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I. - A Cyclic Additivity Theorem of a Functional. (**)

Introduction.

Let $Q \equiv [0 \leqslant u,v \leqslant 1]$ be the unit square in the Euclidean plane E_2 and let E_3 be the Euclidean three space. For T a continuous mapping from Q into E_3, we denote by $L(T,\ Q)$ the LEBESGUE area of the surface represented by the continuous mapping T (T. RADÓ [6], L. CESARI [1]). The LEBESGUE area $L(T,\ Q)$ satisfies remarkable cyclic additivity properties which led to generalization to functionals. Let for T a continuous mapping from Q into E_3,

$$(1) \qquad T = lm, \qquad m : Q \Longrightarrow \mathfrak{M}, \qquad l : \mathfrak{M} \rightarrow E_3$$

be a monotone-light factorization of T (G. T. WHYBURN [7]). The symbols $\Longrightarrow$ and $\rightarrow$ denote that a mapping is *onto* or *into*, respectively. If for C a proper cyclic element of $\mathfrak{M}$ we denote by r_C the monotone retraction from $\mathfrak{M}$ onto C, then the LEBESGUE area $L(T,\ Q)$ is *weakly cyclicly additive*, i.e.,

$$(2) \qquad L(T,\ Q) = \sum L(lr_C m,\ Q), \qquad C \subset \mathfrak{M}.$$

This formula is due to T. RADÓ [5].

In a paper by T. RADÓ and E. J. MICKLE [3], the formula (2) is generalized in several ways. First of all, the writers consider a PEANO space P, a metric space P^*, and the class $\mathfrak{T}$ of all continuous mappings T from P into P^*.

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Instead of using monotone-light factorizations of T, the more general concept of an *unrestricted factorization* of T is introduced. An unrestricted factorization of T consists of a PEANO space $\mathfrak{M}\mathfrak{S}$ and two continuous mappings s, f such that

$$T = sf, \qquad f: P \to \mathfrak{M}\mathfrak{S}, \qquad s: \mathfrak{M}\mathfrak{S} \to P^*. \tag{3}$$

If $\Phi(T)$ is a functional defined for each $T \in \mathfrak{T}$ and satisfying the properties listed in [**3**], then in the same paper the following *strong cyclic additivity* formula has been proved

$$\Phi(T) = \sum \Phi(s\, r_C f), \qquad C \subset \mathfrak{M}\mathfrak{S}, \tag{4}$$

where r_C is the monotone retraction from $\mathfrak{M}\mathfrak{S}$ onto a proper cyclic element $C \subset \mathfrak{M}\mathfrak{S}$. The LEBESGUE area $L(T, Q)$ possesses the properties sufficient for the validity of (4), and hence

$$L(T, Q) = \sum L(s\, r_C f, Q), \qquad C \subset \mathfrak{M}\mathfrak{S}. \tag{5}$$

All the cyclic additivity theorems mentioned so far dealt with functionals defined for continuous mappings from a PEANO space into a metric space. In this paper a cyclic additivity theorem for functionals is studied, where the functionals are now defined for continuous mappings from a *metric* space into a metric space. This investigation was motivated by a recent book of L. CESARI [**1**], in which the writer considers continuous transformations from admissible sets $A \subset E_2$ into E_3. The class of admissible sets consists of all open subsets of E_2, all finite unions of finitely connected disjoint JORDAN regions and their open subsets [**1**; 5.1]. If (T, A) is a continuous mapping from an admissible set $A \subset E_2$ into E_3, then L. CESARI has defined the LEBESGUE area of (T, A), denoted by $L(T, A)$ (see [**1**; 5.8]).

In this paper a class $\mathfrak{A}$ of subsets of a metric space M is considered, which satisfies properties analogous to those of the class of admissible sets (see **I.1**). Let P^* be a metric space, and let $(\mathfrak{T}, \mathfrak{A})$ be the class of all continuous mappings from $A \in \mathfrak{A}$ into P^*. Since A need not be a PEANO space, a more general definition of an unrestricted factorization of (T, A) is necessary. Similar to (3) above, an unrestricted factorization of (T, A) consists of a PEANO space $\mathfrak{M}\mathfrak{S}$, a subset $\mathfrak{M}\mathfrak{S}^*$ of $\mathfrak{M}\mathfrak{S}$ and two continuous mappings s, f such that

$$(T, A) = sf, \qquad f: A \to \mathfrak{M}\mathfrak{S}^*, \qquad s: \mathfrak{M}\mathfrak{S}^* \to P^*. \tag{6}$$

In general, $\mathfrak{M}\mathfrak{S}^*$ is *not* a PEANO subspace of $\mathfrak{M}\mathfrak{S}$.

Instead of dealing with the LEBESGUE area, one considers a functional $\Phi(T, A)$ defined for each $(T, A) \in (\mathfrak{T}, \mathfrak{A})$ satisfying the properties listed in **I.8**. In order to prove a cyclic additivity theorem for $\Phi(T, A)$ using factorizations of the form (6), the following observations may be helpful. If C is a proper cyclic element of $\mathfrak{M}$ and r_c is the monotone retraction from $\mathfrak{M}$ onto C, it may occur that $r_c\, f(A)$ is not contained in $\mathfrak{M}^*$, and hence s need not be defined on $r_c\, f(A)$. The proper cyclic elements C for which $r_c\, f(A)$ does not meet $\mathfrak{M}^*$ can be neglected. With the other proper cyclic elements C of $\mathfrak{M}$ there is associated a non-empty subset G_c of A (see **I.12**). For $\Phi(T, A)$ a functional satisfying the properties of **I.8**, the main result of this paper states that

$$\Phi(T, A) = \sum \Phi(s\, r_c f,\ G_c), \tag{7}$$

where the summation is extended over all proper cyclic elements C of $\mathfrak{M}$ for which G_c is defined (see **I.15**).

To avoid excessive length of this paper, it seemed advisable to deal with the applications to the LEBESGUE area and a surface integral due to L. CESARI in two subsequent papers. This paper is then the first in a series of three papers, which we will designate by **I**, **II**, and **III** (see [**4**, **5**]).

A Cyclic Additivity Theorem.

I.1. – A metric space P which is a continuous image of the unit interval $[0, 1]$ will be termed a *Peano space*. Let $\mathfrak{P}$ be a collection of PEANO spaces P all of which lie in a metric space M. In the applications (see [**4**]) $\mathfrak{P}$ will be the collection of all finitely connected polygonal regions in any open subset M of E_2, where E_2 is the Euclidean plane. The interior of a set $A \subset M$ will be denoted by A^0.

In the sequel we shall be concerned with a collection $\mathfrak{A}$ of subsets of M defined in the following manner. The collection $\mathfrak{A}$ contains all PEANO spaces of $\mathfrak{P}$ and all finite unions of disjoint PEANO spaces of $\mathfrak{P}$. Moreover, $\mathfrak{A}$ contains all those subsets A of M which satisfy the following property. There exists a sequence $\{Q_n\}$, where for each n, Q_n is a finite union of disjoint PEANO spaces of $\mathfrak{P}$, such that

(i) $\qquad Q_n \subset A^0 \qquad (n = 1, 2, \dots),$

(ii) for any compact subset K of A^0 there is an integer $\bar{n} = n(K)$ with the property that $K \subset Q_n^0 \subset A^0$ for all $n \geqslant \bar{n}$.

It should be noted that $Q_n \in \mathfrak{A}$ for all n. In the next four paragraphs we shall prove some lemmas concerning the collection $\mathfrak{A}$.

I.2. – Lemma. If $A \in \mathfrak{A}$ cannot be written as a finite union of disjoint PEANO spaces of $\mathfrak{S}$, then there exists a sequence $\{ Q_n \}$ satisfying (i), (ii) of **I.1** and the additional property that $Q_n \subset Q^0_{n+1}$ $(n = 1, 2, \ldots)$.

Proof. Let $\{ Q'_n \}$ be any sequence satisfying (i), (ii) of **I.1**. Since Q'_1 is compact and contained in A^0, there is by (ii) of **I.1** a first integer $n_2 > 1$ such that $Q'_{n_1} \subset Q'^0_{n_2} \subset A^0$, where $n_1 = 1$. By the same argument, there is a first integer $n_3 > n_2$ such that $Q'_{n_2} \subset Q'^0_{n_3} \subset A^0$. Continuing in this manner, we obtain a sequence $\{ Q'_{n_i} \}$ satisfying (i) and (ii) of **I.1** and $Q'_{n_i} \subset Q'^0_{n_{i+1}}$ $(i = 1, 2, \ldots)$.

I.3. – Lemma. Let G be a component of a set $A \in \mathfrak{A}$ (see **I.1**). Then the following statements hold.

(1) If A is a finite union of disjoint PEANO spaces of $\mathfrak{S}$, then $G \in \mathfrak{A}$.

(2) If A connot be written as a finite union of disjoint PEANO spaces of $\mathfrak{S}$, then $G \in \mathfrak{A}$ provided $G \cap A^0 \neq 0$.

Proof. The statemet (1) is obvious, since in this case G is a PEANO space in $\mathfrak{S}$, and hence G is in $\mathfrak{A}$ (see **I.1**). To prove (2), let $\{ Q_n \}$ be a sequence satisfying (i), (ii) of **I.1** relative to A.

We first establish the equality

$$G \cap A^0 = G^0. \tag{3}$$

Since $G \cap A^0 \supset G^0$ is obvious, let $x \in G \cap A^0$. Since x is a compact subset of A^0, there is an integer n for which $x \in Q^0_n \subset A^0$. In view of the fact that Q_n is a finite union of disjoint PEANO spaces $P_1, \ldots, P_k$, we infer that $Q^0_n = P^0_1 \cup \ldots \cup P^0_k$. Therefore, $x \in P^0_i$ for a unique i. Since P_i is connected and G is a component of A which intersects P_i, there follows that $x \in P^0_i \subset P_i \subset G$, and hence $x \in G^0$. Thus $G \cap A^0 \subset G^0$, and (3) is proved.

Let now $n_1, n_2, \ldots$ be all those integers n for which $Q_n \cap G \neq 0$, and let Q'_{n_i} be the union of all PEANO spaces P in Q_{n_i} $(P \in \mathfrak{S})$ which intersect G. We assert that $Q'_{n_i} \subset G^0$. For, let P be a PEANO space, $P \in \mathfrak{S}$, $P \subset Q_{n_i}$, which intersects G. Then $P \subset G$, and since $P \subset A^0$, we infer from (3), $P \subset G \cap A^0 = G^0$. Next let K be any compact subset of G^0. Then K is also a compact subset of A^0, and hence there is an integer $\bar{n}$ for which $K \subset Q^0_n \subset A^0$, $n \geqslant \bar{n}$. Since $K \subset G^0$, certainly, $K \subset Q'^0_{n_i}$ for all i large enough. Therefore, $G \in \mathfrak{A}$.

I.4. – Lemma. A set $A \in \mathfrak{A}$ (see **I.1**) has at most a denumerable number of components G for which $G \cap A^0 \neq 0$.

Proof. We may exclude the case where A is a finite union of disjoint PEANO spaces in $\mathfrak{S}$, because then the assertion is obvious. There is now a sequence $\{Q_n\}$ $(n=1, 2, \ldots)$, such that each Q_n is a finite union of disjoint PEANO spaces in $\mathfrak{S}$, say $P_n^1, \ldots, P_n^{i_n}$, and such that the conditions (i), (ii) of **I.1** are satisfied. The collection of PEANO spaces $\{P_n^i\}$ $(i = 1, \ldots, i_n;\ n = 1, 2, \ldots)$, is denumerable, and therefore the collection $\{G\}$ of all components G of A which contain at least one P_n^i is denumerable.

We assert now that every component G of A intersecting A^0 is contained in $\{G\}$. For, if G is such a component of A, there is by the same argument as used in the proof of **I.3** (3) a PEANO space P_n^i contained in G. This completes the proof.

I.5. – Lemma 1. Let $A \in \mathfrak{A}$ a finite union of PEANO spaces of $\mathfrak{S}$ (see **I.1**). Then, if $\mathfrak{G}_0$ denotes any collection of components G of A, the set $G_0 = \cup G$, $G \in \mathfrak{G}_0$ is a finite union of disjoint PEANO spaces of $\mathfrak{S}$ and hence is in $\mathfrak{A}$.

Proof: Obvious.

Lemma 2. Assume that $A \in \mathfrak{A}$ cannot be written as a finite union of disjoint PEANO spaces of $\mathfrak{S}$ (see **I.1**). If $\mathfrak{G}_0$ denotes any collection of components G of A which intersect A^0, the set $G_0 = \cup G$, $G \in \mathfrak{G}_0$ is in $\mathfrak{A}$.

Proof. From **I.3** there follows that any such G is in $\mathfrak{A}$ and from **I.4** the collection $\mathfrak{G}_0$ is denumerable. Hence we can write

(1) $$\mathfrak{G}_0 = \{G_i\}_{i \geq 1}.$$

Let us first assume that $\{G_i\}_{i\geq 1}$ is infinite. For each i, there is a sequence $\{Q_n^i\}$ $(n = 1, 2, \ldots)$, such that each Q_n^i is a finite union of disjoint PEANO spaces in $\mathfrak{S}$ and

(2) $$Q_n^i \subset G_i^0 \qquad (n = 1, 2, \ldots);$$

(3) for any compact subset K of G_i^0, there is an integer $\bar{n}_i = n_i(K)$ such that $K \subset Q_n^{i0} \subset G_i^0$ for $n \geqslant \bar{n}_i$.

Define now

(4) $$Q_n = Q_n^1 \cup Q_n^2 \cup \ldots \cup Q_n^n \qquad (n = 1, 2, \ldots).$$

Then for every n, Q_n is a finite union of disjoint PEANO spaces of $\mathfrak{S}$ and $Q_n \subset G_0^0$.

We assert now that

(5) $$G_0^0 = \bigcup_{i \geq 1} G_i^0 .$$

Since $G_0^0 \supset \bigcup_{i \geq 1} G_i^0$, let $x \in G_0^0$. Then $x \in A^0$, and since x is compact, there is a finite union of disjoint PEANO spaces $P_1, \ldots, P_t$ of $\mathfrak{F}$ such that $x \in Q^0 \subset A^0$, where $Q = P_1 \cup \ldots \cup P_t$. Since $x \in G_0^0 \subset \cup G_i$, we have that x is in a unique G_i. Since $x \in Q^0$, there follows that $x \in P_r^0$ for a unique r, $1 \leqslant r \leqslant t$. In view of the connectedness of P_r and $P_r \cap G_i \neq 0$, we infer that $x \in P_r^0 \subset P_r \subset G_i$, and hence $x \in G_i^0$. Thus (5) follows.

Ler now K be any compact subset of G_0^0. From (5), there is an integer i_0 such that

(6) $$K \subset \bigcup_{i \geq 1}^{i_0} G_i^0 .$$

Since the G_i are disjoint, we conclude that $K_i = K \cap G_i^0$ is compact. For, otherwise, there would exist an open covering in G_i^0 of K_i which does not admit of a finite subcovering of K_i. Hence by adjoining the open sets G_j^0 ($i \neq j$; $j = 1, \ldots, i_0$), to this covering one would obtain an open covering of K which does not admit of a finite subcovering of K.

From (3) there is now an integer $\bar{n}_i = \bar{n}_i(K_i)$ available with the property that

(7) $$K_i \subset Q_n^{i0} \subset G_i^0, \qquad n \geqslant \bar{n}_i .$$

Let $\bar{n} = n(K) = \max [\bar{n}_i;\ i = 1, \ldots, i_0]$. Let n be any integer not less than $\bar{n}$. From (4) and (7) we obtain

(8) $$K \subset Q_n^0 .$$

If $\{G_i\}$ is finite, say, $\{G_i\}_{i=1}^{i_0}$, then let $Q_n = Q_n^1 \cup \ldots \cup Q_n^{i_0}$, and proceed as above.

I.6. – Let P be a PEANO space and let P^* be a metric space. In the sequel we shall be concerned with continuous mappings T from P into P^* (written $T\colon P \to P^*$). As a reference for the following definition the reader should consult E. J. MICKLE and T. RADÓ [**3**].

Definition. An *unrestricted factorization* of a continuous mapping $T\colon P \to P^*$ consists of a PEANO space $\mathfrak{M}$, called *middle space*, and two continuous mappings s, f such that $f\colon P \to \mathfrak{M}$, $s\colon \mathfrak{M} \to P^*$, $T = sf$.

This definition of an unrestricted factorization will be generalized in paragraph **I.9**.

I.7. – In the subsequent paragraphs we will have occasion to use some results of the theory of proper cyclic elements of a PEANO space P. The reader is referred to T. RADÓ [**6**], or a forthcoming paper of E. J. MICKLE and C. J. NEUGEBAUER [**2**].

I.8. – Let M and P^* be metric spaces, and let $\mathfrak{A}$ be a collection of sets in M defined in **I.1**. Denote by $(\mathfrak{T}, \mathfrak{A})$ the class of all continuous mappings (T, A) from a set $A \in \mathfrak{A}$ into P^*. The symbol (T, A) is meant to indicate that T operates from A even though T may be defined over a set which contains A properly.

Let $\Phi(T, A)$ be a functional defined for each $(T, A) \in (\mathfrak{T}, \mathfrak{A})$ satisfying the following properties:

(α) $\Phi(T, A)$ is real-valued and non-negative. For certain $(T, A) \in (\mathfrak{T}, \mathfrak{A})$ we may have $\Phi(T, A) = +\infty$.

(β) For every $(T, A) \in (\mathfrak{T}, \mathfrak{A})$, where A is a finite union of disjoint PEANO spaces $P_1, \ldots, P_n$ in $\mathfrak{F}$ (**I.1**), the functional $\Phi(T, A)$ is additive, i.e.,

$$\Phi(T, A) = \sum_{i=1}^{n} \Phi(T, P_i).$$

(γ) For $(T, A) \in (\mathfrak{T}, \mathfrak{A})$ and $\{Q_n\}$ a sequence as in **I.1** (i), (ii),

$$\Phi(T, A) = \lim \Phi(T, Q_n), \qquad \text{as} \qquad n \to \infty.$$

(δ) For A', A'' two sets in $\mathfrak{A}$ for which $A'' \subset A'$ and for $(T, A') \in (\mathfrak{T}, \mathfrak{A})$,

$$\Phi(T, A'') \leqslant \Phi(T, A').$$

(ε) For $P \in \mathfrak{F}$ (**I.1**) and T any continuous mapping from P into P^*, $\Phi(T, P)$ is *strongly cyclicly additive*, i.e., if $T = sf$, $f: P \to \mathfrak{M}$, $s: \mathfrak{M} \to P^*$ is an unrestricted factorization of T (**I.6**), then

$$\Phi(T, P) = \sum \Phi(s\ r_C f, P), \qquad C \subset \mathfrak{M}, \tag{1}$$

where r_C denotes the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element C of $\mathfrak{M}$.

Remark 1. Concerning (1), if $\mathfrak{M}$ contains no proper cyclic elements, i.e., if $\mathfrak{M}$ is a dendrite, then we agree that $\Phi(T, P) = 0$. Moreover, the summation in (1) is understood as follows: if $\Phi(s\, r_C f, P) = +\infty$ for some $C \subset \mathfrak{M}$, or if the series in (1) diverges, then $\Phi(T, P) = +\infty$. Otherwise, $\Phi(T, P)$ is the sum of the series in (1).

Remark 2. Instead of the condition (ε) one can impose upon Φ the following conditions (see [**3**]):

(ε_1) $\Phi(T, P)$ is lower semi-continuous in the following sense. If $(T_n, P) \to (T_0, P)$ uniformly, then $\Phi(T_0, P) \leqslant \liminf \Phi(T_n, P)$ for $n \to \infty$.

(ε_2) $\Phi(T, P)$ is additive under partition, i.e., if the mappings (T', P), (T'', P) constitute a partition of (T, P) (see [**3**]), then $\Phi(T, P) = \Phi(T', P) + \Phi(T'', P)$.

(ε_3) If (T, P) admits of an unrestricted factorization whose middle space is a simple arc, then $\Phi(T, P) = 0$.

The main result of this paper is to show that $\Phi(T, A)$ is cyclicly additive under the definition of an unrestricted factorization given in the next paragraph.

I.9. – In this paragraph we shall generalize the definition of an unrestricted factorization given in **I.6.** Since we are dealing now with continuous mappings from a metric space (see **I.8**), the definition of **I.6** is no longer applicable.

Definition. Let T be a continuous mapping from a metric space S into a metric space P^*. An *unrestricted factorization* of T consists of a PEANO space $\mathfrak{M}$, a subset $\mathfrak{M}^* \subset \mathfrak{M}$, and two continuous mappings s, f such that

(1) $$f : S \to \mathfrak{M}^*,$$

(2) $$s : \mathfrak{M}^* \to P^*,$$

(3) $$T = sf.$$

We shall write $T = sf$, $f : S \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$.

Remark 1. $\mathfrak{M}^*$ may be a proper subset of $\mathfrak{M}$ and need *not* be a PEANO subspace of $\mathfrak{M}$. In general, s does not admit of a continuous extension to $\mathfrak{M}$. It should also be noted that $s(\mathfrak{M}^*)$ may contain $T(S)$ properly.

Remark 2. In view of the generality of the metric spaces S and P^*, there may not exist an unrestricted factorization of a continuous mapping $T: S \to P^*$. However, if the metric space S is a subset of a PEANO space P (this will be the case in the applications), then a continuous mapping $T: S \to P^*$ admits of a *trivial* unrestricted factorization $T = TI$, $I: S \Longrightarrow S$, $T: S \to P^*$, $S \subset P$, where I is the identity mapping.

Remark 3. In the sequel we will be concerned with the collection of mappings $(\mathfrak{T}, \mathfrak{A})$ defined in **I.8**. It will implicitly be assumed that enough conditions are available (e.g., the condition of the previous Remark) to ensure that there is at least one unrestricted factorization of a mapping $(T, A) \in (\mathfrak{T}, \mathfrak{A})$ in the above sense.

I.10. – The following Lemma will be important as it exihibits some relationship between the definitions of unrestricted factorizations given in **I.6**, **I.9**.

Lemma. Let P be a PEANO space in $\mathfrak{F}$, $\mathfrak{F} \subset \mathfrak{A}$ (**I.1**). For T a continuous mapping from P into P^*, let $T = sf$, $f: P \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$, be an unrestricted factorization of T (**I.9**). If Φ is a functional satisfying (α) and (ε) of **I.8**, then

(1) $$\Phi(T, P) = \sum{}^* \Phi(s r_C f, P),$$

where $\sum^*$ denotes the summation extended over all proper cyclic elements C of $\mathfrak{M}$ for which $r_C f(P) \subset \mathfrak{M}^*$. Here r_C denotes the monotone retraction from $\mathfrak{M}$ onto C.

Proof. We first note that (1) can be replaced by

(2) $$\Phi(T, P) = \sum{}' \Phi(s r_C f, P),$$

where $\sum'$ denotes the summation extended over all $C \subset \mathfrak{M}$ for which $f(P) \cap C \neq 0$. To begin with we observe that for such a proper cyclic element C, we have $r_C f(P) = f(P) \cap C \subset \mathfrak{M}^*$ (see T. RADÓ [**6**; II.2.42]). If there is now a proper cyclic element C of $\mathfrak{M}$ such that $r_C f(P) \subset \mathfrak{M}^*$ but $f(P) \cap C = 0$, then $r_C f$ is constant on P and hence $\Phi(s r_C f, P) = 0$.

Let now $f(P) = \mathfrak{M}'$. Then $\mathfrak{M}'$ is a PEANO space in $\mathfrak{M}^*$, and $T = sf$, $f: P \Longrightarrow \mathfrak{M}'$, $s: \mathfrak{M}' \to P^*$ is an unrestricted factorization of T in the sense of **I.6** (the symbol $\Longrightarrow$ is to indicate that a mapping is *onto*). Since Φ satisfies the condition (ε) of **I.8**, we infer

(3) $$\Phi(T, P) = \sum \Phi(s r'_{C'} f, P), \qquad C' \subset \mathfrak{M}',$$

where $r'_{c'}$ denotes the monotne retraction from $\mathfrak{M}'$ onto a proper cyclic element C' of $\mathfrak{M}'$. Since every C' of $\mathfrak{M}'$ is contained in a unique proper cyclic element of $\mathfrak{M}$, (2) follows in case there are no proper cyclic elements C of $\mathfrak{M}$ intersecting $f(P) = \mathfrak{M}'$.

We can therefore assume that there is a proper cyclic element C of $\mathfrak{M}$ for which $C \cap \mathfrak{M}' \neq 0$. For such a C, let K_c be the class of proper cyclic elements of $\mathfrak{M}'$ contained in $C \cap \mathfrak{M}'$. Since, by T. Radó [**6**; II.2.42], $r_c(\mathfrak{M}') = C \cap \mathfrak{M}'$ (and hence $C \cap \mathfrak{M}' = \mathfrak{M}''$ is a Peano space), the proper cyclic elements of $\mathfrak{M}''$ are the sets in K_c. Moreover, each proper cyclic element of $\mathfrak{M}'$ is in a unique K_c.

The mapping $s\, r_c f$ admits of an unrestricted factorization $r_c\, f : P \Longrightarrow \mathfrak{M}''$, $s : \mathfrak{M}'' \rightarrow P^*$. If we denote by $r''_{c'}$ the monotone retraction from $\mathfrak{M}''$ onto a proper cyclic element C' in K_c, we have from the condition (ε) of **I.8**,

$$\Phi(s\, r_c f,\ P) = \sum \Phi(s\, r''_{c'} r_c f,\ P), \qquad C' \in K_c. \tag{4}$$

We will show now that $r''_{c'} r_c = r'_{c'}$ on $\mathfrak{M}'$. Let $x \in \mathfrak{M}'$. If $x \in C'$, there is nothing to prove. Assume then that $x \notin C'$. Let G' be the component of $\mathfrak{M}' - C'$ which contains x. Then the frontier of G' with respect to $\mathfrak{M}'$ is a single point $p' \in C'$ and $r'_{c'}(x) = p'$.

Case 1. $x \notin \mathfrak{M}''$. Then $x \notin C$; for, otherwise, $x \in \mathfrak{M}' \cap C = \mathfrak{M}''$. Let G be the component of $\mathfrak{M} - C$ containing x. Then $r_c(x) = x'$, $x' \in C$, where x' is the frontier of G with respect to $\mathfrak{M}$. It should also be observed that $x' \in \mathfrak{M}''$. Since G' contains x, and $G' \cup p'$ is a connected set intersecting C, we have $G \cap (G' \cup p') \neq 0$, $(\mathfrak{M} - \dot{G}) \cap (G' \cup p') \neq 0$. Hence $x' \in G' \cup p'$. If $x' = p'$, then $r''_{c'} r_c(x) = r'_{c'}(x') = p' = r'_{c'}(x)$. If $x' \in G'$, then $x' \notin C'$, and since $x' \in \mathfrak{M}''$, let G'' be the component of $\mathfrak{M}'' - C'$ containing x'. Then $G'' \subset G'$ and the frontier of G'' with respect to $\mathfrak{M}''$ is p'. Therefore, $r''_{c'}\, r_c(x) = r''_{c'}(x') = p' = r'_{c'}(x)$.

Case 2. If $x \in \mathfrak{M}''$, then $x' = x$ and the relation $r''_{c'}\, r_c(x) = r'_{c'}(x)$ follows from Case 1.

From (4)

$$\Phi(s\, r_c f,\ P) = \sum \Phi\,(s\, r'_{c'} f,\ P), \qquad C' \in K_c. \tag{5}$$

Consequently, from (3),

$$\sum{}' \Phi(s\, r_c f,\ P) = \sum{}' \sum_{C' \in K_c} \Phi(s\, r'_{c'} f,\ P) = \sum_{C' \subset \mathfrak{M}'} \Phi(s\, r'_{c'} f,\ P) = \Phi(T,\ P).$$

Therefore (2) and hence (1) is proved.

I.11. – Let M and P^* be metric spaces, and let $\mathfrak{A}$ be the collection of sets in M defined in **I.1**. Let us denote (as in **I.8**) by $(\mathfrak{T}, \mathfrak{A})$ the class of all continuous mappings from $A \in \mathfrak{A}$ into P^*. For $(T, A) \in (\mathfrak{T}, \mathfrak{A})$, let $(T, A) = sf$, $f : A \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$ be an unrestricted factorization of (T, A) (see **I.9**). In the study of cyclic additivity of a functional $\Phi(T, A)$ satisfying the conditions of **I.8**, the following situation may arise. If for C a proper cyclic element of $\mathfrak{M}$, we denote by r_c the monotone retraction from $\mathfrak{M}$ onto C, then the set $r_c\, f(A)$ may not be contained in $\mathfrak{M}^*$, and $s\, r_c\, f$ need not be defined on A. In order to cope with this occurrence, let us first prove the following Lemma.

Lemma. Let K be a connected subset of A. For C proper cyclic element of $\mathfrak{M}$, the set $r_c\, f(K)$ is either disjoint with $\mathfrak{M}^*$ or else lies entirely in $\mathfrak{M}^*$.

Proof. It suffices to show that,' if $r_c\, f(K)$ is not a single point, then $r_c\, f(K) \subset f(K)$. If $r_c\, f(K)$ is not a single point, then $f(K) \cap C \neq 0$. For, if it were empty, $f(K)$ being connected lies in a component Q of $\mathfrak{M} - C$. Since the frontier of Q is a single point p in C and $r_c(x) = p$ for every $x \in Q$, we conclude that $r_c\, f(K)$ reduces to a single point.

Since $f(K) \cap C \neq 0$ and since $f(K)$ is connected, we infer from T. Radó [**6**; II.2.42] that $r_c\, f(K) = f(K) \cap C \subset f(K)$ This completes the proof.

I 12. – (Continuation). Let now C be proper cyclic element of $\mathfrak{M}$. Then for G a component of A we have from the Lemma in **I.11** that $r_c\, f(G)$ is either disjoint with $\mathfrak{M}^*$ or else lies entirely in $\mathfrak{M}^*$. Using the Lemma of **I.11**, we introduce the following terminology.

For $(T, A) \in (\mathfrak{T}, \mathfrak{A})$, let $(T, A) = sf$, $f : A \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$ be an unrestricted factorization of (T, A).

(T_1) Assume that A can be written as a finite union of disjoint Peano spaces in $\mathfrak{F}$, $\mathfrak{F} \subset \mathfrak{A}$ (see **I.1**). Let $\mathfrak{K}$ be the class of proper cyclic elements C of $\mathfrak{M}$ for which there is at least one component G of A such that $r_c\, f(G) \subset \mathfrak{M}^*$. Then, for each $C \in \mathfrak{K}$, we denote by G_c the union of all components G of A which satisfy $r_c\, f(G) \subset \mathfrak{M}^*$.

(T_2) If A connot be written as a finite union of disjoint Peano spaces in $\mathfrak{F}$, $\mathfrak{F} \subset \mathfrak{A}$, then we let $\mathfrak{K}$ be the class of proper cyclic elements C of $\mathfrak{M}$ for which there exists at least one component G of A such that $G \cap A^0 \neq 0$ and $r_c\, f(G) \subset \mathfrak{M}^*$. For each $C \in \mathfrak{K}$, we denote by G_c the union of all components G of A satisfying $G \cap A^0 \neq 0$ and $r_c\, f(G) \subset \mathfrak{M}^*$.

In both cases (T_1) and (T_2), we shall term $\mathfrak{K}$ *the class of proper cyclic elements associated with* $(T, A) = sf$, and we shall term G_c *the set associated with* $C \in \mathfrak{K}$.

I.13. – Let $(T, A) = sf$, $f: A \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$ be an unrestricted factorization of a mapping $(T, A) \in (\mathfrak{T}, \mathfrak{A})$, and let $\mathfrak{K}$ and G_c be as in **I.12**. We shall proceed to establish a series of lemmas concerning G_c and $\mathfrak{K}$.

Lemma 1. The set G_c is in $\mathfrak{A}$ (see **I.1**).

Proof: This follows from **I.5**.

Lemma 2. Let $A' \subset A^0$, where A' is a finite union of disjoint PEANO spaces of $\mathfrak{P}$, $\mathfrak{P} \subset \mathfrak{A}$ (**I.1**). Then (T, A') admits of an unrestricted factorization of the form $(T, A') = sf$, $f: A' \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$. Let $\mathfrak{K}'$ be the class of proper cyclic elements associated with $(T, A') = sf$. Then a set G_c' is associated with $C \in \mathfrak{K}'$ if and only if G_c' is of the form $G_c \cap A'$.

Proof. Assume that G_c' is the set associated with $C \in \mathfrak{K}'$. Then $G_c' \subset G_c$ since $G_c' \cap A^0 \neq 0$, and hence $G_c' \subset G_c \cap A'$. To prove the complementary inclusion, we note that by definition, $r_c\, f(G_c \cap A') \subset \mathfrak{M}^*$. Since [see **I.12** ($T_1$)], G_c' is the union of all components G' of A' satisfying $r_c\, f(G') \subset \mathfrak{M}^*$, we conclude that $G_c' \supset G_c \cap A'$. Therefore, $G_c' = G_c \cap A'$. Similarly, if $G_c \cap A' \neq 0$, then $G_c' = G_c \cap A'$ is the set associated with $C \in \mathfrak{K}'$.

Remark. The hypothesis $A' \subset A^0$ can be replaced by $A' \subset A$ provided A is also a finite union of disjoint PEANO spaces of $\mathfrak{P}$ [see **I.12** (T_1)].

Lemma 3. Let A be a finite union of disjoint PEANO spaces of $\mathfrak{P}$ (**I.1**), i.e., $A = P_1 \cup \dots \cup P_n$, $P_i \in \mathfrak{P}$ $(i = 1, \dots, n)$. Then (T, P_i) admits of an unrestricted factorization $(T, P_i) = sf$, $f: P_i \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$. Let $\mathfrak{K}_i$ be the class of proper cyclic elements associated with $(T, P_i) = sf$. Then $\bigcup_{i=1}^{n} \mathfrak{K}_i = \mathfrak{K}$.

Proof. Since $\mathfrak{K}_i \subset \mathfrak{K}$ $(i = 1, \dots, n)$, there follows $\bigcup_{i=1}^{n} \mathfrak{K}_i \subset \mathfrak{K}$. To prove the reverse inclusion, let $C \in \mathfrak{K}$ and let G_c be the set associated with $C \in \mathfrak{K}$. Then $G_c \cap A \neq 0$, and hence $G_c \cap P_i \neq 0$ for at least one i, $1 \leqslant i \leqslant n$. By Lemma 2 (Remark), $G_c \cap P_i$ is the set associated with $C \in \mathfrak{K}_i$, and consequently, $\mathfrak{K} \subset \bigcup_{i=1}^{n} \mathfrak{K}_i$. This completes the proof of the Lemma.

Assume now that A cannot be written as a finite union of disjoint PEANO spaces of $\mathfrak{P}$ (**I.1**). Then by **I.2** there exists a sequence $\{Q_n\}$ with the following properties:

(i) Q_n is a finite union of disjoint PEANO spaces of $\mathcal{S}$ $(n = 1, 2, \dots)$,

(ii) $Q_n \subset Q_{n+1}$ $(n = 1, 2, \dots)$,

(iii) $Q_n \subset A^0$ $(n = 1, 2, \dots)$,

(iv) for any compact subset $K \subset A^0$ there is an integer n such that $K \subset Q_n^0 \subset A^0$.

For each n, the mapping (T, Q_n) admits of an unrestricted factorization $(T, Q_n) = sf$, $f: Q_n \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$. Let $\mathcal{K}_n$ be the class of proper cyclic elements associated with $(T, Q_n) = sf$.

Lemma 4. $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ $(n = 1, 2, \dots)$, and $\bigcup_{n \geq 1} \mathcal{K}_n = \mathcal{K}$.

Proof. The assertions $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ $(n = 1, 2, \dots)$ and $\bigcup_{n \geq 1} \mathcal{K}_n \subset \mathcal{K}$ are obvious. To prove that $\mathcal{K} \subset \bigcup_{n \geq 1} \mathcal{K}_n$, let $C \in \mathcal{K}$. Then $G_c \cap A^0 \neq 0$ [see **I.12** (T_2)]. Consequently, in view of (iv), $G_c \cap Q_n \neq 0$ for some n. By Lemma 2, $C \in \mathcal{K}_n$ and hence $\mathcal{K} \subset \bigcup_{n \geq 1} \mathcal{K}_n$.

I.14. – Let M, $\mathfrak{A}$, $\mathcal{S}$ be given as in **I.1**, and denote by $(\mathfrak{T}, \mathfrak{A})$ the class of all continuous mappings (T, A) from $A \in \mathfrak{A}$ into a fixed metric space P^*. Let $\Phi(T, A)$ be a functional defined for each $(T, A) \in (\mathfrak{T}, \mathfrak{A})$ satisfying the properties listed in **I.8**.

Lemma. Let A be a finite union of disjoint PEANO spaces $P_1, \dots, P_n$ of $\mathcal{S}$, and let $(T, A) = sf$, $f: A \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$ be an unrestricted factorization of (T, A) (**I.9**). If for C a proper cyclic element of $\mathfrak{M}$, r_c denotes the monotone retraction from $\mathfrak{M}$ onto C, then

$$\Phi(T, A) = \sum \Phi(s\, r_c f, G_c), \qquad C \in \mathcal{K}, \tag{1}$$

where $\mathcal{K}$ is the class of proper cyclic elements associated with $(T, A) = sf$, and where G_c is the set associated with $C \in \mathcal{K}$ [see **I.12** (T_1)].

Proof. We first assume that $\mathcal{K} \neq 0$. For each i, the mapping (T, P_i) admits of an unrestricted factorization $(T, P_i) = sf$, $f: P_i \to \mathfrak{M}^*$, $s: \mathfrak{M}^* \to P^*$,

$\mathfrak{M}^* \subset \mathfrak{M}$. Let $\mathcal{K}_i$ be the class of proper cyclic elements associated with $(T, P_i) = sf$ [see **I.12** (T_1)]. By Lemma 3 of **I.13**, $\bigcup_{i=1}^{n} \mathcal{K}_i = \mathcal{K}$. For each $C \in \mathcal{K}$, let $n(C)$ be the integers among $i = 1, \ldots, n$ for which $C \in \mathcal{K}_i$. If we set for $C \in \mathcal{K}_i$, $G_C^i = P_i \cap G_C$, then by Lemma 2 (Remark) of **I.13**, G_C^i is the set associated with $C \in \mathcal{K}_i$. Since P_i is connected, $G_C^i = P_i$, and since $G_C = \bigcup_{i \in n(C)} G_C^i$, G_C is a finite union of disjoint PEANO spaces of $\mathcal{S}$. From **I.8**, we have now

$$(2) \qquad \begin{cases} \Phi(T, A) = \sum_{i=1}^{n} \Phi(T, P_i), \\ \Phi(s\, r_C f, G_C) = \sum_{i \in n(C)} \Phi(s\, r_C f, G_C^i) \quad \text{for every } C \in \mathcal{K}. \end{cases}$$

By the Lemma in **I.10**, we have for each i, $1 \leqslant i \leqslant n$, the following relation,

$$(3) \qquad \Phi(T, P_i) = \sum{}^{*} \Phi(s\, r_C f, P_i),$$

where $\sum^*$ denotes the summation extended over all proper cyclic elements C of $\mathfrak{M}$ for which $r_C f(P_i) \subset \mathfrak{M}^*$. Using our new terminology [**I.12** (T_1)], (3) becomes

$$(4) \qquad \Phi(T, P_i) = \sum \Phi(s\, r_C f, G_C^i), \qquad C \in \mathcal{K}_i.$$

From (4) and (2) we obtain now

$$(5) \qquad \Phi(T, A) = \sum_{i=1}^{n} \sum_{C \in K_i} \Phi(s\, r_C f, G_C^i).$$

Since, for a given $C \in \mathcal{K}$, $C \in \mathcal{K}_i$ if and only if $i \in n(C)$, we can rewrite (5) in the form

$$(6) \qquad \Phi(T, A) = \sum_{C \in K} \sum_{i \in n(C)} \Phi(s\, r_C f, G_C^i).$$

From (2) we infer the formula (1).

The above proof was carried out under the assumption that $\mathcal{K} \neq 0$. If $\mathcal{K} = 0$, then it follows from (3) that $\Phi(T, P_i) = 0$ ($i = 1, \ldots, n$) and from (2) that $\Phi(T, A) = 0$. This completes the proof fo the Lemma.

I.15. – We are now able to state and prove the main result.

Let M, $\mathfrak{A}$ be given as in **I.1**, and let $(\mathfrak{T}, \mathfrak{A})$ be the class of all continuous mappings (T, A) from $A \in \mathfrak{A}$ into a fixed metric space P^*. Let $\Phi(T, A)$ be a functional defined for each $(T, A) \in (\mathfrak{T}, \mathfrak{A})$ satisfying the conditions of **I.8**.

Theorem. *Let* $(T, A) = sf$, $f : A \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$ *be an unrestricted factorization of* (T, A) (**I.9**). *If* r_c *denotes the monotone retraction from* $\mathfrak{M}$ *onto a proper cyclic element* C *of* $\mathfrak{M}$, *then*

$$\Phi(T, A) = \sum \Phi(s\, r_c f,\ G_c), \qquad C \in \mathfrak{K}, \tag{1}$$

where $\mathfrak{K}$ *is the class of proper cyclic elements associated with* $(T, A) = sf$, *and where* G_c *is the set associated with* $C \in \mathfrak{K}$ (**I.12**).

Proof. Since the theorem is true in case A is a finite union of disjoint PEANO spaces of $\mathfrak{F}$ (**I.14**), we may assume that A cannot be written as a finite union of disjoint PEANO spaces of $\mathfrak{F}$. By **1.2**, there exists then a sequence $\{Q_n\}$ with the properties listed:

(i) For each n, Q_n is a finite union of disjoint PEANO spaces of $\mathfrak{F}$.

(ii) For each n, $Q_n \subset A^0$.

(iii) For every compact subset K of A^0, there is an integer $\bar{n} = n(K)$ such that $K \subset Q_n^0 \subset A^0$ for all $n \geqslant \bar{n}$.

(iv) For each n, $Q_n \subset Q_{n+1}$.

Assume first that $\mathfrak{K} \neq 0$. The mapping (T, Q_n) admits of an unrestricted factorization $(T, Q_n) = sf$, $f : Q_n \to \mathfrak{M}^*$, $s : \mathfrak{M}^* \to P^*$, $\mathfrak{M}^* \subset \mathfrak{M}$. Let $\mathfrak{K}_n$ be the class of proper cyclic elements associated with $(T, Q_n) = sf$. Then by Lemma 4 in **I.13**, $\mathfrak{K}_n \subset \mathfrak{K}_{n+1}$ $(n = 1, 2, \ldots)$, and $\bigcup_{n \geq 1} \mathfrak{K}_n = \mathfrak{K}$. Hence for each $C \in \mathfrak{K}$, there is an integer $N(C) > 0$ such that $C \in \mathfrak{K}_n$, $n > N(C)$. By Lemma 2 in **I.13**, $G_c^n = G_c \cap Q_n$ is the set associated with $C \in \mathfrak{K}_n$, $n > N(C)$.

From **I.8** we infer that

$$\lim_{n \to \infty} \Phi(T, Q_n) = \Phi(T, A). \tag{2}$$

By Lemma 1 of **I.13**, the set G_c is in $\mathfrak{A}$. The sequence $\{G_c^n\}$, $n > N(C)$ satisfies the following properties:

(v) For each n, G_c^n is a finite union of disjoint PEANO spaces, each of which is in $\mathfrak{F}$, $\mathfrak{F} \subset \mathfrak{A}$ (**I.1**).

(vi) For each n, $G_c^n \subset G_c^0$.

(vii) For every compact subset K of G_c^0 there is an integer $\bar{n} \geqslant n(K)$ such that $K \subset G_c^{n^0}$ for all $n \geqslant \bar{n}$.

(viii) $G_c^n \subset G_c^{n+1}$ $(n = 1, 2, \ldots)$.

We only need to verify (vi) since the other properties are a consequence of (i), (iii), iv). For the proof of (vi), let us first establish the relation $G_c \cap A^0 = G_c^0$. Since, by **I.4**, G_c is a denumerable union of components G of A which intersect A^0 [**I.12** (T_2)], we can write $G_c = \bigcup_{i \geq 1} G_i$. From **I.3** (3) there follows $G_i \cap A^0 = G_i^0$ $(i = 1, 2, \ldots)$, and from the formula (5) of Lemma 2 in **I.5**, we have $G_c^0 = \bigcup_{i \geq 1} G_i^0$. Hence $G_c^0 = \bigcup_{i \geq 1} G_i^0 = \bigcup_{i \geq 1} (G_i \cap A^0) = (\bigcup_{i \geq 1} G_i) \cap A^0 = G_c \cap A^0$. Since $Q_n \subset A^0$ and $G_c^n = G_c \cap Q_n$, we have $G_c^n \subset G_c \cap A^0 = G_c^0$.

From **I.8** we conclude now that

$$\lim_{n \to \infty} \Phi(s r_c f, G_c^n) = \Phi(s r_c f, G_c), \qquad n > N(C), \tag{3}$$

for every $C \in \mathfrak{K}$.

We establish now the following assertion. For $\lambda \geqslant 0$,

$$\Phi(T, A) \leqslant \lambda \quad \text{if and only if} \quad \sum_{C \in K} \Phi(s\ r_c\ f, G_c) \leqslant \lambda. \tag{4}$$

Assume that $\Phi(T, A) \leqslant \lambda$. Then for each n, $\Phi(T, Q_n) \leqslant \lambda$ (see **I.8**). From **I.14**,

$$\Phi(T, Q_n) = \sum_{C \in K_n} \Phi(s r_c f, G_c^n) \leqslant \lambda \qquad (n = 1, 2, \ldots). \tag{5}$$

Let $C_1, \ldots, C_j$ be any finite number of proper cyclic elements in $\mathfrak{K}$. Then by Lemma 4 of **I.13**, we have an integer $N > 0$ such that for $n > N$, $C_i \in \mathfrak{K}_n$ $(i = 1, \ldots, j)$. Thus, from (5),

$$\sum_{i=1}^{j} \Phi(s r_{c_i} f, G_{c_i}^n) \leqslant \lambda, \qquad n > N. \tag{6}$$

Consequently, from (3)

$$\sum_{i=1}^{j} \Phi(s r_{c_i} f, G_{c_i}) \leqslant \lambda. \tag{7}$$

Since $C_1, \ldots, C_j$ was any finite number of proper cyclic elements in $\mathcal{K}$, we deduce from (7)

$$(8) \qquad \sum_{c \in K} \Phi(s\, r_c f,\ G_c) \leqslant \lambda .$$

Conversely, assume that $\sum_{c \in K} \Phi(s\, r_c f,\ G_c) \leqslant \lambda$. Then for each n (see **I.14**, **I.8**),

$$(9) \qquad \Phi(T,\ Q_n) = \sum_{c \in K_n} \Phi(s\, r_c f,\ G_c^n) \leqslant \sum_{c \in K} \Phi\, (s\, r_c f,\ G_c) \leqslant \lambda,$$

and hence from (2)

$$(10) \qquad \Phi(T,\ A) \leqslant \lambda .$$

From (4) we obtain now the desired equality

$$(11) \qquad \Phi(T,\ A) = \sum \Phi(s\, r_c f,\ G_c), \qquad C \in \mathcal{K}.$$

For the above discussion we assumed that $\mathcal{K} \neq 0$. If $\mathcal{K} = 0$, then from (5), $\Phi(T,\ Q_n) = 0$ for all n, and hence from (2), $\Phi(T,\ A) = 0$. This completes the proof of the Theorem.

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