und die Folge* $\{a_n\}$ *nicht monoton. Wenn es eine positive Nullfolge* $\{\varepsilon_n\}$ *gibt, so dass:*

für jedes n für welche $a_n > a_{n-1}$ *auch* $b_n > a_n - \varepsilon_n$,

» » » » » $a_n = a_{n-1}$ » $a_n - \varepsilon_n \leqslant b_n \leqslant a_n + \varepsilon_n$,

» » » » » $a_n < a_{n-1}$ » $b_n < a_n + \varepsilon_n$

ist, so gilt

$$\underline{\lim}\, b_n \leqslant \overline{\lim}\, a_n \leqslant \overline{\lim}\, b_n\,.$$

Einige Anwendungen des Satzes 1 gebe ich in einer Note die in « Publ. Inst. Math. Beograd » erscheinen wird (V. Vučković [**2**]).

Literatur.

[**1**] L. Tanzi Cattabianchi, *Sui teoremi di Mercer e Vijayaraghavan precisati per le successioni oscillanti.* Riv. Mat. Univ. Parma **4** (1953), 337-361.

[**2**] V. Vučković, *Mercersche Sätze für nichtlineäre Mittel.* Publ. Inst. Math. Beograd **20** (1956), 79-84.