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On the subdivision of surfaces into pieces with rectifiable boundaries. (**)

1. – Introduction.

Certain classes of problems in surface area theory can be conveniently treated if it is possible to subdivide the surface into arbitrarily small pieces the sum of whose areas is the area of the surface. In particular, problems which involve definitions of integrals over the surface, for example, certain variational problems can be investigated by making use of techniques in which this type of subdivision is involved. For instance, EWING [5] in a recent paper in which he considers integrals of a WEIERSTRASS type defined over surfaces has made use of representations having the above properties and has shown that the existence for all surfaces S defined over a simply connected JORDAN region and having finite LEBESGUE area of such representations is a consequence of a representation theorem of CESARI [2]. In this paper we prove a somewhat sharper theorem for the case of a surface defined over a simply connected JORDAN region and which is open and non degenerate and has finite LEBESGUE area. It is shown in this case that there exists a decomposition of the surface into arbitrarily small pieces, each of which is the image,

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L. CESARI and R. E. FULLERTON, *On regular representations of surfaces*, Rivista Mat. Univ. Parma 2, 279-288 (1951).

L. CESARI, *Contours of a Fréchet surface*, Rivista Mat. Univ. Parma 4, 173-194 (1953).

R. E. FULLERTON, *On the rectification of contours of a Fréchet surface*, Rivista Mat. Univ. Parma 4, 207-212 (1953).

under a suitably chosen representation, of a rectangle in the unit square in the domain plane and such that the boundary of the rectangle maps into a rectifiable continuous curve. Furthermore, the rectangles can be chosen in such a manner that the ratio of their lengths and widths is less than two. In the case mentioned, the representation theorem of MORREY leads to this result. However the approach to the problem in this paper is from a geometric and topological view-point and is hence closer to the nature of the problem. This approach, based on recent results of CESARI [1, 4] and the author [6] appears to have greater possibility of extension to higher dimensions and also to the case in which the surface is defined over a finitely connected JORDAN region, a case to which MORREY's theorem does not apply.

Les S be a non degenerate open FRÉCHET surface defined over a simply connected JORDAN planar region. S can be represented in various ways by means of maps $T: Q \to E_3$ where Q is the unit square in E_2 and all the representations $\{T\}$ are FRÉCHET equivalent to each other. In case S is non degenerate and open, i.e., the inverse image of any point of the surface under any of the $\{T\}$ separates neither Q nor E_2, it is known that there exists a representation T of S which is light, i.e. the inverse image under T of any point of S_3 is a point of a totally disconnected subset of E_2.

2. – Notations and basic theorems.

The notations are the standard ones used in area theory. We denote by $[S]$ the set of points of E_3 occupied by the surface S. If A is a planar region, its boundary is denoted by A^*, its closure by $\bar{A}$ and its interior by A^0. For the notions of LEBESGUE area $L(S)$, FRÉCHET equivalence, etc. The reader is referred to standard works on the subject, for example, RADÓ [10] and CESARI [4]. We shall make use of vector notation whenever possible and denote by $T: x = x(w)$ the triple of real valued functions of two real variables $x_1 = x_1(u, v)$, $x_2 = x_2(u, v)$, $x_3 = x_3(u, v)$ where $w = (u, v) \in E_2$, $x = (x_1, x_2, x_3) \in E_3$. We denote by Q the unit square in E_2, $0 \leqslant u, v \leqslant 1$. For theorems on the topology of the plane and on the theory of prime ends, see, for example, NEWMAN [9] and KERÉKJÁRTÓ [7]. In particular, for the theory of prime ends, see the paper of URSELL and YOUNG [11] and the first part of the paper of CESARI [1]. We shall follow the notations of CESARI in our discussion of ends and prime ends. It is to be noted that his notation is somewhat different than that employed by URSELL and YOUNG. In particular, if A is an open plane set and γ a portion of its boundary we denote by $\{\eta\}_{\gamma, A}$ the set of its ends corresponding to the boundary points on γ and by w_η the boundary point associated with the end η. The absolute value symbol when used in vector expressions will denote distances in the space involved.

The principal tool used in the proof of the theorem is the recently announced theorem of CESARI [4] in which he extends a classical inequality originally proved for HAUSDORFF measures on metric spaces to an inequality involving LEBESGUE area of FRÉCHET surfaces. Specifically, the theorem of CESARI states that if f is a real valued function defined over the points $[S]$ of a surface S and satisfying the condition $|f(x_1)-f(x_2)|<K|x_1-x_2|$ for some $K>0$ and for every pair x_1, x_2 of points of $[S]$ and if $\lambda(t)$ is the length of the image of the boundary of the open set $\beta_t \subset Q$ for which $f(x(w))<t$ and if t', t'' are respectively the upper and lower bounds of $f(x)$ on $[S]$, then $K\cdot L(S) \geqslant \int_{t'}^{t''} \lambda(t)dt$. The length $\lambda(t)$ of the image of the boundary of β_t is defined in a manner similar to that of defining a length of a curve in terms of a mapping function defined on a simple arc and is discussed in CESARI [1]. CESARI has shown that this generalized length defined on the images of boundaries of open sets of Q which lie on $[S]$ is a FRÉCHET invariant of the surface. The inequality of CESARI shows that in case $L(S)<\infty$, $\lambda(t)<\infty$ for almost all t in the interval $t' \leqslant t \leqslant t''$. The author has proved [6], using CESARI's result, that if $\{t_i\}$ is any countable set of points of the interval $[t', t'']$ for which $\lambda(t_i)<\infty$ there exists a representation T' of S such that for any given $\beta^*_{t_i}$, the set $T'^{-1}[T(\beta^*_{t_i})]$ is a collection of simple arcs and simple closed curves for each i. The substitution of such a map T' for T is called a rectification of the contour $\beta^*_{t_i}$. The principal tool used in the following theorem consists of a rectification of certain contours of finite length to straight line segments.

3. – The subdivision theorem.

Theorem. *Let S be an open non degenerate Fréchet surface of finite Lebesgue area and let T: $x=x(w)$, $w \in Q$, $x \in E_3$, be a representation of S. Then there exists a representation T_0 of S, Fréchet equivalent to T and a nested sequence $\{P_k\}$ of partitions of Q into closed rectangles $P_k=\{R_{ik}\}$ with sides parallel to the axes and with disjoint interiors having the following properties:*

(a) *The ratio of the larger to the smaller side of each R_{ik} does not exceed* 2.

(b) *For each $\varepsilon>0$ there exists an n_ε such that the diameter of each R_{ik} is less than ε if $k>n_\varepsilon$.*

(c) *For each R_{ik}, $T_0(R^*_{ik}-Q^*)$ is a rectifiable continuous curve.*

Proof. Since T is open and non degenerate there exists a light mapping U: $Q \to E_3$ represented by $x=x(w)$ which is FRÉCHET equivalent to T. Let

l_1 be the line segment in Q composed of those points $w = (u, v)$ for which $u = \frac{1}{2}$. Let w be any point of Q and consider the point $x(w) \in [S]$. Let $\{\sigma_j(w)\}$ be a sequence of closed spheres in E_3 with centers $x(w)$ and with radii decreasing to zero. Let $C_{jw} = C_w[U^{-1}\{\sigma_j(w) \cap [S]\}]$ be the component of $U^{-1}\{\sigma_j(w) \cap [S]\}$ in Q which contains w. Then $\{C_{jw}\}$ is a monotone decreasing sequence of closed subsets of Q with $U(C_{jw}) \subset \sigma_j(w)$. Also $\bigcap_{j=1}^{\infty} C_{jw} = w$ since if the intersection contained more than one point it would contain a connected subset of points whose image would be $x(w)$ contrary to the hypothesis that U is light. Let $s(w)$ be a circle around w with a positive radius. Then there esists a j_0 with $C_{jw} \subset s(w)$ for all $j \geqslant j_0$ since if this were not so there would exist $w_1 \neq w$, $w_1 \in \bigcap_{j=1}^{\infty} C_{jw}$ contrary to the assumption that U is light.

Since U is uniformly continuous over Q, there exists for each $\varepsilon > 0$ a $\delta > 0$ such that $|x(w_1) - x(w_2)| < \varepsilon$ if $|w_1 - w_2| < \delta$. Let $\delta_1 > 0$ be chosen so that $|x(w_1) - x(w_2)| < \frac{1}{4}$ for $|w_1 - w_2| < \delta_1$ and $\delta_1 < \frac{1}{4}$. Let L_1 be a rectangular strip of width δ_1 in Q with l_1 as center line, i.e. $w \in L_1$ if and only if $w = (u, v)$ with $\frac{1}{2} - \delta_1/2 \leqslant u \leqslant \frac{1}{2} + \delta_1/2$. For each $w \in l_1$ there exists a sphere $\sigma(w, r_w)$ of radius r_w and center $x(w)$ in E_3 such that $C_w[U^{-1}\{\sigma(w, r_w) \cap [S]\}]$ is of diameter less that δ_1 and hence lies in L_1. For each $w \in l_1$ let a function $f_w(x)$ be defined as follows. Let $\sigma(w, r_w)$ be the sphere above and let $f_w(x) = 1$ if $x \in [S] - \sigma(w, r_w)$, $f_w(x(w)) = 0$ and if $x \in \sigma(w, r_w)$ let $f_w(x) = r/r_w$ where r is the distance between x and $x(w)$ in E_3. Evidently for $K_w > 1/r_w$ we have $|f_w(x_1) - f_w(x_2)| < K_w|x_1 - x_2|$ for any $x_1, x_2 \in S$. Hence the CESARI inequality can be applied the this function. Let $\beta_w(t)$ be the set of points of Q for which $f_w(x(w)) < t$. If $0 < t < 1$, some component of $\beta_w(t)$ lies in L_1. Let $\lambda_w(t)$ be the length of the image of $\beta_w^*(t)$. By the CESARI inequality, $K_w \cdot L(S) \geqslant \int_0^1 \lambda_w(t)\,dt$. Since $L(S) < \infty$ for all $w \in l_1$, $\lambda_w(t)$ is finite for almost all t, $0 \leqslant t \leqslant 1$. For each $w \in l_1$ let t_w be a value of t for which $\lambda_w(t_w) < \infty$ and let $\alpha_w(t_w)$ be the component of $\beta_w(t_w)$ which contains w. Then $\alpha_w(t_w) \subset U^{-1}\{\sigma(w, r_w) \cap [S]\}$. For each $w \in l_1$ let a set $\alpha_w(t_w)$ be so chosen. Then the family of open sets $\{\alpha_w(t_w)\}$ covers the set l_1. Each set of the family lies in L_1 and the image under U of the portion of its boundary interior to Q is of finite length. Since l_1 is compact, there exists a finite subfamily $\alpha_{w_i}(t_{w_i})$, $(i = 1, 2, 3, \ldots, n)$, such that $l_1 \subset \bigcup_{=1}^{n} \alpha_{w_i}(t_{w_i})$. Let each $\alpha_{w_i}(t_{w_i})$ be designated by α_i, $(i = 1, 2, 3, \ldots, n)$. Since each such set has $U(\overline{\alpha_i^* - Q^*})$ of finite length, the sum of the lengths is finite. Since l_1 separates Q, $\bigcup_{i=1}^{n} \alpha_i$ separates Q also. Let A_1 be the set of points of Q which are separated from the points to the left of L_1 by the set $\bigcup_{i=1}^{n} \alpha_i$ or some subset of $\bigcup_{i=1}^{n} \alpha_i$. Then A_1 is simply connected and open in Q and $\mathcal{F}(A_1) = \bar{A}_1 - A_1$ is a single continuum γ, completely contained in L_1, separating A_1 from the segment l_1 in Q. γ has non empty intersections with the portions of the lines $v=0$, $v=1$ included in L_1. Let $w_1=(u_1, 0)$, $w_2=(u_2, 1)$ be the intersection points of maximum abscissa of γ and these segments. Thus the two segments $[v=0,\ u_1 \leqslant u \leqslant 1]$, $[v=1,\ u_2 \leqslant u \leqslant 1]$ characterize two ends η_1, η_2 of γ in A_1 and the collection $\{\eta\}_{\gamma,A_1}$ of all ends of γ in A_1 can be linearly ordered and constitute a closed interval $[\eta_1, \eta_2]$ of ends (see [1]), whose end elements are η_1 and η_2. We shall prove by a number of lemmas that the set γ is a simple arc whose image under U has finite length.

Lemma 1. *For any end* $\eta \in \{\eta\}_{\gamma,A_1}$ *there is an end* $\eta' \in \{\eta\}_{\gamma,\alpha_j}$ *relative to some* α_j *with* $w_\eta = w_{\eta'}$.

Proof. For any end $\eta \in \{\eta\}_{\gamma,A_1}$ let $w_\eta \in \gamma$ be the point on γ corresponding to η. Then $w_\eta \in \alpha_j^*$ for at least one α_j, $j=1, 2, \dots, n$ and hence there exists at least one prime end $\omega \in \{\omega\}_{\alpha_j}$ such that $w_\eta \in E_\omega$ where E_ω is the set of boundary points corresponding to ω. Since $\lambda_j = \lambda_{w_j}(t_{w_j}) < \infty$ and by a theorem of CESARI [1, 4], U is constant on E_ω and since U is light, E_ω is a single point. This implies that E_ω is accessible from α_j and hence corresponds to an end of α_j.

Lemma 2. *If* η', η'' *are any two distinct ends of* γ *in* A_1, *then* $w_{\eta'} \neq w_{\eta''}$.

Proof. Suppose that $w_{\eta'} = w_{\eta''} = w$. Let s', s'' be two arcs defining η', η'' having only w and one other point w^* in common where w^* is interior to A_1. The $s' \cup s''$ is a JORDAN curve defining a closed JORDAN region $J \subset Q$, $J^* = s' \cup s'' \subset A_1 \cup (w)$. $J^0 \cap \gamma \neq 0$, would imply that $J^0 \cap \alpha_j^* \neq 0$ for some j and hence that $J^* \cap \alpha_j^* \neq 0$ for some j, a contradiction since $J^* \subset A_1 \cup (w)$, $J^0 \cap \alpha_j = 0$ and this implies that η' and η'' are not distinct ends of γ in A_1 and yields a contradiction. Thus $w_{\eta'} \neq w_{\eta''}$ for distinct ends η', η'' of γ in A_1.

Lemma 3. *If* $\eta_1, \eta_2, \eta_3, \eta_4$ *are distinct ordered ends of* γ *in* A_1 *and there are four ends* $\eta_1', \eta_2' \eta_3', \eta_4'$ *relative to the same* α_j *with* $w_{\eta_j} = w_{\eta_j'}$, $(j=1, 2, 3, 4)$, *then* $\eta_1', \eta_2', \eta_3', \eta_4'$ *are distinct and ordered ends of* α_j.

Proof. By lemma 2, the points $w_{\eta_i} = w_{\eta_i'}$ are distinct and hence the ends η_i' are distinct. Let s_i, s_i', $i=1, 2, 3, 4$, be arcs defining the ends η_i of γ in A_1 and η_i' of α_j. It may be assumed that the arcs s_i intersect only at some point $w^* \in A_1^0$ and that the arcs s_i' intersect at a single point of α_j. The ordering of the ends $\eta_1, \eta_2, \eta_3, \eta_4$ implies that the cross-cut $s_1 \cup s_3$ separates the arcs $s_2 - (w^*)$, $s_4 - (w^*)$ in A_1. Let J^* be the JORDAN curve $J^* = s_1 \cup s_3 \cup s_1' \cup s_3'$. It is easily seen that J^* separates the arcs $s_2 - (w^*)$, $s_4 - (w^*)$ in Q. Since the arcs s_i lie in $A_1 \cup (w_{\eta_i})$ and the arcs s_i' lie in $\alpha_j \cup (w_{\eta_i})$, and the only intersection point of the arcs s_i is w^* and of the s_i' is a point of α_j, it can be seen that the open arcs $s_2 \cup s_2'$ and $s_4 \cup s_4'$ are separated by J^*. This implies that the half open arcs s_2' and s_4' are separated by J^* and hence by $s_1' \cup s_3'$ in α_j and that the ends $\eta_1', \eta_2', \eta_3', \eta_4'$ are ordered in the same ordering as $\eta_1, \eta_2, \eta_3, \eta_4$.

It will be noted that the above reasoning can be inverted. Also, it can be extended to show that any finite ordered array of ends of A_1 which correspond to boundary points corresponding to ends of some α_j induce the same ordering on the ends of α_j.

Lemma 4. *All points of γ are accessible from A_1.*

Proof. Let ω be any prime end of γ in A_1 and let E_ω be the corresponding continuum $E_\omega \subset \gamma$, E_0 the set of the principal points of E_ω, and E_ω', E_ω'' the left and right wings of E_ω (see [**11**]). E_ω', E_ω'', E_0 are subcontinua of E_ω, $E_0 \subset E_\omega' \cap E_\omega''$, $E_\omega' \cup E_\omega'' = E_\omega$. Consider first the set E_ω'. Let $\{\eta\}'$ be the interval of ends from the first end on γ to ω. For each end in this interval there corresponds an end η' relative to some α_j $(j=1, 2, 3, \dots, n)$. Let $\{\eta'\}_j$ be the subcollection of all the ends relative to α_j corresponding to ends of γ relative to A_1 in the above interval. These n collections are not necessarily disjoint and some may be empty or finite. Assume each class ordered as the corresponding ends of A_1 and let us consider only those classes $\{\eta'\}_j$ which contain ends of any open interval containing ω and which are infinite. Since each point of E_ω' is a limit point of points w_η, $\eta \in \{\eta\}'$ we can denote by E_j' the set of all points $w \in E_\omega'$ which are limit points of $w_{\eta'}$, $\eta \in \{\eta'\}_j$. Then the sets E_j' are closed and cover E_ω'. For each ordered collection $\{\eta'\}_j$ consider also the corresponding ordered collection $\{x(w_{\eta'})\}_j$ of points of E_3. Assume that such a collection has more than one limit point in E_3 and let x', x'' be two distinct limit points with $|x' - x''| = 3\delta > 0$. Consider a finite system of $2N$ ordered ends $\eta_1', \eta_2', \dots, \eta_{2N}' \in \{\eta'\}_j$ such that $|x_{2\nu-1} - x'| < \delta$, $|x_{2\nu} - x''| < \delta$, $(\nu = 1, 2, 3, \dots, N)$, where $x_\nu = x(w_{\eta_\nu})$, $(\nu = 1, 2, 3, \dots, 2N)$. Then the corresponding ends η_ν', $(\nu = 1, 2, 3, \dots, 2N)$, with $w_{\eta_\nu'} = w_{\eta_\nu}$ are ordered in α_j and we have $\infty > \lambda_j \geq \sum_{\nu=1}^{2N} |x_\nu - x_{\nu-1}| > 2(N-1)\delta$ where N can be taken arbitrarily large. This gives a contradiction since $\lambda(\alpha_j^*) < \infty$. Thus the ordered collection $\{x(w_\eta)\}$ has only one limit point x_0 and this implies that U has a constant value x_0 on the set E_j'. Thus U may have at most n distinct values on E_ω' and must be hence constant on E_ω' since E_ω' is a continuum. The same holds for E_ω'' and since $E_\omega' \cap E_\omega'' \neq 0$, U is constant on E_ω. This implies that E_ω is a single point since U is light. This holds for all prime ends ω of γ and, since $\{E_\omega\}$ is a covering of γ, all the points of γ are accessible from A_1.

Lemma 5. *γ is an arc joining the points w_{η_1} and w_{η_2}.*

Proof. It is known [**11**] that there is a mapping $\tau\colon A_1 \to \Gamma$ mapping A_1 into a circle Γ and A_1^* into Γ^*, γ into an arc c of Γ^*, and that τ is one to one and continuous between the collection of the prime ends of A_1^* in A_1 and Γ^* and one to one and continuous between A_1^0 and Γ^0. In the present case all prime ends correspond to simple ends η of γ and distinct ends correspond to different points $w \in \gamma$. Thus τ^{-1} is a homeomorphism between an arc of Γ^* and γ. Hence γ is an arc.

Lemma 6. *Given any two points w', $w'' \in \gamma$. Then there exists a sequence $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_k}$ of the α_j and a sequence of ends $\eta_1', \eta_2', \ldots, \eta_k', \eta_1'', \ldots, \eta_k''$ where η_i', η_i'' are respectively ends of the α_{j_i} and such that $w' = w_{\eta_1'}$, $w'' = w_{\eta_k''}$ and $w_{\eta_i'} = w_{\eta_{i-1}''}$ for $i = 2, 3, \ldots, k$.*

Proof. Evidently the set $\{\alpha_j^* \cap \gamma\}$, $(j=1, 2, \ldots, n)$, are closed and cover γ. Let $w' \in \alpha_{j_1}^*$ and let η_1' be an end corresponding to w'. Consider all the sets α_j^* containing w' and assume that $\alpha_{j_1}^*$ is the set of this family containing points on γ nearest to w''. Let $w_{\eta_1''}$ be the point on $\alpha_{j_1}^* \cap \gamma$ nearest to w''. Then since the $\alpha_j^* \cap \gamma$ are closed and cover γ, if $w_{\eta_1''} \neq w''$, there exist sets α_j^* containing $w_{\eta_1''}$ which do not contain w'. Consider all these sets, let $\alpha_{j_2}^*$ be the set of this family containing points on γ nearest w'' and let $w_{\eta_2'}$ be the point on $\alpha_{j_2}^* \cap \gamma$ nearest w''. Let this process be continued and since the family $\{\alpha_j\}$ is finite we get at most n points. Let η_i', η_i'' be ends of the α_{ji} with $w_{\eta_i'} = w_{\eta_i''}$ for $i = 2, 3, \ldots, k$.

Lemma 7. *The image under U of γ is a continuous curve of finite length $\lambda(\gamma)$ with $\lambda(\gamma) \leqslant \sum_{j=1}^{n} \lambda(\alpha_j^*)$.*

Proof. Let $\{\eta_i\}$ be any finite ordered system of ends of γ in A_1 where $i = 1, 2, \ldots, N$. For each i, the point w_{η_i} lies in some $\alpha_j^* \cap \gamma$. It is to be noted that the points w_{η_i} which lie in the same α_j^* may be non consecutive points in the family $\{w_{\eta_i}\}$. By Lemma 6, it is possible to insert between any two consecutive points w_{η_i}, $w_{\eta_{i+1}}$ lying in different sets α_j^*, convenient finite chains of new points. The final result is a new finite chain $w_{\eta(j)}$, $(j=1, 2, \ldots, M)$, $M \geqslant N$ which is the union of a number K of subfamilies $\{w_{\eta(1)}, \ldots, w_{\eta(m_1)}\}$, $\{w_{\eta(m_1)}, w_{\eta(m_1+1)}, \ldots, w_{\eta(m_2)}\}$, $\ldots$, $\{w_{\eta(m_{k-1})}, \ldots, w_{\eta(m_k)}\}$ where the last element of each family coincides with the first element of the next and where successive points are in the same set α_{jk}^*, $(k=1, 2, \ldots, K)$. The same α_j may occur more than once in this succession. Hence

$$\sum_{i=1}^{N-1} |x(w_{\eta_i}) - x(w_{\eta_{i+1}})| \leqslant \sum_{j=1}^{M-1} |x(w_{\eta(j)}) - x(w_{\eta(j+1)})| = \sum_{s=1}^{K} \sum_{j=m_{s-1}}^{ms} |x(w_{\eta(j)}) - x(w_{\eta(j+1)})| .$$

If the last sum is rearranged so as to associate the various interior sums relative to the same set α_j, we have

$$\sum_{i=1}^{N-1} |x(w_{\eta_i}) - x(w_{\eta_{i+1}})| \leqslant \sum_{j=1}^{n} (\sum |x(w_{\eta(j)}) - x(w_{\eta(j+1)})|)$$

where the interior sum corresponds to ends η_i', relative to α_j ordered in α_j. Since the second sum is always less than $\sum_{j=1}^{n} \lambda(\alpha_j^*) = C < \infty$. The first sum is bounded by C for all possible choices of points w_{η_i} and hence by the definition of the CESARI generalized length, $\lambda(\gamma)$ exists and is less than C.

By the way in which γ was constructed, each point w lies on the boundary of an open set α_j of diameter not exceeding δ_1. Hence each point of γ can be joined to a point of l_1 by a simple arc of diameter less than δ_1. By methods similar to those used in the author's paper [**6**], a homeomorphism φ from A_1 to the set of points for which $u \geqslant \frac{1}{2}$ in Q can be found which is the identity for all points $w = (u, v) \in Q$ with $u \geqslant \frac{1}{2} + \delta_1$ which takes γ into l_1 and is such that the distance between any point of Q and its image under φ is less than δ_1. This homeomorphism can be defined on all of Q in such a way as to be the identity on all points except those of L_1 and which takes Q^* into itself. Let the mapping U_2 be defined from Q to E_3 as $U_2 = U\varphi^{-1}$; $x_2(w) = x(\varphi^{-1}(w))$, $w \in Q$. Thus for $w \in Q$, $|x_2(w) - x(w)| < \frac{1}{4}$ and U_2 is also a light mapping of Q into E_3 which maps l_1 into a continuous curve of finite length.

Let $\delta_2 > 0$ be such that $|x_2(w) - x_2(w')| < \frac{1}{8}$ if $|w - w'| < \delta_2$. Let L_2 be a strip of width δ_2 in Q consisting of all points $w = (u, v)$ with $0 \leqslant u \leqslant \frac{1}{2}$ and $\frac{1}{2} - \delta_2 \leqslant v \leqslant \frac{1}{2} + \delta_2$. Let w_0 be the point $(\frac{1}{2}, \frac{1}{2}) \in Q$ and around $x_2(w_0)$ on $[S]$ construct a sphere such that the inverse image under U_2 of the intersection of this sphere and $[S]$ has its component α_0 which includes w_0 of diameter less than δ_2. Let a function $f_{w_0}(x)$ be defined over $[S]$ as in the first part of the proof. Let β_t be defined as before having a boundary of finite length and let α_0 be the component of which included w_0. Let ξ_0 be a component of the boundary of $\alpha_0 \subset L_2$ which intersects both l_1 and l_2 and let w' be the point of intersection of ξ_0 with l_2 which lies farthest to the right. Let w'' be the point of intersection of ξ_0 and l_1. Evidently $w', w'' \neq w_0$. For all points of l_2 between the point $(0, \frac{1}{2})$ and w' let sets $\alpha_w^{(2)}(t_w)$ be constructed as in the first part of the proof, each having boundaries whose images are of finite length and each of diameter less than δ_2. Let $\{\alpha_j^{(2)}\}$ be a finite subfamily of these sets covering the segment $v = \frac{1}{2}$, $0 \leqslant u \leqslant w'$ and let A_2 be formed in the half of Q to the left of l_1 in a manner similar to that of A_1 bounded by portions of the boundary of $\bigcup_{j=1}^{n_2} \alpha_j^{(2)*}$ and ξ_0. Let l_2' be a line segment, parallel to the u axis joining the point $w'' = (\frac{1}{2}, v'')$ to the point $(0, v'')$. By the same method as was used in the first part of the proof it is possible to find a new mapping U_3; $x_3 = x_3(w)$ which is light and LEBESGUE equivalent to U_2, which rectifies the portion of the boundary of A_2 not on Q^* or l_1 into l_2' such that $x_3(w) = x_2(w)$ on l_1 and on $Q - L_2$ and such that $|x_3(w) - x_2(w)| < \frac{1}{8}$, $w \in Q$.

We continue the same procedure. At the next step the set of points to the right of l_1 is divided in the same manner and a rectifying map U_4 is obtained which is equal to U_3 outside a small strip in Q and such that $|x_4(w) - x_3(w)| < 1/16$, $w \in Q$. Thus at each step of the process, a mapping U_n is obtained which rectifies an arc in Q which divides a rectangle in Q approximately in half and such that at the n-th step in the process U_n is such that $|x_n(w) - x_{n-1}(w)| < 1/2^n$, $w \in Q$. If at any step in the process some of the rectangles have the ratios of their longer to their shorter sides exceeding two, the next step will be taken to divide them approximately in half and to rectify them in such a way that the ratio of the longer to the shorter sides of the resulting rectangles does not exceed two. The partitions P_k will be the subdivisions of Q obtained each time this is realized for all the rectangles of the subdivision of Q. The mapping T_0 will be defined as the uniform limit of the sequence $\{U_n\}$ of mappings as above defined. By the method of their construction all the mappings U_n are LEBESGUE equivalent and converge to a continuous mapping T_0: $Q \to E_3$ where $T_0 = U_n$ over all the $R_{ik} - Q^*$ for any fixed k and all n sufficiently large. Thus for all k and i for which R_{ik} is defined, $T_0(R_{ik}^* - Q^*)$ is a rectifiable continuous curve and the diameters of the R_{ik} approach zero as k becomes infinite.

Corollary. If T is a light mapping, the mapping T_0 of the theorem is LEBESGUE equivalent to T.

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