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On the Geometrie of the Minkowski plane. (**)

1. - Introduction.

MINKOWSKI spaces are the local spaces of the more general FINSLER spaces but are interesting geometrical objects in themselves. This paper deals entirely with the plane case and emphasis is given to the problems which are unsolved in higher dimensions.

Section **2** introduces notation and recalls previous results. Sections **3** and **4** are mainly concerned with extremal problems whose solutions (polygons) require the lack of differentiability assumptions on the MINKOWSKI metric. Most of these are unsolved in higher dimensions. In section **5** the geometrical construction of a curve of constant MINKOWSKI curvature is given and in section **6** it is shown how the MINKOWSKI-FRENET formulas lead naturally to the MINKOWSKI curvature and two others. In section **7** the fundamental MINKOWSKI trigonometric identities are developed with the aid of a calculus of these functions. The main results are contained in section **8**, where the fundamental theorem for curves is given in various but non-equivalent forms. In particular (8.6) gives the fundamental theorem in a form involving a MINKOWSKI motion where two curvatures are required instead of one as in euclidean geometry.

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2. – Basic results.

In the euclidean plane with metric $e(x, y)$, let U be a closed, convex curve with center z at the origin. The MINKOWSKI metric xy with U as unit circle is defined by

$$xy = \frac{2e(x, y)}{e(x', y')}, \tag{2.1}$$

where x', y' is the diameter of U parallel to the euclidean line $g(x, y)$.

The MINKOWSKI area $|\overline{M}|$ of a set $\overline{M}$ with LEBESGUE area $|\overline{M}|^L$ is defined by ([**4**], p. 158) (1)

$$|\overline{M}| = \sigma |\overline{M}|^L, \qquad \sigma = \pi / |\overline{U}|^L. \tag{2.2}$$

BUSEMANN [**6**] solves the isoperimetric problem. A solution (isoperimetrix) is constructed by rotating U through 90° and taking its polar reciprocal with respect to the euclidean unit circle. The normalized isoperimetrix T is uniquely determined by the requirement that its center is at z and twice its area is its perimeter. Its supporting function $T(u) = T(\cos u, \sin u)$ is given by

$$T(u) = \sigma^{-1}/\varrho(u + (\pi/2)), \tag{2.3}$$

where $\varrho(u)$ is the euclidean polar equation of U.

If K is a closed convex curve with supporting function $H(u)$, then its MINKOWSKI perimeter $L(K)$ may be expressed by

$$L(K) = \sigma \int_K T(u)\, d\bar{s} = \sigma \int_T H(u)\, d\bar{s} = \int_T \frac{H(u)}{T(u)}\, ds. \tag{2.4}$$

where $d\bar{s}$, ds are euclidean and MINKOWSKI arclength elements respectively.

Let Δ be the area of a triangle with sides of length a, b lying on lines A, B through a point p respectively, then the MINKOWSKI sine is defined by ([**4**], p. 161)

$$\text{sm}\,(A, B) = 2\Delta/(ab). \tag{2.5}$$

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(1) The numbers in bold face and in brackets refer to References at the end of the paper.

Line A is called normal to B and B transversal to A if

$$\text{(2.6)} \qquad \operatorname{sm}(A^*, B) \leqq \operatorname{sm}(A, B)$$

for any line A^* through p ([**4**], p. 163). In this case we set $\alpha(B) = \operatorname{sm}(A, B)$. The function $\alpha^{-1}(B)$ is the MINKOWSKI polar equation of T ([**4**], p. 171). Furthermore the supporting lines of U at a point q are the lines transversal to the radius $\overline{zq}$ and the supporting lines of T at a point p are the lines normal to the radius $\overline{zp}$ ([**4**], p. 169).

A MINKOWSKI metric is euclidean if and only if U is an ellipse. A euclidean metrization of a MINKOWSKI space R arising in this manner is called *associated* to R.

3. – The α-function.

The α-function plays an important role in investigation of MINKOWSKI spaces. If B is parallel to the direction ω we write $\alpha(\omega) = \alpha(\omega + \pi) = \alpha(B)$. Although $\alpha(\omega)$ is, in general, not constant it is bounded above and below by (see [**5**], p. 283)

$$\text{(3.1)} \qquad \pi/4 \leqq \alpha(\omega) \leqq \pi/2 \, .$$

It is easily seen that if either of these extreme values is attained then U must be a parallelogram. The range for $\max \alpha$ and $\min \alpha$ may also be calculated. First, we prove a theorem concerning the bounds for $|\overline{T}|$.

(3.2) Theorem. *For the Minkowski area $|\overline{T}|$ of the normalized isoperimetrix T we have*

$$8/\pi \leqq |\overline{T}| \leqq \pi,$$

where the left-hand equality is obtained only if U is a parallelogram and the right-hand equality is attained only in euclidean geometry.

Proof. If U^* is the polar reciprocal of U, then the manner in which T is constructed shows that $|\overline{U}|^L \cdot |\overline{U^*}|^L = \pi |\overline{T}|$. But $8 \leqq |\overline{U}|^L \cdot |\overline{U^*}|^L \leqq \pi^2$ where the left-hand equality is obtained only if U is a parallelogram [**8**] and the right-hand equality is obtained only if U is an ellipse [**10**].

Since $\alpha^{-1}(\omega)$ is the MINKOWSKI polar equation of T and $|\overline{T}| \leqq |\overline{U}| = \pi$, we obtain:

(3.3) Corollary. *For the maximum value of* $\alpha(\omega)$ *we have* $\max \alpha \geqq 1$, *where the equality holds only in euclidean geometry.*

In particular we note that *when* T *and* U *coincide* (i.e. $\alpha(\omega) \equiv 1$) *then the geometry must be euclidean.* It is not true, in general, that $\min \alpha \leqq 1$. However, we have in this connection the following result which yields to a standard argument.

(3.4) *If* U *possesses a* $90°$ *rotation in some associated euclidean geometry, then* $\min \alpha \leqq 1$ *and the equality holds only in euclidean geometry.*

The unit circles U for which $\alpha(\omega)$ is constant are precisely the « Radon curves », see [**6**]. They are characterized by the property that the line through z, parallel to a supporting line of U at a point p, intersects U in a point at which there exists a supporting line parallel to $\overline{zp}$. They may be constructed in the following manner. In the euclidean plane with a rectangular coordinate system draw any convex curve s_1 (turning its concave side toward the origin) from (0,1) to (1,0) and remaining within or on the unit square in the first quadrant. The polar reciprocal s_2 of s_1 shares these same properties and the polar reciprocal of s_2 is again s_1. Rotate s_2 through $90°$ and complete U by reflection through the origin. If $E(u)$ is the supporting function of U, then $\alpha(\omega) = \sigma\, \varrho(\omega)\ E(\omega + \pi/2)$ will be constant and consequently U is a Radon curve. Every Radon curve is one of these or an affine transform of one.

We will now calculate the best upper bound for $\min \alpha$. To do this we first find the lower bound for the perimeter $L(U)$ of U. Starting with any point p on U we lay off a chord of unit Minkowski length. At the end-point of this chord we lay off *that* unit chord of U parallel to $\overline{zp}$. Continuing in this manner we obtain a hexagon inscribed in U of perimeter 6 (consecutive unit chord may not be well defined, consider for instance a parallelogram as unit indicatrix). We have then

$$L(U) \geqq 6. \tag{3.5}$$

Now the maximum inscribed and the minimum circumscribed isoperimetrices to U are $(\min \alpha)T$ and $(\max \alpha)T$ respectively. Integrating the inequality

$$(\max \alpha)\ T(u) \geqq E(u) \geqq (\min \alpha)\ T(u)$$

with respect to euclidean arclength of U and multiplying by σ we obtain from (2.2) and (2.4)

$$L(U)\ \max \alpha \geqq 2\pi \geqq L(U)\ \min \alpha\,.$$

Consequently (3.5) with the above inequality yields

$$\min \alpha \leqq \pi/3. \tag{3.6}$$

An affine transform of an ordinary regular polygon is called an affine regular polygon. The inscribed hexagons constructed above are regular affine hexagons. A simple computation shows that equality is obtained in (3·5) and (3·6) for a regular affine hexagon. In the next section we show that this is the *only case* where equality is obtained in these results.

We may obtain a sharpening of the MINKOWSKI isoperimetric inequality analogous to that of BONNESEN ([**2**], p. 63). Let R, r be the expansion factors of translates of T to obtain the minimum circumscribed and maximum inscribed isoperimetrices to a given closed convex curve K. We have then

$$[L(K)]^2 - 4\,|\,\overline{T}\,|\cdot|\,\overline{K}\,| \geqq |\,\overline{T}\,|^2\,(R-r)^2\,. \tag{3.7}$$

A proof of (3·7) follows from the geometric interpretation (2·4) of BLASCHKE's inequality on mixed areas ([**1**], p. 36). Unlike the BONNESEN result, equality may occur in (3·7) without K being homothetic to T. An example is obtained when T is a parallelogram and K is a non-homothetic parallelogram whose sides are parallel to the sides of T. Generalization of BONNESEN's method [**2**] also establishes (3·7) and by this approach sufficient conditions for equality become apparent. Applying (3·7) to U we obtain

$$[L(U)]^2 - 4\pi\,|\,\overline{T}\,| \geqq |\,\overline{T}\,|^2\ (\max \alpha - \min \alpha)^2\,. \tag{3.8}$$

A difficulty which forces the proofs in the next section to be rather complicated is that one convex body $\overline{K}_1$ may be properly contained in another $\overline{K}_2$ and yet both have boundaries of the same length. This is possible when $\overline{U}$ is not strictly convex. For example, if U contains the segment $\overline{pq}$ and we construct the parallelogram $zpdq$ with $\overline{zp}$ and $\overline{zq}$ as adjacent sides, then $zd = zp + pd = 2$. Any convex arc from z to d and lying within the triangle zpd will also have length 2.

In this paper convexity has meant ordinary convexity in an associated euclidean space. One may also define MINKOWSKI convexity of a set M by demanding it contain all shortest (MINKOWSKI) connections between any two points of M. MINKOWSKI convexity implies ordinary convexity but not conversley. The isoperimetrix must necessarily be convex in this more restricted sense. One may show that a closed convex curve H *is Minkowski convex* *if and only if the diameters of U, parallel to the tangent lines of H, cut U in extreme points.* Formulated in terms of the isoperimetrix T, *H is Minkowski convex if and only if its tangent lines are parallel to tangent lines of T.* In the preceding paragraph if $\overline{K}_1$ is MINKOWSKI convex then $L(K_1) < L(K_2)$.

4. – Some extremal problems.

In the preceding section, it was shown that $L(U) \geqq 6$. We now prove

(4.1) Theorem: *$L(U) = 6$ if and only if U is a regular affine hexagon.*

Proof. Let p be a point of U which has the property that $e(z, p)$ is a maximal euclidean radius of U in some associated euclidean geometry. Draw a chord $\overline{ab}$ of U parallel to $g(z, p)$ of unit MINKOWSKI length. A regular affine hexagon of perimeter 6 may be inscribed in U using $\overline{ab}$ as one of its sides. Since $L(U) = 6$, the arc ab of U has unit length. Consequently, if q is a point of arc ab, then $e(a, b)/e(z, p) = ab = aq + qb$. Using the fact that $e(z, p)$ is maximal together with the euclidean triangular inequality, we obtain $e(a, b) = e(a, q) + e(q, b)$ and, therefore, q lies on the chord $\overline{ab}$.

Since the LOEWNER ellipse [unique ellipse wich contains U with center z and minimum area [**4**], p. 160)] touches U in at least 4 points, there are at least two pair of such directions. Suppose U has exactly two pair of such points. By a central affine transformation we may assume the four ponts have coordinates $(\pm 1, \pm 1)$. Every ellipse in standard form which passes through one of these points is necessarily a circumscribing ellipse to this transform U^* of U. It follows that U^* is a square which gives a contradiction since $L(U) \neq 8$. Therefore, there are at least three pair of such directions and consequently U must be a central hexagon with the property that its diagonals are parallel to sides. It is easily shown that this property characterizes a regular affine hexagon.

This result gives another characterization of a regular affine hexagon, namely: if a central convex hexagon has the property that each side is half the length of the parallel diameter, then it is a regular affine hexagon.

If $\min \alpha = \pi/3$, then since $2\pi \geqq L(U) \min \alpha$, it follows that $L(U) = 6$ and we obtain the Corollary:

(4.2) *The* $\min \alpha = \pi/3$ *if and only if* $L(U) = 6$. *In this case U is a regular affine hexagon and $\alpha(\omega)$ is constant.*

Our next goal is to obtain a geometric interpretation for $\min \alpha$ in terms of a circumscribing quadralateral to U of minimum area. First the reader may prove:

(4.3) Lemma. *Let h^+, g^+ be two rays issuing from a point q and let a_1 separate q and a_2 on h^+. Similarly let b_1 separate q and b_2 on g^+. If p is the intersection point of $\overline{a_1b_2}$ and $\overline{a_2b_1}$, then*

$$pa_i + pb_i \leqq qa_i + qb_i \qquad (i = 1, 2).$$

(4.4) Lemma. *The perimeter of a quadralateral containing U is equal to or greater than* 8.

Proof. We may assume that the quadralateral is convex and that each side touches U.

If two sides of the quadralateral Q are parallel and a parallel is drawn through z, then twice the length of the segment subtended by Q is equal to the sum of the lengths of the parallel sides and the proof follows in this case.

We now assume that no two sides of Q are parallel. If h and g are two lines then we will denote by $h \cdot g$ their point of intersection.

Let the lines containing opposite sides denoted by h_1, h_2 and g_1, g_2 respectively where h_1, g_1 separate $g_1 \cdot g_2$, $h_1 \cdot h_2$ from U respectively. If a vertex $h_i \cdot g_j$ of Q lies on U, we may assume that h_i and g_j are right- and left-hand supporting lines of U at $h_i \cdot g_j$.

If $h_2 \cdot g_2$ lies on U, the parallelogram formed by drawing lines parallel to h_2 and g_2 through the reflection of $h_2 \cdot g_2$ in z will contain U and will lie within Q. We assume then that $h_2 \cdot g_2$ does not lie on U.

Let g_i', h_j' be the opposite parallel supporting lines of U to g_i, h_j respectively. The line g_2' intersects g_1 on the segment from $h_1 \cdot g_1$ to $h_2 \cdot g_1$ but not in $h_1 \cdot g_1$ for otherwise g_1 would not be a limiting supporting line to U. Consequently $h_1 \cdot g_1$ separates $h_1 \cdot g_2$ and $h_1 \cdot g_2'$ and we have

(1) $$m(h_1 \cdot g_2,\ h_1 \cdot g_1) + m(h_1 \cdot g_1,\ h_1 \cdot g_2') = m(h_1 \cdot g_2,\ h_1 \cdot g_2').$$

Furthermore $g_1 \cdot g_2'$ separates $h_1 \cdot g_2'$ and $h_1' \cdot g_2'$.

In a similar manner we find

(2) $$m(h_2 \cdot g_1,\ h_1 \cdot g_1) + m(h_1 \cdot g_1,\ h_2' \cdot g_1) = m(h_2 \cdot g_1,\ h_2' \cdot g_1)$$

and $h_2' \cdot h_1$ separates $h_2' \cdot g_1$ and $h_2' \cdot g_1'$.

Now a reflection through z preserves both the separation property and distance. Also since $h_2 \cdot g_2$ does not lie on U it follows that $h_1' \cdot g_2$ separates $g_1' \cdot g_2$ and $h_2 \cdot g_2$ and $h_2 \cdot g_1'$ separates $h_2 \cdot h_1'$ and $h_2 \cdot g_2$. By (4·3) we have

(3) $$m(h_1 \cdot g_2',\ h_1 \cdot g_1) + m(h_1 \cdot g_1,\ h_2' \cdot g_1) \leqq m(h_1' \cdot g_2,\ h_2 \cdot g_2) + m(h_2 \cdot g_2,\ h_2 \cdot g_1').$$

Using relations (1)–(3), we find that $L(Q) \geqq 8$.

(4.5) Theorem. *The Minkowski area of any quadralateral Q circumscribing U is equal to or greater than* $4 \min \alpha$.

Proof. We may assume Q is convex and that each side a_i of Q touches U. Let a_i with length $\overline{a}_i$ lie on the line h_i. By joining the vertices of Q to z and adding the areas of the 4 triangles we obtain

$$|\overline{Q}| = \frac{1}{2} \sum_{i=1}^{4} \overline{a}_i \, \alpha(h_i) \geqq \frac{1}{2} \min\alpha \, L(Q) \geqq 4 \min\alpha$$

which proves the Theorem.

There always exists a parallelogram P circumscribing U such that $|\overline{P}| = 4 \min\alpha$. To see this, let ω be a direction such that $\alpha(\omega)$ is minimal and construct a parallelogram P by drawing supporting lines to U parallel to ω and also parallel supporting lines at the points of intersection of the rays $\lambda\omega$ and $-\lambda\omega$ with U. It follows from the definition of $\alpha(\omega)$ that $|\overline{P}| = 4 \min\alpha$. We have then

(4.6) *If P is the quadralateral (or parallelogram) of minimum area circumscribing U, then*

$$\min\alpha = \frac{\pi}{4} \frac{|\overline{P}|^L}{|\overline{U}|^L} = \frac{|\overline{P}|}{4}.$$

The results (3.6), (4.2), and (4.6) solve the following extremal problem.

(4.7) Theorem. *If U is any closed, convex curve with center z and P is a quadrilateral (or parallelogram) of minimum area which circumscribes U, then*

$$\frac{|\overline{P}|^L}{|\overline{U}|^L} \leqq \frac{4}{3}$$

and the equality holds only for a regular affine hexagon.

A similar relation to (4.6) may be obtained for $\max\alpha$. Let Q be an inscribed quadralateral to U of maximum area. Q is convex and has the property that through each vertex there exists a supporting line of U parallel to a diagonal of Q. Consequently by parallel displacement of opposite vertices of Q we may obtain an inscribed parallelogram P' to U with the same area as Q and whose center is z. The vertices of P' must be points for which $\max\alpha$ is attained and $|\overline{P}'| = 2 \max\alpha$. We have then

(4.8) *If P' is a quadralateral (or parallelogram) of maximum area inscribed in U, then*

$$\max \alpha = \frac{\pi}{2} \frac{|\overline{P'}|^L}{|\overline{U}|^L} = \frac{|\overline{P'}|}{2}.$$

(4.9) Corollary. *If U is a closed convex curve with center z, then twice the area of a maximal inscribed quadralateral is equal to the area of a minimum circumscribed quadralateral if and only if U is a Radon curve.*

5. – Curves of constant curvature.

Following BUSEMANN ([4], p. 173), we define the MINKOWSKI curvature $\varkappa$ of a curve S of class C^2. Let p_0, p_1, p_2 be 3 distinct points of S determining a triangle with area Δ and sides with lengths a, b, c. When $p_i \to p_0$ we define

$$\varkappa = \lim \frac{4\Delta}{abc}. \tag{5.1}$$

In an associated euclidean geometry, $\varkappa$ may be expressed by

$$\varkappa = \bar{\varkappa}\sigma^{-2}[T(\theta)]^{-3} = \bar{\varkappa}\sigma\varrho^3\left(\theta + \frac{\pi}{2}\right), \tag{5.2}$$

where θ is the angle, with respect to the reference axis, of the perpendicular to the tangent line of S at p_0 with euclidean curvature $\bar{\varkappa}$.

The relation (5.2) shows that a simple closed curve V of unit constant curvature is necessarily convex, has a center, and is uniquely determined up to a translation. We determine V exactly by insisting its center is at z. If $H(\theta)$ is the supporting function of V, then $H(\theta)$ must satisfy the differential equation

$$H''(\theta) + H(\theta) = \sigma\varrho^3(\theta + \pi/2). \tag{5.3}$$

Solving (5.3) by variation of parameters and using the periodicity conditions we obtain

$$H(\theta) = \frac{\sigma}{2}\int_{\theta}^{\theta+\pi} \varrho^3\{\varphi + (\pi/2)\}\sin(\varphi - \theta)\,d\varphi. \tag{5.4}$$

We seek now the geometrical construction of V. A line through z cuts the unit MINKOWSKI disc U into pieces of equal area. The centroids of all such pieces constitute a closed convex curve with center z ([**3**], p. 10), which we will call the centroid curve of $\overline{U}$. A calculation shows that the supporting function $H^*(\theta)$ of the centroid curve is given by

$$H^*(\theta) = \frac{2\sigma}{3\pi}\int_{\theta}^{\theta+\pi} \varrho^3\{\varphi + (\pi/2)\} \sin(\varphi - \theta)\, d\varphi .$$

Comparison with (5.4) yields

(5.5) Theorem. *The curve V of unit constant curvature is identical to the centroid curve of the unit Minkowski disc U expanded by the factor $3\pi/4$.*

Concerning the perimeter $L(V)$ and area $|\overline{V}|$ of V we have

(5.6) Theorem. *If V is the curve of unit constant curvature, then $L(V) = 2\pi$, $|\overline{V}| \geqq \pi$ where equality occurs in the latter only in euclidean geometry.*

Proof. From (5.3) and (2.4) it follows that $L(V) = 2\pi$. By BLASCHKE's affine isoperimetric inequality ([**10**], p. 158), we have

$$(1) \qquad [L_a(V)]^3 \leqq (2\pi)^2\, 2\, |\overline{V}|^L,$$

where $L_a(V)$ is the affine perimeter of V. But from the definition of affine perimeter [3] we find that $L(V) = \sigma^{1/3}L_a(V) = 2\pi$. Equality occurs in (1) only if V is an ellipse, but the uniqueness of the solution to the integral equation (5.4) shows that if V is an ellipse, U is an ellipse and the theorem follows.

One may also introduce the analogue to the osculating circle of curvature to a curve S of class C^2. This rests on the following result which is extendable to n-dimensions.

(5.7) *If H is a closed, strictly convex, differentiable curve with center z, then there exists one and only one curve homothetic to H passing through any 3 non-collinear points.*

Differentiability ensures existence and strict convexity ensures uniqueness. Briefly, in definition (5.1) when $p_i \to p_0$ the curve constant curvature through

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[3] In general, there is no close relation between affine perimeter and MINKOWSKI perimeter, except for curves of constant curvature.

p_0, p_1, p_2 converges to a translate of RV, where $R = 1/\varkappa$ is the MINKOWSKI radius of curvature of S at p_0.

In general neither a MINKOWSKI circle nor an isoperimetrix has constant curvature. Indeed we have

(5.8) Theorem. *If an isoperimetrix has constant curvature then the geometry is euclidean.*

Proof. The supporting function $E(\theta)$ of an ellipse in standard form is given by

$$E^2(\theta) = a^2(1 - e^2 \sin^2\theta), \tag{1}$$

where a is the semi-major axis and e the eccentricity.

It is sufficient to show that T is an ellipse since T determines U. By (5.2) we have

$$T^3(\theta)[T''(\theta) + T(\theta)] = C, \tag{2}$$

where C is a positive constant. If we choose the reference axis in the direction of a maximal euclidean radius of T, then $T'(0) = 0$ and $T''(0) \leqq 0$. Under this condition, if we solve (2) we obtain

$$T^2(\theta) = T^2(0)\left[1 - \frac{|T''(0)|}{T(0)}\sin^2\theta\right]$$

and T must be an ellipse with semi-major axis $T(0)$ and eccentricity $e = [|T''(0)|/T(0)]^{1/2}$.

It is also true that if a MINKOWSKI circle has constant curvature then the geometry is euclidean. However, this will be shown elsewhere.

6. – Differential geometry of plane curves.

Due to the narrow group of motions, it is natural to expect the need for more than one curvature in order to extend various theorems to MINKOWSKI spaces. Two addition curvatures, which are natural ones to consider, are obtained when U and T are taken as « carriers of the circular image ». We assume that U is of class C^2 with positive euclidean curvature; T must necessarily share these same properties. In order to attach a sign to our curvatures we assign a sense of increasing arclength to U. As a point p on U moves in the positive sense the ray $\overrightarrow{zp}$ by definition will assign to V and T a sense of increasing arclength. A neighborhood with signed MINKOWSKI length Δs of a point p of an oriented curve S of class C^2 may be mapped by oriented parallel tangents (³) onto arcs with signed lenghts Δs_1, Δs_2, Δs_3 of V, T, U respectively. We then define

(6.1) $$\varkappa_i = \lim_{\Delta s \to 0} \frac{\Delta s_i}{\Delta s} \qquad (i = 1, 2, 3),$$

where the MINKOWSKI curvature $\varkappa = |\varkappa_1|$ and $\varkappa_2$, $\varkappa_3$ are called the isoperimetric and circular curvatures respectively. The three curvatures are necessarily all zero (then euclidean curvature in any associated geometry is also zero), all positive or all negative. In general the three curvatures are different. If $\varkappa_1$ and $\varkappa_2$ (or $\varkappa_3$) differ only by a constant factor for an arbitrary curve then the geometry is euclidean and if $\varkappa_2$ and $\varkappa_3$ differ only by a constant factor U is a RADON curve.

We may attach a sign to sm (A, B), but first we introduce a notation which will prove useful in the next section.

(6.2) **Definition.** Let the oriented line A^+ through z intersect U, T in p_1, p_2 respectively where p_i follows z on A^+. The oriented lines A_1^+, A_2^+ will denote the transversal and normal respectively to A^+ where A_i^+ has the same sense as the tangent vector at p_i. It follows that $A_{12}^+ = A_{21}^+ = A^-$.

(6.3) Let B^+ through z intersect U in b (follows z on B^+) and let the parallel to A_1^+ through b intersect A^+ in a, then we define:

$$\operatorname{sm}(A^+, B^+) = \begin{cases} \operatorname{sm}(A, B) & \text{if } \overrightarrow{ab} \text{ has same sense as } A_1^+, \\ -\operatorname{sm}(A, B) & \text{if } \overrightarrow{ab} \text{ has opposite sense to } A_1^+, \\ 0 & \text{if } a, b \text{ coincide.} \end{cases}$$

It follows that $\operatorname{sm}(A^+, B^+) = -\operatorname{sm}(B^+, A^+)$.

Let A^+, B^+ through z intersect U in a, b respectively, where a, b follow z on A^+, B^+. Following BUSEMANN ([5], p. 285), we introduce measure $\theta(A^+, B^+)$ of the angle between two directed lines A^+, B^+ $(\neq A^-)$ as numerically equal

(³) Although these curvatures exist and may be continuous as a function of arclength under weaker assumptions on S, we insist [except in (7.14), b.] on the continuously turning oriented tangent [or continuity of the circular image and (6.1) finite] since otherwise the uniqueness part of the existence theorems (section **8**) collapses.

to twice the area of the sector of $\overline{U}$ bounded by A^+, B^+ and the directed arc $\widehat{ab}$ of U $[|\widehat{ab}| < L(U)/2]$. That is

$$\theta(A^+, B^+) = \sigma \int_{\widehat{ab}} \varrho^2(\varphi)\, d\varphi, \tag{6.4}$$

where θ is positive or negative in agreement with the sign of $\widehat{ab}$. Clearly $\theta(A^+, B^+) = -\theta(B^+, A^+)$ and $-\pi < \theta(A^+, B^+) < \pi$ for $B^+ \neq A^-$. By (2.5), (6.3), and (6.4) we have

$$\lim_{B^+ \to A^+} \frac{\mathrm{sm}(A^+, B^+)}{\theta(A^+, B^+)} = 1. \tag{6.5}$$

Furthermore if A^+, B^+ are the directions of the oriented tangents to S at s_0, $s_0 + \Delta s$ respectively, then by (6.4), (5.3) we have

$$\varkappa_1 = \lim_{\Delta s \to 0} \frac{\theta(A^+, B^+)}{\Delta s}. \tag{6.6}$$

For a closed convex curve K we will always assume the orientation is such that the $\varkappa_i$ are non-negative. If R is the MINKOWSKI radius of curvature of K at point, where tangent has direction A^+, then by (6.6) we have

$$L(K) = \int_{-\pi}^{\pi} R \,\mathrm{d}\,[\theta(A^+)]. \tag{6.7}$$

We now obtain the FRENET formulas for the MINKOWSKI plane. Let $\boldsymbol{X}(s)$ be a MINKOWSKI vector representation of S where initial point of $\boldsymbol{X}(s)$ is at z. Addition and subtraction of MINKOWSKI vectors follows the parallelogram law and differentiation is defined in the natural way by

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}s} = \lim_{\Delta s \to 0} \frac{\boldsymbol{X}(s + \Delta s) - \boldsymbol{X}(s)}{\Delta s}. \tag{6.8}$$

(6.9) **Theorem.** *Let S be a curve of class C^2 with vector representation* (4) *$\boldsymbol{X}(s)$ and let A^+ be oriented tangent at s, then*

a. $$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}s} = \boldsymbol{t}(s), \qquad |\boldsymbol{t}(s)| = 1.$$

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(4) Even if S is analytic, its MINKOWSKI vector representation $\boldsymbol{X}(s)$ may not have derivatives higher than the first since it involves the MINKOWSKI metric.

If U is of class C^1, then

b. $$\frac{dt}{ds} = \varkappa_1(s)\ n(s), \qquad |n(s)| = \alpha^{-1}(A_1).$$

If U is of class C^2 with positive curvature, then,

c. $$\frac{dn}{ds} = -\varkappa_2(s)\, t(s),$$

where $t(s)$ is unit tangent vector to S at s and vector $n(s)$ is parallel to and has the same sense as A_1^+.

Proof. To prove b., let $t(s + \Delta s)$, $t(s)$ with initial point z terminate in p and p_0 on U respectively. By the MINKOWSKI law of sines ([**4**], p. 162) we have

$$pp_0 = \frac{\mathrm{sm}(A, \overline{zp})}{\mathrm{sm}(A, \overline{p_0 p})}$$

and consequently by (6.5), (6.6) we obtain

$$\left|\frac{dt}{ds}\right| = \alpha^{-1}(A_1)\,\varkappa(s).$$

Statement c. follows from definition (6.1) for $i = 2$. We *keep in mind that $t(s)$ is normal to $n(s)$ and $n(s)$ transversal to $t(s)$.* If S is a closed convex curve the vectors $t(s)$, $n(s)$ fill out $\overline{U}$ and $\overline{T}$ respectively.

To illustrate the use of (6.9) we consider MINKOWSKI involutes and evolutes. We define an involute S_1 to a given curve S by the property that the tangents to S are *normal* to S_1. If $Y(s^*)$ is the vector representation of S_1, we have

(1) $$Y(s^*) = X(s) + \lambda(s)\, t(s).$$

Since $\frac{dY}{ds}$ is parallel to $n(s)$ we conclude from (1) that $\lambda(s) = c - s$ and we have

(6.10) $$Y = X(s) + (c - s)t(s)$$

which shows the « unwinding thread property ». On the other hand the evolute S_2, with representation $W(s^*)$, of S is expressed by

(2) $$W(s^*) = X(s) + \mu(s)\ n^*(s), \qquad |n^*| = 1,$$

where $\boldsymbol{n}^*(s)$ is unit vector *normal* to tangent A^+ [or $\boldsymbol{t}(s)$] of S and having the same sense as A_2^+ [notation (6.2)]. By a proof similar to (6.9), c. we have

$$\frac{\mathrm{d}\boldsymbol{n}^*(s)}{\mathrm{d}s} = -\varkappa_3(s)\,\boldsymbol{t}(s)\,. \tag{6.11}$$

Since $\dfrac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}s}$ is parallel to $\boldsymbol{n}^*$ we have $\mu(s) = \varkappa_3^{-1}(s)$ and hence

$$\boldsymbol{W} = \boldsymbol{X}(s) + \varkappa_3^{-1}(s)\,\boldsymbol{n}^*(s), \qquad |\boldsymbol{n}^*| = 1\,. \tag{6.12}$$

Therefore, *to construct the Minkowski evolute of a given curve S we lay off on the concave side Minkowski distance equal to the reciprocal of the circular curvature of S along the line normal to the tangent.*

The curvature theorem on curves of constant width also use the circular curvature. If $H(u)$ is the supporting function of curve K of constant width D and $E(u)$ is the supporting function of U, then

$$H(u) + H(u + \pi) = D\,E(u)\,. \tag{6.13}$$

(6.14) Theorem. *If K has constant width D, then:*

a. *The perimeter of K is $D\,L(U)/2$.*

b. *The line joining two corresponding points is normal to the tangents.*

c. *The sum of the reciprocals of the circular curvature at corresponding points is constant (equal to D).*

d. *The evolute is a curve of zero width* (i. e. *one and only one tangent in given direction*).

Proof. Statement a. follows from (6.13) and (2.4) and b. yelds to a standard synthetic argument. Neither a. nor b. require the concept of curvature for their proof and are true without differentiability assumptions. For instance U (and hence K) may be a polygon. Statement c. follows from (6.1) and (6.13) by differentiation and d. is proved by b., c., and (6.12).

Construction of Minkowski curves of constant width may proceed by Euler's method using d. or by the related method of construction of the Reuleaux-polygons [**11**]. Since the « Vektorkorper » of a curve of constant width is a Minkowski circle, among all curves of constant width D the circle has maximal area ([**3**], p. 105). Also a Reuleaux triangle has least area [**9**]. However, one must choose a particular Reuleaux triangle since those of a given width do not necessarily have the same area. If $a_1, \ldots, a_6$ are the vertices of a maximal regular affine hexagon inscribed in U, then the REULEAUX triangle of width D homothetic to the convex domain bounded by arcs a_1a_2, a_3a_4, a_5a_6 will solve the problem.

7. – Calculus of Minkowski trigonometric functions.

In addition to our sine function (6.3) we choose FINSLER's cosine function [7].

(7.1) Definition. Assume U is differentiable. Let A^+, B^+ be two oriented lines through z and let a on U follow z on A^+. If the tangent A_1 at a intersects B^+ in b, then we define

$$\operatorname{cm}(A^+, B^+) = \begin{cases} 1/(zb) & \text{if } b \text{ follows } z \text{ on } B^+, \\ -1/(zb) & \text{if } b \text{ precedes } z \text{ on } B^+, \\ 0 & \text{if } A_1 \text{ is parallel to } B^+. \end{cases}$$

In general $\operatorname{cm}(A^+, B^+) \neq \operatorname{cm}(B^+, A^+)$, in fact we have:

(7.2) *If* $\operatorname{cm}(A^+, B^+) = \operatorname{cm}(B^+, A^+)$ *for all* A^+, B^+, *then the geometry is euclidean.*

Proof. Let a, b on U follow z on A^+, B^+ respectively. Also let A_1, B_1 intersect B^+, A^+ in c, d respectively. By hypothesis $zc = zd$ and the lines $g(a, b)$, $g(c, d)$ are parallel. Consequently z, midpoint of $\overline{ab}$ and intersection e of A_1, B_1 are collinear. Let ray $\overrightarrow{ze}$ intersect U in q and let the tangent at q intersect A^+, B^+ in a', b' respectively. As before $za' = ze = zb'$ and $g(a', b')$ is parallel to $g(a, b)$. Therefore, the midpoints of almost every family of parallel chords (corresponding to a regular tangent) are collinear and U must be an ellipse.

We now establish the fundamental trigonometric identities for the MINKOWSKI plane.

(7.3) Theorem. *If* U *is of class* C^1, *then for any three oriented lines* A^+, B^+, C^+ *we have:*

a. $\operatorname{sm}(A^+, B^+) = \operatorname{sm}(C^+, B^+)\operatorname{cm}(C^+, A^+) - \operatorname{sm}(C^+, A^+)\operatorname{cm}(C^+, B^+)$,

b. $\operatorname{cm}(A^+, B^+) = \operatorname{cm}(A^+, C^+)\operatorname{cm}(C^+, B^+) +$

$$+ \alpha^{-1}(C_1)\,\alpha^{-1}(A_1)\operatorname{sm}(C^+, B^+)\operatorname{sm}(C_1^+, A_1^+),$$

where other forms may be obtained by use of $\operatorname{sm}(Q^+, R^+) = -\operatorname{sm}(R^+, Q^+)$.

Proof. Let c on U follow z on C^+. There exists a unique ellipse with center z, passing through c with tangent C_1^+ (same orientation as U) and area equal to that of $\overline{U}$. The metric determined by this ellipse will be called the *associated euclidean metric with respect to* C^+. Using this euclidean metric ($\sigma = 1$) we have:

$$(1) \quad \begin{cases} \mathrm{sm}\,(C^+,\ A^+) = \varrho(A)\ \sin\,(C^+,\ A^+) \\ \mathrm{cm}\,(C^+,\ A^+) = \varrho(A)\ \cos\,(C^+,\ A^+) \\ \mathrm{sm}\,(A^+,\ B^+) = \varrho(A)\,\varrho(B)\ \sin\,(A^+,\ B^+) \end{cases}$$

which hold in sign as well as magnitude. Statement a. follows from (1). We will prove b. presently.

If we use the associated euclidean metric with respect to A^+ and apply L'Hospital rule and the cotangent representation of ϱ'/ϱ we obtain

(7.4) Lemma. *If U is of class C^1, then*

$$\lim_{B^+ \to A^+} \frac{\mathrm{cm}(A^+, B^+) - 1}{\mathrm{sm}(A^+, B^+)} = 0.$$

(7.5) Definition. Let $F(A_1^+, \ldots, A_n^+)$ be a function of the oriented lines $A_1^+, \ldots, A_n^+$, then we define

$$\frac{\mathrm{d}F(A_1^+, \ldots, A_n^+)}{\mathrm{d}(\theta(A_i^+))} = \lim_{B^+ \to A_i^+} \frac{F(A_1^+, \ldots, B^+, \ldots, A_n^+) - F(A_1^+, \ldots, A_i^+, \ldots, A_n^+)}{\theta(A_i^+, B^+)}.$$

(7.6) Theorem. *If U is of class C^1, then*

a. $$\frac{\mathrm{d}\ \mathrm{sm}(A^+, B^+)}{\mathrm{d}(\theta(B^+))} = \mathrm{cm}\,(B^+,\ A^+),$$

b. $$\frac{\mathrm{d}\ \mathrm{sm}(A^+, B^+)}{\mathrm{d}(\theta(A^+))} = -\,\mathrm{cm}\,(A^+,\ B^+).$$

Proof. By (7.3), a. we have

$$\frac{\mathrm{sm}(A^+, C^+) - \mathrm{sm}(A^+, B^+)}{\mathrm{sm}(B^+, C^+)} = \mathrm{sm}\,(A^+,\ B^+) \left[\frac{\mathrm{cm}(B^+, C^+) - 1}{\mathrm{sm}(B^+, C^+)}\right] + \mathrm{cm}\,(B^+,\ A^+)\,.$$

Use of (7.4) and (6.5) proves a.. Statament b. follows from a. by $\mathrm{sm}\,(A^+,\ B^+) = -\,\mathrm{sm}\,(B^+,\ A^+)$.

When U is of class C^1 we may introduce *a Minkowski vector coordinate system respect to A^+ with the origin at z.* Let $\boldsymbol{a}_0$, $\boldsymbol{a}_1$ be parallel to and have the same sense as A^+, A_1^+ respectively where $|\boldsymbol{a}_0| = 1$, $|\boldsymbol{a}_1| = \alpha^{-1}(A_1)$.

If $\boldsymbol{X}(s)$ is a vector representation of a curve S of class C^1 with tangent B^+ at s, then we have

(7.7) $$\boldsymbol{X}(s) = x(s)\boldsymbol{a}_0 + y(s)\boldsymbol{a}_1 \quad (^5),$$

(7.8) $$\boldsymbol{t}(s) = \text{cm}\,(A^+,\ B^+)\,\boldsymbol{a}_0 + \text{sm}\,(A^+,\ B^+)\boldsymbol{a}_1, \qquad |\boldsymbol{t}| = 1,$$

(7.9) $$\frac{dx}{ds} = \text{cm}\,(A^+,\ B^+), \qquad \frac{dy}{ds} = \text{sm}\,(A^+,\ B^+),$$

(7.10) $$\boldsymbol{n}(s) = \alpha^{-1}(B_1)\,[\text{cm}\,(A^+,\ B_1^+)\boldsymbol{a}_0 + \text{sm}\,(A^+,\ B_1^+)\boldsymbol{a}_1].$$

For the curve V of unit constant MINKOWSKI curvature we have $\dfrac{ds_1}{d(\theta(B^+))} \equiv 1$ and from (6.9), b., (7.8), (7.10), and (7.6), a. we obtain the formulas

(7.11) $$\frac{d\,\text{cm}(A^+,\ B^+)}{d(\theta(B^+))} = \alpha^{-1}(B_1)\ \text{cm}\,(A^+,\ B_1^+),$$

(7.12) $$\frac{d\,\text{sm}(A^+,\ B^+)}{d(\theta(B^+))} = \text{cm}\,(B^+,\ A^+) = \alpha^{-1}(B_1)\ \text{sm}\,(A^+,\ B_1^+).$$

Differentiating (7.3), a. keeping A^+, C^+ fixed and using (7.11) and (7.12) we prove (7.3), b.. Also we have the following Corollary to (7.3), b..

(7.13) *If U is of class C^1, then for any A^+, B^+ we have:*

a. $\alpha(A_1)\ \text{cm}\,(A^+,\ B^+) = \text{sm}\,(B^+,\ A_1^+)$,

b. $\text{cm}\,(A^+,\ B^+)\,\text{cm}\,(B^+,\ A^+) +$

$$+\ \alpha^{-1}(A_1)\,\alpha^{-1}(B_1)\,\text{sm}\,(A^+,\ B^+)\,\text{sm}\,(A_1^+,\ B_1^+) = 1.$$

We next prove

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(5) $X(s)$ is MINKOWSKI distance to $\boldsymbol{a}_1$-axis, but $y(s)$ is only proportional to distance to $\boldsymbol{a}_0$-axis unless $\boldsymbol{a}_1$ is normal to $\boldsymbol{a}_0$ and $\alpha^{-1}(A_1) = 1$.

(7.14) Theorem. *If U is of class C^1, then*

$$\text{a.} \qquad \frac{\mathrm{d}\,\mathrm{cm}(A^+, B^+)}{\mathrm{d}(\theta(B^+))} = -\alpha^{-1}(A_1)\ \alpha^{-1}(B_1)\ \mathrm{sm}\,(A_1^+,\ B_1^+).$$

If U is of class C^2, then

$$\text{b.} \qquad \frac{\mathrm{d}\,\mathrm{cm}(A^+, B^+)}{\mathrm{d}(\theta(A^+))} = R(T,\ A)\ \mathrm{sm}\,(A^+,\ B^+),$$

where $R(T, A)$ in the Minkowski radius of curvature of T at point with tangent parallel to A [since we allow U to have zero curvature, $R(T, A)$ exists but may be zero].

Proof. Statement a. follows from (7.11) and (7.13), a.. If we apply (7.3), b. to (7.5) we obtain

$$\frac{\mathrm{d}\,\mathrm{cm}(A^+, B^+)}{\mathrm{d}(\theta(A^+))} = \alpha^{-3}(A_1)\ \varkappa(U,\ A)\ \mathrm{sm}\,(A^+,\ B^+), \tag{7.15}$$

where $\varkappa(U, A)$ is MINKOWSKI curvature of U at point where A through z cuts U. To obtain the more useful form b., let B^+ be oriented supporting line (6) to T at q, then $y(s)$ in (7.7) is given by $y(s) = \alpha^{-1}(B_1)\ \mathrm{sm}\,(A^+,\ B_1^-) = -\,\mathrm{cm}\,(B^+,\ A^+)$. By (6.6) and (7.9) and adjusting notation we obtain b..

One may now compute the higher derivatives; for instance we find that $\mathrm{sm}\,(A^+,\ B^+)$ satisfies the differential equation

$$\frac{\mathrm{d}^2\mathrm{sm}(A^+, B^+)}{\mathrm{d}(\theta(B^+))^2} + R(T,B)\ \mathrm{sm}\,(A^+,B^+) = 0\,. \tag{7.16}$$

8. – Fundamental theorems for curves.

For a discussion of the relative strength of the fundamental theorem in various forms for MINKOWSKI spaces see ([4], pp. 174-176). We first prove

(8.1) Theorem. *If U is of class C^1 and $\varkappa_1^*(s)$ is an arbitrary single-valued continuos function, $0 \leqslant s \leqslant s_0$, then there exists one and only one curve S with initial point p and oriented tangent A^+ at p with $\varkappa_1^*(s)$ as its Minkowski curvature as a function of Minkowski arclength s of S.*

(6) If U is differentiable, T is strictly convex and therefore q is unique. The case where B^+ is not a tangent line may be handled separately.

Proof. We choose the MINKOWSKI vector coordinate system with respect to A^+ with p at the origin z and let initial point of V have oriented tangent A^+. Set

$$(1) \qquad s_1(s) = \int_0^s \varkappa_1^*(s)\ \mathrm{d}s, \qquad s_1(0) = 0$$

and let B^+ be oriented tanget to V at s_1. We may then write

$$x(s) = \int_0^s \mathrm{cm}\,(A^+,\ B^+)\ \mathrm{d}s, \qquad x(0) = 0, \quad x'(0) = 1,$$

$$y(s) = \int_0^s \mathrm{sm}\,(A^+,\ B^+)\ \mathrm{d}s, \qquad y(0) = 0, \quad y'(0) = 0$$

and let $\boldsymbol{X}(s^*)$ be vector representation of the curve S determined by $\boldsymbol{X}(s^*) = x(s)\boldsymbol{a}_0 + y(s)\boldsymbol{a}_1$, $s^*(0) = 0$ and arclength s^* is increasing function of s. Therefore

$$\boldsymbol{t}(s^*) = \mathrm{cm}\,(A^+,\ B^+)\ \frac{\mathrm{d}s}{\mathrm{d}s^*}\ \boldsymbol{a}_0 + \mathrm{sm}\,(A^+,\ B^+)\ \frac{\mathrm{d}s}{\mathrm{d}s^*}\ \boldsymbol{a}_1\,.$$

Since $|\boldsymbol{t}| = 1$, $s^*(s) \equiv s$ and B^+ is the oriented tangent to S at s. Also

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \alpha^{-1}(B_1)\left[\mathrm{cm}\,(A^+,\ B_1^+)\ \frac{\mathrm{d}s_1}{\mathrm{d}s}\ \boldsymbol{a}_0 + \mathrm{sm}\,(A^+,\ B_1^+)\ \frac{\mathrm{d}s_1}{\mathrm{d}s}\ \boldsymbol{a}_1\right].$$

Consequently, by (6.9), b., (7.10), and (1) we have $\frac{\mathrm{d}s_1}{\mathrm{d}s} = \varkappa_1(s) = \varkappa_1^*(s)$. The curve S is of class C^2 since its oriented tangent vector turns continously. The uniqueness follows from initial conditions, (6.9), a. and the fact that the tangent vectors must be identical by (1).

(8.2) Theorem. *Let A^+ be an oriented line for which A_1^+ is defined (is unique) and let $R(s_1)$ be a positive, single-valued continuous function defined on V for which*

$$\int_0^{2\pi} \mathrm{cm}\,(A^+,\ B^+)\ R(s_1)\ \mathrm{d}s_1 = \int_0^{2\pi} \mathrm{sm}\,(A^+,\ B^+)\ R(s_1)\ \mathrm{d}s_1 = 0,$$

where B^+ is oriented tangent to V at s_1, then there exists a closed convex curve uniquely determined up to a translation for wich $R(s_1)$ is its Minkowski radius of curvature at the point with oriented tangent B^+.

Proof. Consider the curve S given by

$$X(s) = \int_0^{s_1} \mathrm{cm}\,(A^+, B^+)\, R(s_1)\, \mathrm{d}s_1\, \boldsymbol{a}_0 + \int_0^{s_1} \mathrm{sm}\,(A^+, B^+)\, R(s_1)\, \mathrm{d}s_1\, \boldsymbol{a}_1,$$

where $s(0) = 0$ and s is increasing function of s_1 for $0 \leqq s_1 \leqq 2\pi$. Consequently,

$$\boldsymbol{t}(s) = \mathrm{cm}(A^+, B^+)\, R(s_1)\, \frac{\mathrm{d}s_1}{\mathrm{d}s}\, \boldsymbol{a}_0 + \mathrm{sm}\,(A^+, B^+)\, R(s_1)\, \frac{\mathrm{d}s_1}{\mathrm{d}s}\, \boldsymbol{a}_1\,.$$

Since both $R(s_1)$, $\dfrac{\mathrm{d}s_1}{\mathrm{d}s}$ are positive, B^+ is oriented tangent to S at s and therefore by (6.1) $R(s_1) = \dfrac{\mathrm{d}s_1}{\mathrm{d}s}$ is its MINKOWSKI radius of curvature at s. By hypothesis $\boldsymbol{X}(s(0)) = \boldsymbol{X}(s(2\pi))$ and S is a closed curve whose tangent vector simply and continuously sweeps out the unit MINKOWSKI disc. S is therefore convex and of class C^2. If S^* is any other closed convex curve with these properties, then we may translate it such that the point with directed tangent A^+ is at the origin. Since $s = \int_0^{s_1} R(s_1)\, \mathrm{d}s_1$, the uniqueness follows from (6.9), a.

The MINKOWSKI curvature also permits the four vertex theorem without differentiability assumptions on the MINKOWSKI metric.

(8.3) Theorem. *If K is a closed convex curve with positive Minkowski curvature $\varkappa$, then $\varkappa$ has at least four extrema.*

Proof. Consider $\varkappa$ as a function of arclength s_1 of V obtained by parallel mappings of oriented tangents. Suppose $\varkappa$ has exactly two extrema and let a, b on V correspond to the maximal and minimal values of $\varkappa$ respectively. Along the two arcs of V from a to b, $\varkappa$ is strictly monotone decreasing and consequently there is at most one pair of opposite points on V at which the values of $\varkappa$ are equal. But by (5.2) this implies there is at most one pair of corresponding points on K at which the euclidean curvatures are equal. However, this is impossible by the SZEGÖ-SÜSS result [**12**] and since the number of extrema (if finite) must be even there are at least four.

We now define isometry of two plane curves.

(8.4) Definition. Let S, S^* be two curves with arclength s $[a \leqslant s \leqslant b]$. If $p(s_i)$, $p^*(s_i)$ (for $i = 1, 2$) lie on S and S^* respectively and if $p(s_1)\ p(s_2) = p^*(s_1)\ p^*(s_2)$ for any s_1, s_2 on $[a, b]$, then S, S^* are said to be isometric.

(8.5) Theorem. *If U is of class C^2 with positive curvature and if S, S^* are two curves of class C^2 with identical curvature functions $\varkappa_1(s)$, $\varkappa_2(s)$, then they are isometric.*

Proof. We assume the initial points of S, S^*, which may be any two corresponding points, are at the origin z and let A^+, A^{*+} be the oriented tangents to S, S^* respectively at z. Choosing the MINKOWSKI vector coordinate system with respect to A^+, the vectors $\boldsymbol{t}(s)$, $\boldsymbol{n}(s)$ of S may be expressed by

$$\boldsymbol{t}(s) = \alpha_1(s)\ \boldsymbol{a}_0 + \alpha_2(s)\ \boldsymbol{a}_1, \qquad \boldsymbol{n}(s) = \beta_1(s)\ \boldsymbol{a}_0 + \beta_2(s)\ \boldsymbol{a}_1, \tag{1}$$

where $\alpha_1(0) = \beta_2(0) = 1$, $\alpha_2(0) = \beta_1(0) = 0$. Applying our FRENET formulas $(6 \cdot 9)$ we obtain the differential system

$$\begin{cases} \dfrac{d\alpha_1}{ds} = \varkappa_1 \beta_1, & \dfrac{d\alpha_2}{ds} = \varkappa_1 \beta_2, \\ \dfrac{d\beta_1}{ds} = -\varkappa_2 \alpha_1, & \dfrac{d\beta_2}{ds} = -\varkappa_2 \alpha_2 . \end{cases} \tag{2}$$

Similarly, with the MINKOWSKI vector coordinate system with respect to A^{*+} we obtain for S^* the same differential system with the same initial conditions. Consequently, by the existence theorem for system (2) and (7.8) we have

$$\begin{aligned} \mathrm{cm}\,(A^+,\ B^+) &= \mathrm{cm}\,(A^{*+},\ B^{*+}), \\ \mathrm{sm}\,(A^+,\ B^+) &= \mathrm{sm}\,(A^{*+},\ B^{*+}), \end{aligned} \tag{3}$$

where B^+, B^{*+} are oriented tangents to S and S^* respectively at points with arclength s.

Now let E_1, E_2 be the associated euclidean metrics with respect to A^+, A^{*+} respectively. From (3) we have

$$\varrho_1(B) \cdot \cos\,(A^+,\ B^+) = \varrho_2(B^*) \cdot \cos\,(A^{*+},\ B^{*+}),$$

$$\varrho_1(B) \cdot \sin\,(A^+,\ B^+) = \varrho_2(B^*) \cdot \sin\,(A^{*+},\ B^{*+})$$

and consequently $\varrho_1(B) = \varrho_2(B^*)$. With respect to the euclidean metric E_i the relation (5.2) holds in sign as well as magnitude. Therefore, $\bar{\varkappa}^{(1)}(s)\ \varrho_1^3(B) =$

$=\bar{\varkappa}^{(2)}(s)\ \varrho_2^3(B^*)$ and $\bar{\varkappa}^{(1)}(s)=\bar{\varkappa}^{(2)}(s)$. Also if $\bar{s}$, $\bar{s}^*$ are euclidean arclenths of S, S^* in E_1, E_2 respectively, then $\bar{s}=\bar{s}^*=\int_0^s \varrho_1(B)\,ds=\int_0^s \varrho_2(B^*)\,ds$ and it follows that $e_1(z,\ p)=e_2(z,\ p^*)$, where p, p^* are points of S, S^* corresponding to arclength s. Since S is of class C^2, there exists a tangent to S parallel to $g(z,\ p)$ at a point q between z and p on S for which $|\bar{s}|$ is minimal. The same may be said for S^* with the metric E_2 and it follows that both tangents correspond to the same value of MINKOWSKI arclength s. Therefore $zp=zp^*$ by (2.1) and the theorem is proved.

The above theorem is, in general, incorrect with only one of the two curvature $\varkappa_1$, $\varkappa_2$. For example, if (8.5) were true with $\varkappa_1$ alone then any two semi-perimeters of V would be isometric, but this implies V is a circle. Similarly, by considering T we see that $\varkappa_2$ will not suffice alone. BUSEMANN in ([**4**], p. 176) gives an example of two curves which are isometric but the curvatures $\varkappa_1(s)$ are not the same.

(8.6) Theorem. *If U is of class C^2 with positive curvature and S, S^* are two curves of class C^2 such that*: (a) *the curvature functions $\varkappa_1(s)$, $\varkappa_2(s)$ are identical for S, S^**; (b) *the tangent of S varies through at least π, then there exists a motion of the Minkowski plane carrying S into coincidence with S^*.*

Proof. Our proof is indirect in order to give a method of computing the motion. By (8.1) it is sufficient to show that the initial oriented tangent A^+ of S can be carried by a motion into the initial oriented tangent A^{*+} of S^*. Let the initial point of T have oriented tangent A^+ and let A^{*+} be oriented tangent of T at c^*. By (6.1), if B^+, B^{*+} are oriented tangents to S, S^* respectively at s, then their images on T are at s_2 and s_2+c^* respectively. For $\varkappa_1(s)\neq 0$, $\varkappa_2(s)/\varkappa_1(s)=R(s_2)=R(s_2+c^*)$, where R is the MINKOWSKI radius of curvature of T. Since B^+ varies through at least π, the relation holds for almost all s_2, and by continuity all s_2. Also since $L(T)/2$ is a period of $R(s_2)$, if $c^*/L(T)$ is irrational then, by continuity of R, T has constant MINKOWSKI curvature and the geometry is euclidean (5.8). Otherwise, there exists a smallest $c>0$ such that $nc=L(T)$ and $R(s_2)=R(s_2+c)$ for all s_2. The proof of (8.7) implies there exists a central affine transformation carrying s_2 into s_2+c on T. This is necessarily a MINKOWSKI motion Φ and the group generated by Φ contains a motion carrying A^+ into A^{*+}.

To compute Φ, let the parallels A^+, C^+ through z to the oriented tangents of T at 0, c cut U in p_1, p_2 respectively. If F_i is the FINSLER ellipse (see [**4**], p. 179) to U at p_i, then since $R(0)=R(c)$ it follows that $|F_1|=|F_2|$. Therefore Φ is given by the central, area and orientation preserving affine transformation sending p_1 into p_2 and the FINSLER ellipse at p_1 into the FINSLER ellipse at p_2.

Theorem (8.1) shows that (8.6) is true with $\varkappa_1(s)$ alone only in Euclidean geometry and by considering T the same may be said for $\varkappa_2(s)$ alone. A condition of type (b) is also essential no matter how many curvatures we use. For a simple example, consider a straight line for which any reasonable curvature is zero, but only in euclidean geometry does there exist a motion sending an arbitrary line into another.

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