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A strong cyclic additivity theorem of a surface integral. (**)

Introduction.

Let Q be the unit square in the (u, v)-plane, $Q \equiv [0 \leqslant u, v \leqslant 1]$, and let E_3 be the Euclidean (x, y, z)-space.

Let S be a FRÉCHET surface of the type the 2-cell represented by the continuous transformation $T: Q \rightarrow E_3$,

$$(1) \qquad T: \quad x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \qquad (u, v) \in Q.$$

Denote by $L(T)$ the LEBESGUE area of S. For $L(T) < +\infty$ L. CESARI [4] introduced a surface integral $J(T) = \int_S F\, d\sigma$ as a WEIERSTRASS integral. Recently, J. CECCONI [1] has shown that the integral $J(T)$ is *weakly cyclicly additive* in the following sense.

If

$$T = lm, \qquad m: Q \Longrightarrow \mathfrak{M}, \quad l: \mathfrak{M} \rightarrow E_3$$

is a monotone-light factorization of T, then

$$(2) \qquad J(T) = \sum J(lr_c\, m), \qquad C \subset \mathfrak{M},$$

where C is a proper cyclic element of $\mathfrak{M}$, and r_c is the monotone retraction from $\mathfrak{M}$ onto C.

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The formula (2) extends a cyclic addivity theorem due to T. RADÓ; namely, if $L(T)$ is the LEBESGUE area of the surface represented by (1), then we have the following weak cyclic additivity formula (see RADÓ [**9**]),

$$L(T) = \sum L(lr_c\, m), \qquad C \subset \mathfrak{M}. \tag{3}$$

In a paper by E. J. MICKLE and T. RADÓ [**8**], the writers established a *strong cyclic additivity* theorem for the LEBESGUE area in the following sense. If

$$T = sf, \qquad f : Q \to \mathfrak{M}, \qquad s : \mathfrak{M} \to E_3$$

is an *unrestricted factorization* of T (see **II·1**), then

$$L(T) = \sum L(sr_c\, f), \qquad C \subset \mathfrak{M}, \tag{4}$$

where r_c is again the monotone retraction from $\mathfrak{M}$ onto a proper cyclic element C of $\mathfrak{M}$. The question arises whether such a strong cyclic additivity theorem does also hold for the surface integral $J(T)$. It is the purpose of this paper to answer this question affirmatively, i.e., we will prove that

$$J(T) = \sum J(sr_c\, f), \qquad C \subset \mathfrak{M}. \tag{5}$$

It is clear that the formula (2) of J. CECCONI is a special case of the formula (5) to be established in **II·17**.

The approach followed to prove (5) is different from the one followed by J. CECCONI [**1**], and does not make use of his formula (2). By application of a theorem of L. CESARI concerning convergence of $J(T)$ it was possible to reduce (5) to essentially a topological consequence of (4). This remark accounts for the fact that the first part of this paper deals with some of the topological issues that are needed later on.

I. – A-sets and proper cyclic elements.

I·1. – In this paragraph we will recall to the reader some results of the theory of *A-sets* and *proper cyclic elements* of a PEANO space P. As a general reference the reader is referred to T. RADÓ [**9**] or G. T. WHYBURN [**10**]. Let A be the generic notation of an A-set and let C be the generic notation of a proper cyclic lement of P. We remark that an A-set is always a non-degenerate subset of P, i.e., A contains at least two distinct points.

(1) C is a cyclic A-set of P, and every A-set of P is a PEANO subspace of P.

(2) There is a unique continuous and monotone retraction r_A from P onto A.

(3) If G is a component of $P - A$, then the frontier of G, denoted by $\mathrm{Fr}(G)$, reduces to a single point, which is, of course, in A. Moreover, the closure of G, denoted by $\mathrm{c}(G)$, is an A-set of P. If G is a component of $P - A$, and r_A is the continuous and monotone retraction from P onto A, then $r_A(x) = \mathrm{Fr}(G)$ for every $x \in G$.

(4) The proper cyclic elements of A coincide with the proper cyclic elements of P which are subsets of A.

(5) If $P - A$ decomposes into an infinite number of components $\{ G_i \}$, then $\varrho(G_i) \to 0$ as $i \to \infty$, where $\varrho(G_i)$ denotes the diameter of G_i.

(6) Let $\{ G_i \}$ be the sequence of components of $P - A$, and let them be divided into two disjoint classes $\{ G_{i_n} \}$, $\{ G_{j_n} \}$. Then $A^* = A \cup \bigcup_{n \geqslant 1} G_{i_n}$ is an A-set of P, and the components of $P - A^*$ are $\{ G_{j_n} \}$.

(7) If $\mathfrak{A}$ is a collection of A-sets of P such that $H = \cap A$, $A \in \mathfrak{A}$, contains at least two points, then H is an A-set of P.

(8) Let K be a connected subset of P. Then $K \cap A$ is connected (possibly empty).

(9) Let P^* be a PEANO subspace of a PEANO space P, and let A be an A-set of P. If $P^* \cap A$ is non-degenerate, then $A^* = P^* \cap A$ is an A-set of P^*. If P^* is an A-set of P and A^* is an A-set of P^*, then A^* is also an A-set of P.

(10) There is only a denumerable number of proper cyclic elements of P, and two distinct proper cyclic elements of P are either disjoint or else have a single point in common.

(11) If the sequence $\{ C_n \}$ of proper cyclic elements of P is infinite, then $\varrho(C_n) \to 0$ as $n \to \infty$.

(12) If E is a cyclic subset of P, i.e., $E - x$ is connected for every $x \in P$, and if E contains more than one point, then E is a subset of a unique proper cyclic element C of P.

(13) A point x will be termed a *cut-point* of P provided $P - x$ is not connected. If C is a proper cyclic element of P, then a point $x \in C$ is a cut-point of P if and only if there is a component G of $P - C$ such that $\mathrm{Fr}(G) = x$.

(14) If C is a proper cyclic element of P and A is an A-set of P such that $A \cap C$ contains more than one point, then $C \subset A$.

In what follows we shall make free use of the properties listed above. The main result of this part of the paper is the Theorem in **I·12.**

I·2. – Lemma. Let P be a PEANO space, and let C_0 be a proper cyclic element of P such that $P - C_0$ has at least three components G_1, G_2, G_3 with $\mathrm{Fr}(G_i) \neq \mathrm{Fr}(G_j)$ $(i \neq j;\ i, j = 1, 2, 3)$. Let $C \neq C_0$ be a proper cyclic element of P. Then at least two of the sets G_1, G_2, G_3 lie in a component of $P - C$.

Proof. Since $C \neq C_0$, there is a component G of $P - C_0$ such that $C \subset c(G)$. Then at least two of the components G_1, G_2, G_3, say G_1, G_2 are distinct from G and $Fr(G_i) \neq \mathrm{Fr}(G)$ $(i = 1, 2)$. Then $[C_0 - \mathrm{Fr}(G)] \cup G_1 \cup G_2$ is a connected set in $P - c(G) \subset P - C$. Hence G_1, G_2 lie in a component of $P - C$.

I·3. – Definition. Let P be a PEANO space and let $x_1, \ldots, x_n$ be $n > 1$ distinct points of P. We define $H(x_1, \ldots, x_n)$ to be *the smallest A-set containing* $x_1, \ldots, x_n$, i.e., $H(x_1, \ldots, x_n)$ *is the intersection of all A-sets of P containing* $x_1, \ldots, x_n$.

I·4. – Lemma. Let $x_1, \ldots, x_n$ be $n > 1$ points of a PEANO space P. For any A-set A of P in $H(x_1, \ldots, x_n)$, the set $H(x_1, \ldots, x_n) - A$ decomposes into at most n components and each component contains at least one of the points $x_1, \ldots, x_n$.

Proof. It follows from the properties listed in **I·1**, that A is also an A-set of $H(x_1, \ldots, x_n)$. Excluding the trivial case $x_i \in A$ $(i = 1, \ldots, n)$, we may assume that $x_{i_1}, \ldots, x_{i_j}$, $j \leqslant n$, are not in A. Let $G_1, \ldots, G_k$, $k \leqslant n$, be the components of $H(x_1, \ldots, x_n) - A$ containing $x_{i_1}, \ldots, x_{i_j}$. Now $A^* = A \cup G_1 \cup \ldots \cup G_k$ is an A-set of $H(x_1, \ldots, x_n)$ containing $x_1, \ldots, x_n$. Hence, by Definition **I·3**, $A^* = H(x_1, \ldots, x_n)$. Thus the components of $H(x_1, \ldots, x_n) - A$ are $G_1, \ldots, G_k$, $k \leqslant n$, and $G_i \cap (x_1 \cup \ldots \cup x_n) \neq 0$ $(i = 1, \ldots, k)$.

I·5. – Lemma. There is at most a finite number of proper cyclic elements of $H(x_1, \ldots, x_n)$ having more than two cut-points with respect to $H(x_1, \ldots, x_n)$.

Proof. Denying the assertion, we assume that there is an infinite number of proper cyclic elements $C_1, \ldots, C_m, \ldots$ of $H(x_1, \ldots, x_n)$ having more than two cut-points with respect to $H(x_1, \ldots, x_n)$. Then for each m there are at least three components G_1^m, G_2^m, G_3^m of $H(x_1, \ldots, x_n) - C_m$ such that

(1) $$\mathrm{Fr}(G_i^m) \neq \mathrm{Fr}(G_j^m) \qquad (i \neq j;\ i, j = 1, 2, 3);$$

(2) there is an $x_i^m \in (x_1 \cup \ldots \cup x_n)$ which is in G_i^m $(i = 1, 2, 3)$ (see **I·4**).

Since there is only a finite number of points $x_1, \ldots, x_n$, there are integers m_1, m_2 such that $x_i^{m_1} = x_i^{m_2}$ $(i = 1, 2, 3)$. Then, by renumbering if necessary, we have by **I·2** that $G_1^{m_1} \cup G_2^{m_1}$ lies in a component of $H(x_1, \ldots, x_n) - C_{m_2}$. This contradicts that $x_1^{m_2}, x_2^{m_2}$ lie in different components of $H(x_1, \ldots, x_n) - C_{m_2}$.

I·6. – Lemma. Let P be a PEANO space and let $\varepsilon > 0$ be given. Then there exists a finite number of distinct points $x_1, \ldots, x_n$ in P such that if G is a component of $P - H(x_1, \ldots, x_n)$, $\varrho(G) < \varepsilon$.

Proof. Since P is uniformly locally connected, we have a $\delta > 0$ such that all pairs of points x', x'' of P with $\varrho(x', x'') < \delta$ lie in a connected subset of P whose diameter is less than $\varepsilon/3$. In the PEANO space P there is now a finite number of distinct points $x_1, \ldots, x_n$ $(n > 1)$, such that for each $x \in P$, $\varrho(x, x_i) < \delta$ for some i $(1 \leqslant i \leqslant n)$. If $P - H(x_1, \ldots, x_n) \neq 0$, let G be a component of $P - H(x_1, \ldots, x_n)$. For $x \in G$ we have a $x_i \in H(x_1, \ldots, x_n)$ such that $\varrho(x, x_i) < \delta$. By the choice of $\delta > 0$, there is a connected set E of P with $\varrho(E) < \varepsilon/3$ containing x, x_i. But then $\mathrm{Fr}(G) \in E$ [note that $\mathrm{Fr}(G)$ reduces to a single point], and hence $\varrho[x, \mathrm{Fr}(G)] < \varepsilon/3$. From the triangle inequality there follows now that $\varrho(G) < \varepsilon$.

I·7. – Lemma. Let P be a PEANO space, and let $\{C_n\}$ be the sequence of proper cyclic elements of P. Assume that there is only a finite number of proper cyclic elements of P having more than two cut-points of P. For $\varepsilon > 0$ given there exists a PEANO space $P^* \subset P$ with the following properties.

(1) P^* contains only a finite number of proper cyclic elements each of which is a proper cyclic element of P.

(2) There is a continuous retraction t^* from P onto P^* satisfying $\varrho[x, t^*(x)] < \varepsilon$ for $x \in P$.

(3) If C is a proper cyclic element of P^*, and r_c^*, r_c denote the monotone retractions from P^* onto C and from P onto C, respectively, then $r_c = r_c^* t^*$.

Proof. If $\{C_n\}$ is finite, then set $P^* = P$, and let t^* be the identity on P. We may then assume that $\{C_n\}$ is infinite. Then $\varrho(C_n) \to 0$ as $n \to \infty$. Therefore there is an integer $N_1(\varepsilon) > 0$ such that $\varrho(C_n) < \varepsilon$ for $n > N_1$. By hypothesis, there is an integer $N_2 > 0$ such that for $n > N_2$, C_n contains at most two cut-points of P. Let n_0 be a fixed integer greater than $\max(N_1, N_2)$.

Let now n be any integer greater than n_0. Then C_n contains either one cut-point x_n or two cut-points x_n, y_n of P. In the first case let $E_n = x_n$, and in the second case let E_n be a simple arc in C_n with endpoints x_n, y_n. Since C_n is a PEANO space we have that E_n is a continuous retract of C_n (WHYBURN [**10**]). Hence there is a continuous transformation $t_n : C_n \Longrightarrow E_n$ such that $t_n(x) = x$ for $x \in E_n$.

I·8. – (Continuation.) Let $P^* = (P - \bigcup_{n > n_0} C_n) \cup \bigcup_{n > n_0} E_n$, and define a mapping $t^* : P \Longrightarrow P^*$ *by*

$$t^*(x) = \begin{cases} x, & \text{if } x \in P - \bigcup_{n > n_0} C_n, \\ t_n(x), & \text{if } x \in C_n \quad (n > n_0). \end{cases}$$

We assert that t^* is a continuous retraction from P onto P^*. Thus P^* is a PEANO subspace of P.

Proof. Let $P_i = (P_i - \bigcup_{n=n_0+1}^{i} C_n) \cup \bigcup_{n=n_0+1}^{i} E_n \qquad (i = n_0+1, \; n_0+2, \ldots)$.
Define a mapping $t_i^* : P \Longrightarrow P_i$ by

$$t_i^*(x) = \begin{cases} x, & \text{if } x \in P - \bigcup_{n=n_0+1}^{i} C_n = K_i, \\ t_n(x), & \text{if } x \in C_n \quad (n_0 < n \leqslant i). \end{cases}$$

We observe that t_i^* is continuous on K_i, and on C_n $(n_0 < n \leqslant i)$. Since

(i) $\quad c(K_i) \cap C_n$ is either empty or else in E_n $(n_0 < n \leqslant i)$,

(ii) $\quad C_j \cap C_k$ $(j \neq k)$ is either empty or else in $E_j \cap E_k$ $(n_0 < j, \; k \leqslant i)$,

(iii) $\quad t_i^*(x) = x, \qquad x \in E_n$ $(n_0 < n \leqslant i)$,

we have that t_i^* is continuous on a finite number of closed sets and agrees on the intersection of any two. Hence t_i^* is continuous on P.

In order to show that t^* is continuous on P it suffices to demonstrate that $t_i^* \to t^*$ uniformly on P. But this is a ready consequence of the definition of t_i^*, t^* and the fact that $\varrho(C_n) \to 0$ as $n \to \infty$. Since $t^*(x) = x$ for $x \in P^*$, t^* is a continuous retraction from P onto P^*. We note that t^* need not be monotone.

I·9. – (Continuation.) Let $t^* : P \Longrightarrow P^*$ be defined as in **I·8**. Then

(1) $\quad \varrho[x, \; t^*(x)] < \varepsilon$ for $x \in P$,

(2) $\quad$ if C is a proper cyclic element of P and G is a component of $P - C$, then $t^*(G) \subset c(G)$.

Proof. To prove (1), we note that, if $x \in P - \bigcup_{n > n_0} C_n$, then $t^*(x) = x$. If $x \in C_n$ for some $n > n_0$, then $t^*(x) \in C_n$. Hence $\varrho[x, \; t^*(x)] \leqslant \varrho(C_n) < \varepsilon$. In order to verify (2), let us take $x \in G$. If x is in no C_n $(n > n_0)$, then $t^*(x) = x$. If $x \in C_n$ for some $n > n_0$, then $C_n \subset c(G)$ (see **I·1**), and $t^*(x) \in C_n \subset c\,(G)$. Thus $t^*(G) \subset c(G)$.

I·10. – (Continuation.) Let P^* be defined as in **I·8**. Then P^* is a Peano subspace of P, and $C_1, \ldots, C_{n_0}$ are the proper cyclic elements of P^*.

Proof. Since $C_i \subset P^*$ $(i = 1, \ldots, n_0)$, we have that C_i is a proper cyclic element of P^*. To show that P^* does not contain any other proper cyclic elements, assume that there is a proper cyclic element C^* of P^* not equal to C_i $(i = 1, \ldots, n_0)$. Since C^* is a cyclic subset of P, we have an integer $n > n_0$ such that $C^* \subset C_n$. But then $C^* \subset E_n$, which is impossible since E_n is either a single point or a simple arc.

I·11. – (Continuation.) Let C be a proper cyclic element of P^*. Denote by r_c^*, r_c the monotone retractions from P^* onto C, and from P onto C, respectively. Then

$$r_c^* \, t^* = r_c \, . \tag{1}$$

Proof. If $x \in C$, then (1) is obvious. For $x \notin C$, let G be the component of $P - C$ that contains x. Then $\mathrm{Fr}\,(G) = y$, $y \in C$ and $r_c(x) = y$. From **I·9** we have that $t^*(x) \in c(G)$. If $t^*(x) = \mathrm{Fr}\,(G)$, there is nothing to prove. If then $t^*(x) \in G$, let G^* be the component of $P^* - C$ containing $t^*(x)$. Then $G^* \subset G$ and $\mathrm{Fr}^*\,(G^*) = y$, where Fr^* denotes the frontier operation with respect to P^*. Hence $r_c^* \, t^*(x) = y$ and (1) follows.

This completes the proof of **I·7**.

I·12. – The following theorem is the main result of this section. It will be used in **II·18** to approximate a given continuous mapping which makes it possibile to apply a result of L. Cesari to complete the proof of the assertion made in the Introduction.

Theorem. *Let P be a Peano space, and let $\varepsilon > 0$ be given. Then there exists a Peano space $P^* \subset P$ with the following properties.*

(1) P^* contains only a finite number of proper cyclic elements each of which is a proper cyclic element of P, and if C is a proper cyclic element of P^*, then $P^* - C$ decomposes into a finite number of components.

(2) There is a continuous retraction t^* from P onto P^* satisfying $\varrho[x, t^*(x)] < \varepsilon$ for $x \in P$.

(3) If for C a proper cyclic element of P^*, $r_c{}^*$ and r_c denote the monotone retractions from P^* onto C and from P onto C, then $r_c = r_c^* t^*$.

Proof. From the paragraphs **I·4**, **I·5** and **I·6** we have an A-set $P' \subset P$ satisfying the following conditions.

(i) If G is a component of $P - P'$, then $\varrho(G) < \varepsilon/2$.

(ii) If C is a proper cyclic element of P', then $P' - C$ decomposes into a finite number of components. Moreover, there is at most a finite number of proper cyclic elements of P' containing more than two cut-points of P'.

Since P' is an A-set of P, the proper cyclic elements of P' are those of P which are subsets of P'. Let r' denote the monotone retraction from P onto P'. In view of (i) we have (see **I·1**)

$$\varrho[x, \; r'(x)] < \varepsilon/2 \quad \text{for } x \in P. \tag{iii}$$

The PEANO space P' satisfies the conditions of **I·7**. Therefore there is a PEANO space $P^* \subset P'$ with the following properties.

(iv) P^* contains only a finite number of proper cyclic elements each of which is a proper cyclic element of P' and hence of P.

(v) There is a continuous retraction t'' from P' onto P^* (defined as in **I·8**) satisfying $\varrho[x, t''(x)] < \varepsilon/2$ for $x \in P'$.

(vi) If for C a proper cyclic element of P^*, r_c^*, r_c' denote the monotone retractions from P^* onto C, and from P' onto C, respectively, then $r_c' = r_c^* t''$.

Define $t^* = t'' r'$. Then t^* is a continuous retraction from P onto P^*, and from (iii) and (v),

$$\varrho[x, t^*(x)] \leqslant \varrho[x, r'(x)] + \varrho[r'(x), t''r'(x)] < \varepsilon \quad \text{for } x \in P.$$

We also have $r_c^* t^* = r_c^* t'' r' = r_c' r' = r_c$, from (vi) and the uniqueness of monotone retractions onto A- sets (see **I·1**).

I·13. – (Continuation.) It remains to show that for C a proper cyclic element of P^*, $P^* - C$ decomposes into only a finite number of components. From (ii) in **I·12** we know that $P' - C$ possesses only a finite number of components. In order to prove our assertion it suffices to show that for G' a component $P' - C$, $G' \cap P^*$ is connected (possibly empty). Let us deny this and assume that $G' \cap P^*$ is not connected. There are then at least two distinct components G_1^*, G_2^* of $P^* - C$ such that $G_1^* \subset G'$, $G_2^* \subset G'$. If Fr$'$, Fr* denote the frontier operations with respect to P', P^*, respectively, we have $\mathrm{Fr}^*(G_1^*) = \mathrm{Fr}^*(G_2^*) = \mathrm{Fr}'(G') = x^*$, $x^* \in C$.

Since x^* is an accessible boundary point of G_1^*, G_2^* there are two simple arcs γ_1^*, γ_2^* such that $\gamma_1^* \subset G_1^* \cup x^*$ joins x^* to a point $x_1^* \in G_1^*$, and $\gamma_2^* \subset G_2^* \cup x^*$ joins x^* to a point $x_2^* \in G_2^*$. Since G' is acrwise connected (WHYBURN [**10**]), we have a simple arc $\gamma' \subset G'$ joining x_1^* to x_2^*.

Since $\gamma_1^* \cup \gamma_2^* \cup \gamma'$ contains a simple closed curve s with the property that $x^* \in s$, we may for the sake of notational simplification assume that $s = \gamma_1^* \cup \gamma_2^* \cup \gamma'$. In view of the fact that s is cyclic, there is a unique proper cyclic element C' of P' containing s. Now C$'$ cannot be a proper cyclic element of P^*, since $C' \cap G_1^* \neq 0$, $C' \cap G_2^* \neq 0$. We also have that $C \cap C' = x^*$, and $\gamma_1^* \cup \gamma_2^* = \alpha$ is a simple arc in $C' \cap P^*$ which contains x^* in its interior. Since $t'' : P' \Longrightarrow P^*$ is defined as in **I·8**, and since $\alpha = t''(\alpha) \subset t''(C')$, we infer that C' has exactly two cut-points relative to P'. Clearly, x^* is one of the cut-points. Let $y' \in C'$ be the other cut-point. But then by the definition of t'' (see **I·8**), $t''(C')$ is a simple arc β of C' with endpoints x^*, y'. Since $\alpha \subset t''(C') = \beta$, x^* cannot be an interior point of α. This is a contradiction. The proof of **I·12** is therefore complete.

The writer is indebted to Professor E. J. MICKLE for many kelpful suggestions concerning this part of the paper. For other recent independent research on A-sets and proper cyclic elements see E. J. MICKLE and C. J. NEUGEBAUER [7].

II. – A cyclic additivity Theorem.

II·1. – Let P be a PEANO space and let P^* be a metric space. Let T be a continuous mapping from P into P^*.

Definition. An *unrestricted factorization* of T consists of two mappings f, s and a PEANO space $\mathfrak{M}$ such that

(a) f is a continuous mapping from P *onto* $\mathfrak{M}$,

(b) s is a continuous mapping from $\mathfrak{M}$ *into* P^*,

(c) $T = sf$.

We shall write $T = sf$, $f : P \Longrightarrow \mathfrak{M}$, $s : \mathfrak{M} \rightarrow P^*$, and we shall term $\mathfrak{M}$ the *middle space* of the unrestricted factorization.

Definition. We shall term T a *partial mapping* of a continuous mapping T_0 from P into P^* if and only if the following holds. There exists an unrestricted factorization $T_0 = sf$, $f : P \Longrightarrow \mathfrak{M}$, $s : \mathfrak{M} \rightarrow P^*$ such that $T = sr_A f$, where r_A is the monotone retraction from $\mathfrak{M}$ onto an A-set A of $\mathfrak{M}$ (1).

II·2. – Let $\mathfrak{T}$ be a class of continuous mappings from P into P^* such that if $T \in \mathfrak{T}$, then also all the partial mappings of T are in $\mathfrak{T}$. We remark that we do not require that $\mathfrak{T}$ is the class of all continuous mappings from P into P^*. (For the applications see **II·7**, **II·8**).

Let $\Phi(T)$ be a functional defined for each $T \in \mathfrak{T}$ satisfying the following contions.

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(1) In [8] an unrestricted factorization is defined as above except for the condition (a) which is replaced by

(a′) f is a continuous mapping from P *into* $\mathfrak{M}$.

The restriction that f is a continuous mapping from P onto $\mathfrak{M}$ has been made to assure that $T(P) \subset T_0(P)$, where T is a partial mapping of T_0.

(1) $\Phi(T)$ is real-valued and finite for each $T \in \mathfrak{T}$. In particular $\Phi(T)$ could be negative, but for no $T \in \mathfrak{T}$ we have either $\Phi(T) = +\infty$ or $\Phi(T) = -\infty$.

(2) If $T \in \mathfrak{T}$ admits of an unrestricted factorization whose middle space is a dentrite, i.e., a PEANO space containing no proper cyclic elements, then $\Phi(T) = 0$.

(3) Assume $T \in \mathfrak{T}$ admits of an unrestricted factorization $T = sf$, $f: P \Longrightarrow \mathfrak{M}$, $s: \mathfrak{M} \to P^*$, where $\mathfrak{M} = A_1 \cup A_2$, $A_1 \cap A_2 = x$, $x \in \mathfrak{M}$, A_1, A_2 being A-sets of $\mathfrak{M}$. Let r_1, r_2 be the monotone retractions from $\mathfrak{M}$ onto A_1 and from $\mathfrak{M}$ onto A_2, respectively. Then $\Phi(T) = \Phi(sr_1 f) + \Phi(sr_2 f)$. Note that $sr_1 f$, $sr_2 f$ are in $\mathfrak{T}$.

Remark. If $T \in \mathfrak{T}$ is constant, then $\Phi(T) = 0$. For then T admits of an unrestricted factorization whose middle space is a simple arc.

II·3. – Lemma. Let $T = sf$, $f: P \Longrightarrow \mathfrak{M}$, $s: \mathfrak{M} \to P^*$ be an unrestricted factorization of a mapping $T \in \mathfrak{T}$. Assume there is a finite number of A-sets $A_1, \ldots, A_n$ of $\mathfrak{M}$ such that

(i) $\mathfrak{M} = A_1 \cup \ldots \cup A_n$,

(ii) $(A_1 \cup \ldots \cup A_i) \cap A_{i+1}$ is a single point of $\mathfrak{M}$ $(i = 1, \ldots, n-1)$.

Let r_i be the monotone retraction from $\mathfrak{M}$ onto A_i $(i = 1, \ldots, n)$. If Φ is a functional satisfying the conditions of **II·2**, then

$$\Phi(T) = \sum_{i=1}^{n} \Phi(sr_i f). \tag{1}$$

Proof. The proof is by induction on n. If $n = 2$, then $\Phi(T) = \Phi(sr_1 f) + \Phi(sr_2 f)$ by condition (3) of **II·2**. Suppose now the formula (1) is valid in case the number of A-sets satisfying (i), (ii) is not greater that $n-1$. In order to establish (1), write $\mathfrak{M} = (A_1 \cup \ldots \cup A_{n-1}) \cup A_n$. Let $A = A_1 \cup \ldots \cup A_{n-1}$. Then A is an A-set of $\mathfrak{M}$, and by (ii), $A \cap A_n$ is a single point. If r denotes the monotone retraction from $\mathfrak{M}$ onto A, we have by the case treating $n = 2$, $\Phi(T) = \Phi(srf) + \Phi(sr_n f)$.

The transformation $srf: P \to P^*$ admits of an unrestricted factorization $rf: P \Longrightarrow A_1 \cup \ldots \cup A_{n-1} = A$, $s: A \to P^*$. Since A_i is an A-set of A $(i = 1, \ldots, n-1)$, let $\bar{r}_i$ be the monotone retraction from A onto A_i. By the induction hypothesis, $\Phi(srf) = \sum_{i=1}^{n-1} \Phi(s\bar{r}_i rf)$. The mapping $\bar{r}_i r: \mathfrak{M} \Longrightarrow A_i$ is monotone and hence by the uniqueness of monotone retractions onto A-sets, $r_i = \bar{r}_i r$. Thus we obtain finally $\Phi(T) = \Phi(srf) + \Phi(sr_n f) = \sum_{i=1}^{n-1} \Phi(s\bar{r}_i rf) + \Phi(sr_n f) = \sum_{i=1}^{n} \Phi(sr_i f)$.

II·4. — Lemma. Let $T = sf$, $f : P \Longrightarrow \mathfrak{M}$, $s : \mathfrak{M} \dashrightarrow P^*$ be an unrestricted factorization of $T \in \mathfrak{T}$. For C a proper cyclic element of $\mathfrak{M}$, let r_c denote the monotone retraction from $\mathfrak{M}$ onto C. If $\mathfrak{M}$ contains only a finite number of proper cyclic elements, $C_1, \ldots, C_n$, with the property that for each i, $\mathfrak{M} - C_i$ decomposes into only a finite number of components, then

$$(1) \qquad \Phi(T) = \sum_{i=1}^{n} (sr_{c_i} f),$$

where Φ is a functional satisfying the conditions of **II·2**.

Proof. The proof is by induction on n. If $n = 1$, $\mathfrak{M}$ contains only one proper cyclic element C_1. By assumption, there is only a finite number of components $G_2, \ldots, G_k$ of $\mathfrak{M} - C_1$. Let $A_1 = C_1$ and $A_i = c(G_i)$ $(i = 2, \ldots, k)$. Then for each i, $1 \leqslant i \leqslant k$, A_i is an A-set of $\mathfrak{M}$, and $A_1, \ldots, A_k$ satisfy the conditions of **II·3**. Hence

$$(2) \qquad \Phi(T) = \Phi(sr_{c_1} f) + \sum_{i=2}^{k} \Phi(sr_i f),$$

where r_i is the monotone retraction from $\mathfrak{M}$ onto A_i $(i = 2, \ldots, k)$. But $sr_i f$ admits of an unrestricted factorization whose middle space is A_i. By **I·1**, A_i is a dendrite $(i = 2, \ldots, k)$, and hence by condition (2) of **II·2**, $\Phi(sr_i f) = 0$ $(i = 2, \ldots, k)$.

Suppose now that (1) is valid in case the number of proper cyclic elements of $\mathfrak{M}$ is not greater than $n-1$. In order to prove (1), assume $\mathfrak{M}$ contains exactly n proper cyclic elements $C_1, \ldots, C_n$. If $G_2, \ldots, G_k$ are the components of $\mathfrak{M} - C_1$, then using the same argument and notation as above, we have

$$(3) \qquad \Phi(T) = \Phi(sr_{c_1} f) + \sum_{i=2}^{k} \Phi(sr_i f).$$

Let K_i be the class of proper cyclic elements of $A_i = c\,(G_i)$ $(i = 2, 3, \ldots, k)$. K_i may be empty for some i. Then $K_i \cap K_j = 0$ for $i \neq j$, and if $C \in K_i$, C is one of the sets $C_1, \ldots, C_n$ (see **I·1**). Thus the number of proper cyclic elements in each A_i, $2 \leqslant i \leqslant k$, is not greater than $n-1$, and if $C \subset A_i$, $A_i - C$ decomposes into a finite number of components. If $\bar{r}_c$ denotes the monotone retraction from A_i onto $C \subset A_i$, then in view of the observation that $sr_i f$ admits of an unrestricted factorization with middle space A_i, we obtain by the induction hypothesis,

$$(4) \qquad \Phi(sr_i f) = \sum_{C \subset A_i} \Phi(s\bar{r}_c r_i f) = \sum_{C \subset A_i} \Phi\,(sr_c f) \qquad (i = 2, \ldots, k).$$

The formula (1) follows now from (3) and (4).

II.5. – In what follows we will restrict the PEANO space P to be the unit square $Q \equiv [0 \leqslant u, \ v \leqslant 1)$, and the metric space P^* to be the Euclidean (x, y, z)-space E_3. Let T be a continuous mapping from Q into E_3,

$$(1) \quad T: \qquad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in Q.$$

We introduce now three plane transformations by the formulas:

$$T_1: \qquad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in Q,$$

$$T_2: \qquad z = z(u, v), \quad x = x(u, v), \quad (u, v) \in Q,$$

$$T_3: \qquad x = x(u, v), \quad y = y(u, v), \quad (u, v) \in Q.$$

Let K_i be a square with sides parallel to the coordinate axes such that $T_i(Q) \subset K_i$ $(i = 1, 2, 3)$. For π a simple polygonal region in Q we denote by π^* the counterclockwise oriented boundary curve of π. Let $\mathcal{O}(p_i; T_i, \pi)$ be the topological index of a point $p_i \in K_i$ with respect to $T_i(\pi^*)$ if $p_i \notin T_i(\pi^*)$. We set $\mathcal{O}(p_i; T_i, \pi) = 0$ if $p_i \in T_i(\pi^*)$.

Let α be an open subset of Q and denote by σ a finite system of non-overlapping simple polygonal regions $[\pi_k; k = 1, \ldots, n]$ belonging to α. We define according to L. CESARI **[3]** the following quantities:

$$g_i(T, \pi) = \iint_{K_i} |\mathcal{O}(p_i; T_i, \pi)|, \quad \tau_i(T, \pi) = \iint_{K_i} \mathcal{O}(p_i; T_i, \pi), \quad (i = 1, 2, 3),$$

$$t(T, \pi) = (\tau_1^2 + \tau_2^2 + \tau_3^2)^{1/2},$$

$$\Psi_i(p_i; T, \alpha) = \underset{\sigma}{\text{l.u.b.}} \sum_{k=1}^{n} |\mathcal{O}(p_i; T_i, \pi_k)| \quad (i = 1, 2, 3),$$

$$G_i(T, \alpha) = \underset{\sigma}{\text{l.u.b.}} \sum_{k=1}^{n} g_i(T, \pi_k) \quad (i = 1, 2, 3),$$

$$G(T, \alpha) = \underset{\sigma}{\text{l.u.b.}} \sum_{k=1}^{n} g(T, \pi_k),$$

where $g(T, \pi_k) = (g_1^2 + g_2^2 + g_3^2)^{1/2}$. The least upper bound in the above expressions is taken with respect to all systems σ.

For the proof of the following two theorems the reader is referred to L. CESARI [**2**].

(i) For α an open subset of Q we have

$$\iint_{K_i} \Psi_i(p_i;\ T,\ \alpha) = G_i(T,\ \alpha) \qquad [\mathbf{2};\ 12\cdot 8,\ (\text{ii})].$$

(ii) Let E be a closed subset of Q and for α an open subset of Q let $\{\delta\}$ be the collection of components of $\alpha - E$. Under the assumption that $|T_i(E)| = 0$ (measure of $T_i(E)$) for $i = 1, 2, 3$, we have

$$\Psi_i(p_i;\ T,\ \alpha) = \sum \Psi_i(p_i;\ T,\ \delta),\quad \text{a.e. in } K_i, \tag{1}$$

$$G(T,\ \alpha) = \sum G(T,\ \delta), \tag{2}$$

where the summation in (1) and (2) is extended over all $\delta \in \{\delta\}$ [**2**; 21·4, (i)].

II·6. – For later application we shall state in this paragraph two immediate corollaries of **II·5**, (ii).

(i) Lemma. Given T, T_i ($i = 1, 2, 3$) as in **II·5**. Let E be a closed subset of Q such that $Q - E = \alpha \mid \beta$, i.e., α, β, are non-empty disjoint open subsets of $Q - E$ whose union is $Q - E$. Assume that $|T_i(E)| = 0$ ($i = 1, 2, 3$). Then $G_i(T, Q) = G_i(T, \alpha) + G_i(T, \beta)$ ($i = 1, 2, 3$).

Proof. If we denote by $\{\delta'\}$, $\{\delta''\}$ the components of α, β, respectively, we have as a consequence of **II·5**, (ii),

$$\Psi_i(p_i;\ T,\ \alpha) = \sum{}' \Psi_i(p_i;\ T,\ \delta'), \qquad \Psi_i(p_i;\ T,\ \beta) = \sum{}'' \Psi_i(p_i;\ T,\ \delta''),$$

$$\Psi_i(p_i;\ T,\ Q) = \sum{}' \Psi_i(p_i;\ T,\ \delta') + \sum{}'' \Psi_i(p_i;\ T,\ \delta'') \qquad \text{a.e. in } K_i,$$

where $\sum'$, $\sum''$ are extended over all $\delta' \in \{\delta'\}$, $\delta'' \in \{\delta''\}$, respectively. Thus $\Psi_i(p_i;\ T,\ Q) = \Psi_i(p_i;\ T,\ \alpha) + \Psi_i(p_i;\ T,\ \beta)$, a.e. in K_i. From **II·5**, (i) we deduce the desired equality.

(ii) Lemma. Assuming the conditions of (i), we have the following formula:

$$G(T,\ Q) = G(T,\ \alpha) + G(T,\ \beta).$$

Proof. The proof follows immediately from **II·5**, (ii).

Remark. If the conditions of (i) apply,

$$G(T, Q) = G(T, Q - E), \qquad G_i(T, Q) = G_i(T, Q - E) \ (i = 1, 2, 3).$$

II·7. – In this paragraph we shall summarize some results of a surface integral introduced by L. Cesari [**4**].

Let X be a compact subset of E_3, and let $F(x, y, z, u, v, w)$ be a function defined for each $(x, y, z) \in X$ and for each triple $(u, v, w) \neq (0, 0, 0)$ satisfying, moreover, the following conditions.

(1) $F(x, y, z, u, v, w)$ is continuous for each $(x, y, z) \in X$ and for each triple $(u, v, w) \neq (0, 0, 0)$.

(2) $F(x, y, z, u, v, w)$ is positively homogeneous of degree one with respect to u, v, w, i.e.,

$$F(x, y, z, ku, kv, kw) = kF(x, y, z, u, v, w) \quad \text{for each } k > 0.$$

If we put $F(x, y, z, 0,0,0) = 0$ for each $(x, y, z) \in X$, we have as a consequnece of (2) that F is also continuous at each point $(x, y, z, 0, 0, 0)$ where $(x, y, z) \in X$.

Let T, T_i $(i = 1, 2, 3)$ be given as is **II·5**. Assume that $T(Q) \subset X$ and that the Lebesgue area $L(T)$ is finite. Let α be an open subset of Q. For $[\pi_k; k = 1, \ldots, n]$ a finite system of non-overlapping polygonal regions in α, let us consider the following quantities:

$$m = \max_{i=1,2,3} \left| \sum_{k=1}^{n} T_i(\pi_k^*) \right|, \qquad \delta = \max_{k=1,\ldots,n} \varrho[T(\pi_k)],$$

$$\mu = \max \left[G(T, \alpha) - \sum_{k=1}^{n} t(T, \pi_k), \ \ G_i(T, \alpha) - \sum_{k=1}^{n} |\tau_i(T, \pi_k)| \ \ (i = 1, 2, 3) \right],$$

where $\varrho[T(\pi_k)$ denotes the diameter of $T(\pi_k)_i$.

The numbers m, δ, μ are termed the indices of the system of polygons $[\pi_k; k = 1, \ldots, n]$ with respect to (T, α), and are $\geqslant 0$. In view of the hypothesis $L(T) < \infty$, it is possibile to determine for $\gamma > 0$ given, a system of non-overlapping polygonal regions $[\pi_k; k = 1, \ldots, n]$ in α with indices less than γ with respect to (T, α) (see L. Cesari [**3**]).

Select a point (u_k, v_k) from each π_k and consider the sum

$$\sum_{k=1}^{n} F[x(u_k, v_k), \ y(u_k, v_k), \ z(u_k, v_k), \ \tau_1(T, \pi_k), \ \tau_2(T, \pi_k), \ \tau_3(T, \pi_k)].$$

L. Cesari [**4**] has shown that

$$\lim_{m,\delta,\mu \to 0} \ \sum_{k=1}^{n} F[\ \ldots\ ]$$

exixts and is finite, and we shall denote this limit by $J(T, \alpha) = \int_{(T,\alpha)} F\, d\sigma$. In case $\alpha = Q$, we write $J(T) = J(T, Q)$.

The following three theorems are due to L. CESARI [4].

Theorem 1. $J(T)$ is independent of the representation of the surface S.

Let now $\mathfrak{F}$ denote the class of all continuous mappings $T: Q \to E_3$ such that (i) $T(Q) \subset X$, (ii) $L(T) < +\infty$.

Theorem 2. There exists a positive number M such that $|J(T)| < < ML(T)$, and M does not depend upon $T \in \mathfrak{F}$.

Theorem 3. Let T, T_n $(n = 1, 2, \ldots)$ be in $\mathfrak{F}$. If $T_n \to T$ uniformly and $L(T_n) \to L(T)$, then $J(T_n) \to J(T)$.

II·8. – Let $\mathfrak{F}$ be the class of all continuous mappings $T: Q \to E_3$ as defined in **II·7**, i.e., $T(Q) \subset X$ and $L(T) < +\infty$.

Lemma. If $T_0 \in \mathfrak{F}$, then all the partial mappings of T_0 are also in $\mathfrak{F}$ (see **II·1**).

Proof. Let T be a partial mapping of T_0. Then there exists an unrestricted factorization $T_0 = sf$, $f: Q \Longrightarrow \mathfrak{M}$, $s: \mathfrak{M} \to E_3$ such that $T = sr_A f$, where r_A is the monotone retraction from $\mathfrak{M}$ onto an A-set A of $\mathfrak{M}$. We only need to verify that $L(T) < \infty$. If r_c denotes the monotone retraction from $\mathfrak{M}$ onto C, and if $\bar{r}_c$ denotes the monotone retraction from A onto a proper cyclic element C of A, we have in view of the strong cyclic additivity theorem of $L(T)$ (see [8]):

$$L(T_0) = \sum L(sr_c f), \qquad C \subset \mathfrak{M}, \tag{1}$$

$$L(T) = \sum L(s\bar{r}_c r_A f), \qquad C \subset A. \tag{2}$$

Since A is an A-set of $\mathfrak{M}$, a proper cyclic element C of A is also a proper cyclic element of $\mathfrak{M}$ and $\bar{r}_c r_A = r_c$ (see **I·1**). Hence

$$L(T) = \sum_{C \subset A} L(sr_c f) \leqslant \sum_{C \subset \mathfrak{M}} L(sr_c f) = L(T_0) < +\infty.$$

The class $\mathfrak{F}$ satisfies therefore the conditions of **II·2**. We proceed to verify that $J(T)$ also satisfies the three conditions of **II·2**.

II·9. – Lemma. $J(T)$ satisfies the condition (1) of **II·2**.

Proof. This is an immediate consequence of **II·7**.

Lemma. $J(T)$ satisfies the condition (2) of **II·2**, i.e., if $T \in \mathfrak{T}$ admits of an unrestricted factorization whose middle space is a dendrite, then $J(T) = 0$.

Proof. In view of the strong cyclic additivity theorem of $L(T)$ mentioned in **II·8**, we have $L(T) = 0$. From Theorem 2 in **II·7**, $J(T) = 0$.

II·10. – (Continuation.) In order to establish that $J(T)$ satisfies the condition (3) of **II·2**, let us first discuss two lemmas.

Lemma 1. Given T, T_i $(i = 1, 2, 3)$ as in **II·5**. Assume that $T \in \mathfrak{T}$. If E is a closed subset of Q such that $Q - E = \alpha \mid \beta$ and $T_i(E)$ is of measure zero $(i = 1, 2, 3)$, then

$$J(T) = J(T, Q) = J(T, \alpha) + J(T, \beta).$$

Proof. This Lemma is an immediate consequence of the definition of $J(T, \alpha)$ and of the section **II·6**.

Lemma 2. Let α be an open subset of Q. If for $T \in \mathfrak{T}$, $G(T, \alpha) = 0$, then $J(T, \alpha) = 0$.

Proof. It follows from the definitions in **II·5**, that then for every polygonal region $\pi \subset \alpha$, $\tau_i(T, \pi) = 0$ $(i = 1, 2, 3)$. From the definition of $F(x, y, z, u, v, w)$ there follows then that $F = 0$. Hence $J(T, \alpha) = 0$.

II·11. – (Continuation.) We are now ready to verify that $J(T)$ satisfies the condition (3) of **II·2**. Assume that $T \in \mathfrak{T}$ admits of an unrestricted factorization $T = sf$, $f : Q \Rightarrow \mathfrak{M}$, $s : \mathfrak{M} \to E_3$, where $\mathfrak{M}$ is the union of two A-sets A_1, A_2 and $A_1 \cap A_2$ reduces to a single point x of $\mathfrak{M}$.

Lemma. $J(T) = J(sr_1 f) + J(sr_2 f)$, where r_1, r_2 denote the monote retractions from $\mathfrak{M}$ onto A_1 and from $\mathfrak{M}$ onto A_2, respectively.

Proof. Let $E = f^{-1}(x)$. Then E is a closed subset of Q such that $T_i(E)$ is of measure zero $(i = 1, 2, 3)$, where T_i are the mappings introduced in **II·5**. In view of $f(Q) \cap (A_i - x) \neq 0$ $(i = 1, 2)$ we have, setting $\alpha = f^{-1}(A_1 - x)$, $\beta = f^{-1}(A_2 - x)$ that $Q - E = \alpha \mid \beta$. Now from the Lemmas 1 and 2 in **II·10** we have that

$$(1) \qquad J(sr_1 f) = J(sr_1 f, \alpha) + J(sr_1 f, \beta) = J(sr_1 f, \alpha),$$

since $G(sr_1 f, \beta) = 0$. Similarly,

$$(2) \qquad J(sr_2 f) = J(sr_2 f, \alpha) + J(sr_2 f, \beta) = J(sr_2 f, \beta).$$

We also have

$$(3) \qquad J(T) = J(T, \alpha) + J(T, \beta) = J(sr_1 f, \alpha) + J(sr_2 f, \beta).$$

Combining (1), (2) and (3), we obtain the desired conclusion.

II·12. – Lemma. Assume $T \in \mathfrak{T}$ admits of an unrestricted factorization $T = sf$, $f : Q \Longrightarrow \mathfrak{M}$, $s : \mathfrak{M} \to E_3$ such that $\mathfrak{M}$ contains only a finite number of proper cyclic elements $C_1, \ldots, C_n$ with the property that for each i, $\mathfrak{M} - C_i$ decomposes into only a finite number of components. If r_{c_i} denotes the monotone retraction from $\mathfrak{M}$ onto C_i $(i = 1, \ldots, n)$, then

$$J(T) = \sum_{i=1}^{n} J(sr_{c_i} f).$$

Proof. This formula follows from **II·4**, since in the preceding paragraph we have shown that $J(T)$ satisfies the conditions of **II·2**.

II·13. – We are now ready to prove our main result.

Theorem. *$J(T)$ is strongly cyclicly additive in the following sense. Let $T = sf$, $f : Q \Longrightarrow \mathfrak{M}$, $s : \mathfrak{M} \to E_3$ be an unrestricted factorization of $T \in \mathfrak{T}$. For C a proper cyclic element of $\mathfrak{M}$, let r_c denote the monotone retraction from $\mathfrak{M}$ onto C. Then*

$$(1) \qquad J(T) = \sum J(sr_c f), \qquad C \subset \mathfrak{M}.$$

Proof. From [**8**] there follows that

$$(2) \qquad L(T) = \sum L(sr_c f), \qquad C \subset \mathfrak{M}.$$

II·14. – (Continuation.) We proceed now to exhibit a sequence of mappings T_n in $\mathfrak{T}$ which converge uniformly to T on Q. Each T_n satisfies, moreover, the following properties. Each T_n admits of an unrestricted factorization $T_n = s_n f_n$, $f_n : Q \Longrightarrow \mathfrak{M}_n$, $s_n : \mathfrak{M} \to E_3$ such that

(i) $\mathfrak{M}_n \subset \mathfrak{M}$ $(n = 1, 2, \ldots)$;

(ii) $\mathfrak{M}_n$ contains only a finite number of proper cyclic elements each of which is a proper cyclic element of $\mathfrak{M}$;

(iii) if C is a proper cyclic element of $\mathfrak{M}_n$, $\mathfrak{M}_n - C$ decomposes into a finite number of components;

(iv) if for C a proper cyclic element of $\mathfrak{M}_n$, $r_c^{(n)}$ and r_c denote the monotone retractions from $\mathfrak{M}_n$ onto C and from $\mathfrak{M}$ onto C, then $s_n r_c^{(n)} f_n = s r_c f$.

Proof. Let n be a positive integer. Since $s : \mathfrak{M} \to E_3$ is uniformly continuous, we have a number $\varepsilon = \varepsilon(n) > 0$ such that for all sets $E \subset \mathfrak{M}$ with $\varrho(E) < \varepsilon$ we have $\varrho[s(E)] < 1/n$. From **I·12** there exists a PEANO space $\mathfrak{M}_n \subset \mathfrak{M}$ satisfying the following properties.

(1) Properties (ii) and (iii).

(2) There is a continuous retraction t_n from $\mathfrak{M}$ onto $\mathfrak{M}_n$ satisfying $\varrho[x, t_n(x)] < \varepsilon$ for $x \in \mathfrak{M}$.

(3) If for C a proper cyclic element of $\mathfrak{M}_n$, $r_c^{(n)}$ and r_c denote the monotone retractions from $\mathfrak{M}_n$ onto C and from $\mathfrak{M}$ onto C, respectively, then $r_c = r_c^{(n)} t_n$.

Define $T_n = s t_n f$. If we let $s_n = s$, $f_n = t_n f$, it follows from (3) that, $s_n r_c^{(n)} f_n = s r_c^{(n)} t_n f = s r_c f$. Now let w be any point of Q. Then $f(w) \in \mathfrak{M}$ and in view of (2), $\varrho[f(w), t_n f(w)] < \varepsilon$. Hence $\varrho[T_n(w), T(w)] = \varrho[s t_n f(w), s f(w)] < 1/n$ for every $w \in Q$.

If T_n is defined in this manner for every positive integer n, we have that $T_n \to T$ uniformly.

II·15. – (Continuation.) Let $\{T_n\}$ be the sequence of mappings in $\mathfrak{T}$ as defined in **II·14**. We assert that $L(T_n) \to L(T)$.

Proof. Since $T_n \to T$ uniformly,

$$(1) \qquad L(T) \leqslant \liminf_{n \to \infty} L(T_n) .$$

From (2) in **II·13**, however, $L(T) \geqslant \sum_{C \subset \mathfrak{M}_n} L(s r_c f) = \sum_{C \subset \mathfrak{M}_n} L(s_n r_c^{(n)} f_n) = L(T_n)$. Hence

$$(2) \qquad L(T) \geqslant \limsup_{n \to \infty} L(T_n).$$

(1) and (2) imply $L(T_n) \to L(T)$.

II·16. – (Continuation.) From Theorem 2 in **II·7**,

$$(1) \qquad |J(T_n)| < M L(T_n) \qquad (n = 1, 2, \ldots) .$$

From **II·12** and **II·14** there results

$$(2) \qquad J(T_n) = \sum_{C \subset \mathfrak{M}_n} J(s_n r_c^{(n)} f_n) = \sum_{\substack{C \subset \mathfrak{M} \\ C \subset \mathfrak{M}_n}} J(s r_c f).$$

Finally, Theorem 3 of **II·7** implies

$$(3) \qquad J(T_n) \to J(T).$$

Let now $\varepsilon > 0$ be given. Then there exists a positive integer N such that

$$(4) \qquad |J(T) - J(T_n)| < \varepsilon/2 \quad \text{for} \quad n > N,$$

$$(5) \qquad |L(T) - L(T_n)| = |\sum_{C \notin \mathfrak{M}_n} L(sr_c f)| < \varepsilon/(2M) \quad \text{for} \cdot n > N.$$

Now from (2),

$$|J(T) - \sum_{C \subset \mathfrak{M}} J(sr_c f)| \leqslant |J(T) - JT_n)| + |J(T_n) - \sum_{C \subset \mathfrak{M}} J(sr_c f)| \leqslant$$

$$\leqslant |J(T) - J(T_n)| + |J(T_n) - \sum_{C \subset \mathfrak{M}_n} J(sr_c f)| + \sum_{C \notin \mathfrak{M}_n} |J(sr_c f)| \leqslant$$

$$\leqslant |J(T) - J(T_n)| + M \sum_{C \notin \mathfrak{M}_n} L(sr_c f) < \varepsilon/2 + M\varepsilon/(2M) = \varepsilon$$

for $n > N$. Since $\varepsilon > 0$ was arbitrary, we finally obtain

$$J(T) = \sum_{C \subset \mathfrak{M}} J(sr_c f).$$

This completes the proof of Theorem in **II·13**.

II·17. – In the next two paragraphs we shall discuss a generalization of the Theorem in **II·13**.

Let $T \in \mathfrak{T}$ and let $T = sf$, $f: Q \to \mathfrak{M}$, $s: \mathfrak{M} \to E_3$ be an unrestricted factorization of T. It should be noted that we assume now that f is a mapping *into* a PEANO space $\mathfrak{M}$ instead of *onto* $\mathfrak{M}$. If for C a proper cyclic element of $\mathfrak{M}$ we denote by r_c the monotone retraction from $\mathfrak{M}$ onto C, then it may happen that $sr_c f(Q)$ is not contained in X. Consequently, $J(sr_c f)$ need not be defined since the function $F(x, y, z, u, v, w)$ is assumed to be defined only for $(x, y, z) \in X$ (**II·7**). This difficulty can be overcome as follows. First let us observe that the function $F(x, y, z, u, v, w)$, $(x, y, z) \in X$ and (u, v, w) any triple of numbers, can be extended to a function $F_0(x, y, z, u, v, w)$ defined for all $(x, y, z) \in E_3$ and any triple of numbers (u, v, w) preserving uniform continuity and positive homogeneity of degree one with respect to (u, v, w). Moreover, for $u^2+v^2+w^2=1$, $F_0(x, y, z, u, v, w)$ possesses the same bound as $F(x, y, z, u, v, w)$, $(x, y, z)\in X$, $u^2+v^2+w^2=1$. Such an extension can easily be carried out by a result of E. J. McShane [*Extension of range of functions*, Bull. Amer. Math. Soc. **40**, 837-842(1934)]. For the actual computations see L. Cesari [*An existence theorem of Calculus of variations for integrals on parametric surfaces*, Amer. J. Math. **74**, p. 281 (1952)].

II·18. – (Continuation.) With any continuous mapping T from Q into E_3 for which $L(T)<\infty$ we can now associate a surface integral $J_0(T)$ defined as in **II·7** by using the function $F_0(x, y, z, u, v, w)$. For $T(Q)\subset X$ and $L(T)<\infty$, we have $J_0(T)=J(T)$.

The Theorem in **II·13** can now be formulated as follows.

Theorem. *Let $T\in\mathfrak{T}$ and let $T=sf$, $f: Q\rightarrow\mathfrak{M}$, $s:\mathfrak{M}\rightarrow E_3$ be an unrestricted factorization of T. If for C a proper cyclic element of $\mathfrak{M}$ we denote by r_c the monotone retraction from $\mathfrak{M}$ onto C, then*

$$J(T)=\sum J_0(sr_c f), \qquad C\subset\mathfrak{M}. \tag{1}$$

Proof. The proof is entirely analogous to the proof given for the Theorem in **II·13**.

Remark. If we denote by $\sum'$ the summation in (1) over all proper cyclic elements C of $\mathfrak{M}$ for wich $C\cap f(Q)\neq 0$, then

$$J(T)={\sum}' J(sr_c f)\,. \tag{2}$$

Proof. If $C\cap f(Q)=0$, then in view of the connectedness of $f(Q)$ we have that $r_c f(Q)$ is a single point in C, and hence $sr_c f$ is constant on Q. Consequently, $J_0(sr_c f)=0$. If now C is a proper cyclic element intersecting $f(Q)$, then it follows from T. Radó [9, II·2·42] that $r_c f(Q)=f(Q)\cap C$. Consequently, $sr_c f(Q)\subset X$ and $J_0(sr_c f)=J(sr_c f)$. From (1) we infer now (2).

Bibliography.

1. J. CECCONI, *Sulla additività ciclica degli integrali sopra una superficie,* Rivista Mat. Univ. Parma **4**, 43-67 (1953).

2. L. CESARI, *Surface Area,* Princeton University Press, Princeton 1956.

3. L. CESARI, *Sui fondamenti geometrici dell'integrale classico per l'area delle superficie in forma parametrica,* Mem. Accad. Italia **13**, 1323-1481 (1943).

4. L. CESARI, *La nozione di integrale sopra una superficie in forma parametrica,* Ann. Scuola Norm. Super. Pisa (2) **13**, 77-117 (1944).

5. L. CESARI, *Condizioni necessarie per la semicontinuità degli integrali sopra una superficie in forma parametrica,* Ann. Mat. Pura Appl. (4) **29**, 199-224 (1950).

6. L. CESARI, *Condizioni sufficienti per la semicontinuità degli integrali sopra una superficie in forma parametrica,* Ann. Scuola Norm. Super. Pisa (2) **14**, 47-79 (1948).

7. E. J. MICKLE and C. J. NEUGEBAUER, *Weak and strong cyclic additivity* (to appear).

8. E. J. MICKLE and T. RADÓ, *On cyclic additivity theorems,* Trans. Amer. Math. Soc. **66**, 347-365 (1949).

9. T. RADÓ, *Length and Area,* Amer. Math. Soc. Col. Pub., Vol. **30**, 1948.

10. G. T. WHYBURN, *Analytic Topology,* Amer. Math. Soc. Col. Pub., Vol. **27**, 1942.