JOHN M. DANSKIN, Jr. (*)

On the existence of minimizing surfaces in parametric double integral problems of the Calculus of Variations. (**)

1. - Introduction.

The object of this paper is the demonstration of the existence of a surface minimizing the integral

$$\mathcal{J}(S) = \iint f(\boldsymbol{x}, \boldsymbol{X})\, \mathrm{d}u\, \mathrm{d}v$$

among the class of all parametric FRÉCHET surfaces bounded by a fixed JORDAN curve in space, under general conditions on the function f. This is a generalization of a theorem obtained in 1936 by E. J. MCSHANE [6], in which the function f depended only on the Jacobian vector $\boldsymbol{X}$ and not on the coordinate vector $\boldsymbol{x}$. L. CESARI has recently obtained (oral communication, August 1950; see also Abstract No. 26, Evanston meeting, Bulletin A.M.S., January 1951) a very similar theorem, and so has A. G. SIGALOV ([10], 1950). Professor CESARI knew of but has not seen my proof, and I have not yet seen his. The existence of SIGALOV's proof was brought to my attention in August 1950 by Professor MCSHANE. All three proofs are thus completely independent, and employ different methods.

The existence theorem is stated and proved in Section 3. Section 2 is devoted to notation and two preliminary lemmas.

(*) Address: The RAND Corporation, 1500 Fourth Street, Santa Monica, California, U.S.A..

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The results of J. W. CALKIN in [1] and C. B. MORREY in [7] and [9], as well as those of Professor CESARI in his many papers, are used freely.

I take this opportunity to thank my teacher, Professor CHARLES B. MORREY, Jr., for his very active guidance and stimulation during the carrying out of this work.

2. - Preliminaries.

2.1. - Notation, definitions, and general remarks.

Boldface letters indicate vectors; $\boldsymbol{x}$ in general will refer to the vector (x^1, x^2, x^3). If $\boldsymbol{x}(u, v)$ is a representation of a surface S the vector $\boldsymbol{X}(u, v)$ will denote the generalized Jacobian in the sense of CESARI at the point (u, v). If S is of finite area the quantity $\boldsymbol{X}(u, v)$ exists almost everywhere and is integrable L_1. Further, it is equal almost everywhere ([3], Theorem VII et seq.) in value and in sign to the classical Jacobian $J(u, v) = \boldsymbol{x}_u \times \boldsymbol{x}_v$ whenever $\boldsymbol{x}$ has partial derivatives almost everywhere. Since no other kind of Jacobian appears in this paper, we refer to $\boldsymbol{X}$ as the Jacobian without further specification.

We frequently write $f(\boldsymbol{x}, \boldsymbol{X})$ for $f(x^1, x^2, x^3, X^1, X^2, X^3)$.

FRÉCHET surfaces are understood as, for example, in [12].

The integral $\mathcal{J}(S)$ need not always exist in the LEBESGUE sense. But under very general conditions on f — weaker than those we assume here — it does always exist as a WEIERSTRASS integral in the sense of CESARI [2]. Whenever the area of S is given by the classical (LEBESGUE) integral, then this WEIERSTRASS integral equals the LEBESGUE integral. $\mathcal{J}(S)$ as defined by CESARI does not depend on the particular representation employed for S. By $\mathcal{J}(S)$ we shall always understand this WEIERSTRASS integral. As each surface of the final minimizing sequence employed in Section 3, as well as the limit surface, is A.C.T., then $\mathcal{J}(S)$ there coincides with the LEBESGUE integral.

See for the definition of absolute continuity in the sense of TONELLI (A.C.T.) [1] or [7] or [11].

A representation $\boldsymbol{x}$ is said to be generalized conformal provided

1) $\boldsymbol{x}$ is A.C.T.;

2) $\boldsymbol{x}_u \cdot \boldsymbol{x}_v = 0$ and $\boldsymbol{x}_u^2 = \boldsymbol{x}_v^2$ almost everywhere;

3) $\iint (\boldsymbol{x}_u^2 + \boldsymbol{x}_v^2)\, du\, dv < \infty$.

A surface is said to be nondegenerate provided it possesses a nondegenerate representation. A representation $\boldsymbol{x}$ is nondegenerate provided that there are no continua, containing more than one point, over which $\boldsymbol{x}$ is constant.

The DIRICHLET integral $D_2(\boldsymbol{x}, R)$ of $\boldsymbol{x}$ over the region R is defined by the equation

$$D_2(\boldsymbol{x}, R) = \frac{1}{2}\iint_R (\boldsymbol{x}_u^2 + \boldsymbol{x}_v^2)\, du\, dv\,.$$

$H(\boldsymbol{x}, R)$ denotes the harmonic (vector) function on R coinciding with $\boldsymbol{x}$ on R^*, the boundary of R.

If E is a set $m(E)$ will denote its LEBESGUE measure. $|\;|$ will denote length of vectors, and $\|\;\|$ FRÉCHET distance.

For the definition of a region of class $\mathfrak{K}$, see [7], p. 195. For the definition of a function of class $\mathfrak{P}_2$, see [1].

2.2. – Two lemmas.

Lemma I. *Let $\boldsymbol{x}(u, v)$ be of class $\mathfrak{P}_2$ on a region G of class $\mathfrak{K}$. Then, given $\varepsilon > 0$, there exists a region S contained with its closure in the interior of G, such that*

$$|D_2[H(\boldsymbol{x}, G), G] - D_2[H(\boldsymbol{x}, R), R]| < \varepsilon$$

for all regions R of class $\mathfrak{K}$ satisfying $S \subseteq R \subseteq G$.

Proof: For any R of class $\mathfrak{K} \subseteq G$, the DIRICHLET principle implies that

$$D_2[H(\boldsymbol{x}, R), R] + D_2(\boldsymbol{x}, G - R) \geqq D_2[H(\boldsymbol{x}, G), G]\,.$$

Hence the inequality

$$D_2[H(\boldsymbol{x}, R), R] > D_2[H(\boldsymbol{x}, G), G] - \varepsilon$$

will follow on taking S appropriately.

To obtain the inequality in the opposite direction, put

$$\boldsymbol{w} = \boldsymbol{x} - H(\boldsymbol{x}, G)$$

at each point (u, v) of G. According to a theorem of MORREY ([7], Theorem 7.1 (iv)) we can choose S satisfying the conditions already imposed on it and find a $\boldsymbol{w}'$ vanishing outside S with $D_2(\boldsymbol{w} - \boldsymbol{w}', G)$ arbitrarily small. Then for any R of class $\mathfrak{K}$ with $S \subseteq R \subseteq G$,

$$D_2[H(\boldsymbol{w}, R), R] = D_2[H(\boldsymbol{w} - \boldsymbol{w}', R), R] \leqq D_2(\boldsymbol{w} - \boldsymbol{w}', R)\,,$$

so that $D_2[H(\boldsymbol{w}, R), R]$ is arbitrarily small. Accordingly, as strong convergence in L_2 implies convergence in norm (the norm here being the square root of the DIRICHLET integral), the quantity $D_2[H(\boldsymbol{w}, R) + H(\boldsymbol{x}, G), R]$ is arbitrarily close to $D_2[H(\boldsymbol{x}, G), R]$ and therefore to $D_2[H(\boldsymbol{x}, G), G]$. But again according to the DIRICHLET principle, as $H(\boldsymbol{x}, R)$ and $\{H(\boldsymbol{w}, R) + H(\boldsymbol{x}, G)\}$ take on the same values on R^*, then

$$D_2[H(\boldsymbol{x}, R), R] \leqq D_2[H(\boldsymbol{w}, R) + H(\boldsymbol{x}, G), R] .$$

Taking account of the above remarks, the Lemma is proved.

Lemma II. *Let R be a region which is the sum of a finite number of convex polygonal regions $R_1, \ldots, R_q$. Let $\{\boldsymbol{x}_n(u, v)\}$ be a sequence of functions satisfying the following conditions:*

1) *there is a function $\boldsymbol{x}_0(u, v)$ on R such that along each boundary R_i^* the sequence $\{\boldsymbol{x}_n\}$ converges uniformly to $\boldsymbol{x}_0(u, v)$;*

2) *each $\boldsymbol{x}_n$ is absolutely continuous one-dimensionally along each R_i^*;*

3) *the generalized derivatives of each $\boldsymbol{x}_n$ in the sense of Evans (1) exist almost everywhere on the R_i^* and coincide almost everywhere there with the ordinary partial derivatives;*

4) $\sum_{i=1}^{p} \int_{R_i^*} \boldsymbol{x}_{n_s}^2 \, ds < A$ *uniformly in n.*

Then

$$\lim_{n\to\infty} D_2[H(\boldsymbol{x}_n, R), R] = D_2[H(\boldsymbol{x}_0, R), R] .$$

Proof: We break the proof into sections.

A. *The lemma is true if R is replaced by a circle and if $\boldsymbol{x}_0(u, v)$ is identically zero on C^*.*

Let us write, regarding $\boldsymbol{x}_n$ as a function of θ on C^*,

$$\boldsymbol{x}_n(\theta) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} (\boldsymbol{a}_{nm} \cos m\theta + \boldsymbol{b}_{nm} \sin m\theta) .$$

Here $\sim$ indicates that convergence of the series on the right to the function

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(1) For the definition of generalized derivatives in the sense of EVANS, see [1], p. 175. This condition is required so that in the course of proof we may employ a change of variable theorem due to EVANS (see [7], pp. 189-190).

on the left is in L_2. According to known theorems (cf. for example [9], Lemma 6.2, p. 40), the DIRICHLET integral of the harmonic function taking on the boundary values of x_n is given by

$$D_2[H(\boldsymbol{x}_n, C), C] = \pi \sum_{m=1}^{\infty} m(\boldsymbol{a}_{nm}^2 + \boldsymbol{b}_{nm}^2)\,.$$

On differentiating the above FOURIER series term by term –it can easily be shown from the facts that $\boldsymbol{x}_n(\theta)$ is A.C. in θ and $\boldsymbol{x}_{n\theta}$ is integrable L_2 that this gives the FOURIER series for the derivative– we get

$$\boldsymbol{x}_{n\theta}(\theta) \sim \sum_{m=1}^{\infty} (-m\boldsymbol{a}_{nm} \sin m\theta + m\boldsymbol{b}_{nm} \cos m\theta)\,,$$

so that

$$\int_R \boldsymbol{x}_{n\theta}^2 \, \mathrm{d}\theta = \sum_{m=1}^{\infty} m^2(\boldsymbol{a}_{nm}^2 + \boldsymbol{b}_{nm}^2)\,.$$

As the quantity on the left is uniformly bounded by hypothesis, and since

$$\sum_{m=N}^{\infty} m(\boldsymbol{a}_{nm}^2 + \boldsymbol{b}_{nm}^2) \leqq \frac{1}{N} \sum_{m=N}^{\infty} m^2(\boldsymbol{a}_{nm}^2 + \boldsymbol{b}_{nm}^2)\,,$$

then $\sum_{m=N}^{\infty} m(\boldsymbol{a}_{nm}^2 + \boldsymbol{b}_{nm}^2)$ can be made arbitrarily small independently of n merely by taking N large enough. Since for each separate m we have

$$\lim_{n\to\infty} \boldsymbol{a}_{nm} = 0\,; \qquad \lim_{n\to\infty} \boldsymbol{b}_{nm} = 0$$

then it is clear that

$$\lim_{n\to\infty} \pi \sum_{m=1}^{\infty} m(\boldsymbol{a}_{nm}^2 + \boldsymbol{b}_{nm}^2) = 0$$

as required.

B. *The lemma is true if R is a single convex polygon and $\boldsymbol{x}_0(u, v) \equiv 0$ on R^*.*

We choose a point in the interior of R and from that point triangulate R. Let C be the unit circle in the (u', v') plane. Each triangle can be mapped in a 1-1 manner onto a sector of C in such a way that

1) the mappings, when pieced together, form a 1-1 continuous transformation T of R onto C;

2) this transformation and its inverse is uniformly Lipschitzian throughout, and almost everywhere analytic.

Accordingly, using EVANS' change of variable theorem (see footnote (1), p. 46), the condition that

$$\int_{R^*} x_{ns}^2 \, ds ,$$

be uniformly bounded in n transforms into a condition that

$$\int_{C^*} y_{n\theta}^2 \, d\theta ,$$

be uniformly bounded in n, y_n being the transformed function

$$y_n(P') = x_n(T^{-1}P') , \qquad P' \in C , \qquad (n = 1, 2, \ldots).$$

Also, y_n is A.C. along the boundary C^*; and $y_n(u', v')$ converges uniformly to zero there; hence by Remark A we have

$$\lim_{n\to\infty} D_2[H(y_n, C), C] = 0 .$$

Accordingly, taking account of the uniform Lipschitz conditions,

$$\lim_{n\to\infty} D_2[w_n, R), R] = 0 ,$$

where

$$w_n(P) = H(y_n, C)(TP) , \qquad P \in R , \qquad (n = 1, 2, \ldots).$$

By the DIRICHLET principle,

$$D_2[H(x_n, R), R] \leqq D_2(w_n, R) , \qquad (n = 1, 2, \ldots)$$

the values of w_n and $H(x_n, R)$ coinciding on R^*. Accordingly

$$\lim_{n\to\infty} D_2[H(x_n, R), R] = 0$$

as required.

C. *The lemma is true if R is as in the statement of the Lemma.*

We prove this first for $x_0(u, v) \equiv 0$ on the boundary. Now for each i, $(i = 1, \ldots, q)$, we have

$$\lim_{n\to\infty} D_2[H(x_n, R_i), R_i] = 0$$

by Remark B. By the DIRICHLET principle,

$$D_2[H(x_n, R), R] \leqq \sum_{i=1}^{q} D_2[H(x_n, R_i), R_i] ,$$

so that

$$\lim_{n\to\infty} D_2[H(\boldsymbol{x}_n, R), R] = 0$$

as required.

Now consider the general case and put

$$\boldsymbol{y}_n(u, v) = \boldsymbol{x}_n(u, v) - \boldsymbol{x}_0(u, v), \qquad (u, v) \in R.$$

Then

$$\lim_{n\to\infty} D_2[H(\boldsymbol{x}_n, R) - H(\boldsymbol{x}_0, R), R] = \lim_{n\to\infty} D_2[H(\boldsymbol{y}_n, R), R] = 0,$$

so that $H(\boldsymbol{x}_n, R)$ converges strongly to $H(\boldsymbol{x}_0, R)$ in the space with norm given by the square root of the DIRICHLET integral. Hence $H(\boldsymbol{x}_n, R)$ converges in that space to $H(\boldsymbol{x}_0, R)$ in norm also, i.e.

$$\lim_{n\to\infty} D_2[H(\boldsymbol{x}_n, R), R] = D_2[H(\boldsymbol{x}_0, R), R].$$

The proof of the lemma is thus complete.

3. - The existence theorem.

Theorem: *Let the function* $f(x^1, x^2, x^3, p^1, p^2, p^3)$ *satisfy the following conditions*:

1) f *is of class* C^2 *in all six variables, if* $\boldsymbol{p} \neq \boldsymbol{0}$;
2) $f(\boldsymbol{x}, \boldsymbol{0}) = 0$ *for all* $\boldsymbol{x}$;
3) $f(\boldsymbol{x}, k\boldsymbol{p}) = |k| f(\boldsymbol{x}, \boldsymbol{p})$ *for all* $\boldsymbol{x}$, $\boldsymbol{p}$, *and* k;
4) *there exist positive constants* m *and* M, *with* $m < M$, *such that*

$$m|\boldsymbol{p}| \leqq f(\boldsymbol{x}, \boldsymbol{p}) \leqq M|\boldsymbol{p}|$$

for all $\boldsymbol{x}$ *and all* $\boldsymbol{p} \neq \boldsymbol{0}$;

5) $f_{p_i p_j}(\boldsymbol{x}, \boldsymbol{p})\xi^i\xi^j > 0$ *unless* $\boldsymbol{\xi} \times \boldsymbol{p} = \boldsymbol{0}$.

Let g be a Jordan curve in space bounding at least one surface of finite area. Then, among the class of all surfaces bounded by g, there is one for which the integral

$$\mathcal{J}(S) = \iint f(\boldsymbol{x}, \mathbf{X})\, \mathrm{d}u\, \mathrm{d}v$$

attains its minimum.

In Section 3.1 through Section 3.5 we shall find a minimizing sequence with very special properties; in Section 3.7 we shall modify some of its terms, and in Section 3.8 we shall use the modified sequence to obtain the theorem.

3.1. – **The minimizing sequence.**

Let d be the greatest lower bound of $\mathcal{J}(S)$ for surfaces S bounded by g. Obviously $d \geqq 0$; and, because of condition 4) and the fact that g is bounded by at least one surface of finite area, $d < \infty$.

There exists, then, a sequence of surfaces of finite area and bounded by g whose integrals tend to d. Fix attention on one of these surfaces, say S. It is the limit of a sequence of polyhedra converging to it in the sense of Fréchet and whose areas converge to its area. Neither type of convergence will be disturbed by moving the vertices of each polyhedron sufficiently to insure that it is nondegenerate and that its boundary is simple. According to a theorem of Cesari ([2], Theorem III), the integrals taken over the polyhedra also converge, to the integral taken over S.

It follows that the original minimizing sequence may be replaced by a minimizing sequence of nondegenerate polyhedra, with simple boundaries which converge to g in the sense of Fréchet. We denote this sequence by $\{S_n\}$.

Choose on g three distinct points p, q, and r. On taking account of the Fréchet convergence of the boundaries S_n^* to g, choose distinct points p_n, q_n, and r_n on each S_n^* in such a way that $p_n \to p$, $q_n \to q$, and $r_n \to r$. Fix three points a, b, and c on the unit circle, 120° apart. According to a classical theorem, each S_n possesses a generalized conformal representation on the unit circle C, which sends a, b, and c into p_n, q_n, and r_n respectively. We denote this representation by $\boldsymbol{x}_n(u, v)$.

Observe that the Dirichlet integrals

$$D_2(\boldsymbol{x}_n,\ C) = \frac{1}{2}\iint\limits_C (\boldsymbol{x}_{nu}^2 + \boldsymbol{x}_{nv}^2)\, du\, dv$$

of the representations $\boldsymbol{x}_n$ are uniformly bounded in n. For since $\boldsymbol{x}_n$ is generalized conformal on C,

$$|\boldsymbol{X}_n| = \frac{\boldsymbol{x}_{nu}^2 + \boldsymbol{x}_{nu}^2}{2},$$

almost everywhere. Hence

$$\iint f(\boldsymbol{x}_n, \boldsymbol{X}_n)\, du\, dv \geqq m \iint |\boldsymbol{X}_n|\, du\, dv = m D_2(\boldsymbol{x}_n,\ C),$$

and the assertion follows from the fact that $\{\boldsymbol{x}_n\}$ is a minimizing sequence.

The proof will proceed as follows. First we shall find a countable everywhere dense (along the interval $(-1, 1)$) set of lines parallel to the u axis and a countable everywhere dense set of lines parallel to the v axis, whose intersections with the unit circle form a net $\mathfrak{N}$ possessing certain special properties with respect to the sequence $\{\boldsymbol{x}_n(u, v)\}$. We then prove that the sequence $\{\boldsymbol{x}_n(u, v)\}$ is equicontinuous on the boundary of the unit circle, and from this and the uniform boundedness of the DIRICHLET integrals that a subsequence converges weakly in a certain HILBERT space to a function $\boldsymbol{x}_0(u, v)$, which does not a priori have to be continuous. With the aid of the net $\mathfrak{N}$, however, we shall show that x_0 is LEBESGUE-equivalent to a continuous function and may therefore be taken to be continuous. Again employing $\mathfrak{N}$, we find a further subsequence which can be so modified as to converge uniformly to $\boldsymbol{x}_0$, while the integrals still converge to d. At this stage a known semicontinuity theorem can be applied and the theorem proved.

3.2. – The net $\mathfrak{N}$.

Let K be an upper bound for the DIRICHLET integrals $D_2(\boldsymbol{x}_n, C)$. Put

$$E_{np} = \mathop{\mathsf{E}}_{-1 \leq u \leq 1} \left[\int_{[u]} (\boldsymbol{x}_{nu}^2 + \boldsymbol{x}_{nv}^2) \, \mathrm{d}v \leq 2pK \right],$$

where p is a positive integer and the integral is taken over the intersection $[u]$ of the line of abcissa u with the unit circle. Then it is easily seen that

$$m(E_{np}) \geq 2 - \frac{1}{p},$$

so that

$$m(\limsup_n E_{np}) \geq \limsup_n m(E_{np}) \geq 2 - \frac{1}{p}.$$

Put

$$F_p = \limsup_n E_{np}$$

and

$$F = \sum F_p .$$

Then the F_p are increasing and

$$m(F) = 2 .$$

Thus if u is not in a certain set of measure zero, there exists an infinite subsequence of $\{x_n\}$ for which

$$\int\limits_{[u]} (x_{nu}^2 + x_{nv}^2)\, dv ,$$

is. uniformly bounded. The same holds if we replace $\{x_n\}$ by any one of its subsequences; that is, given the subsequence then there is a set of measure zero such that if u is not in that set the integrals over the segment $[u]$ of abcissa u are uniformly bounded for some further infinite subsequence.

The same remarks hold for integrals taken over segments $[v]$; we are now ready to define the net $\mathfrak{N}$.

Observe that there is a set Z_u' of measure zero on $(-1, 1)$ such that if $u \notin Z_u'$, then $x_n(u, v)$ is absolutely continuous in v on the open segment $[u]$ for all n, a set Z_u'' of measure zero such that if $u \notin Z_u''$ then the ordinary partial derivatives x_{nu} and x_{nv} are equal almost everywhere on $[u]$ to the generalized derivatives in the sense of EVANS (this property is needed in the application of Lemma II) and a set Z_u''' of measure zero on $(-1, 1)$ such that if $u \notin Z_n'''$ then the integral

$$\int\limits_{[u]} (x_{nu}^2 + x_{nv}^2)\, dv ,$$

exists and is finite for each n. Put now $Z_u = Z_u' + Z_u'' + Z_u'''$. Define correspondingly a set Z_v.

Choose u_1, not in the set of measure zero corresponding to the original sequence, and not in Z_u. We obtain a subsequence for which the integrals over $[u_1]$ are uniformly bounded. Choose u_2 not in the set of measure zero corresponding to that subsequence and also not in Z_u; we obtain a further subsequence on which the integrals over $[u_2]$, as well as $[u_1]$, are uniformly bounded. Continue this process, choosing the numbers $u_1, u_2, \ldots$ in such a way that their totality is everywhere dense on $(-1, 1)$. Employing the diagonal process of CANTOR we obtain a single subsequence whose integrals are uniformly bounded over each segment $[u_1], [u_2], \ldots$. We subject this subsequence to the same process with respect to v. Thus we obtain finally an everywhere dense collection of segments

$$[u_1],\ [u_2],\ \ldots$$

parallel to the v axis and an everywhere dense collection of segments

$$[v_1],\ [v_2],\ \ldots$$

parallel to the u axis, and an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ for which the integral of $(x_{n_k u}^2 + x_{n_k v}^2)$ is uniformly bounded in n_k over each segment of the collection. Furthermore, each x_{n_k} is absolutely continuous over each segment of the set. This collection of segments is called the net $\mathfrak{N}$, and is fixed throughout the sequel.

3.3. – Equicontinuity on the boundary.

In this section we prove that the minimizing sequence $\{x_n\}$ is equicontinuous on the boundary C^* of the unit circle. We first observe that the boundaries S_n^* are equi-uniformly locally connected in the following sense:

For any $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that for every* n, *if* A *and* B *are two points on* S_n^* *distant less than* δ *then they lie in an arc of* S_n^* *of diameter* $< \varepsilon$.

To prove this, take $\varepsilon > 0$. Since g is uniformly locally connected there is a σ satisfying $0 < \sigma < \varepsilon$ such that any two points on g distant less than σ lie on an arc of g of diameter less than $\varepsilon/3$. Since the boundaries S_n^* converge in the sense of FRÉCHET to g, there is an n_0 such that if $n > n_0$ then $\|S_n^*, g\| < \sigma/3$. Corresponding to each of the curves S_n^* with $1 \leqq n \leqq n_0$ there is a $\delta_n > 0$ such that any two points of S_n^* distant less than δ_n lie on an arc of S_n^* of diameter less than ε. Take $0 < \delta < \min(\sigma/3, \delta_1, ..., \delta_{n_0})$. Let n be any positive integer and A and B two points on S_n^* distant less than δ. If $n \leqq n_0$ the result is immediate. Suppose $n > n_0$. Then there is a homeomorphism of the circumference of the unit circle into itself under which the image on S_n^* and g, respectively, of corresponding points, are distant less than $\sigma/3$. If A' and B' on g correspond to A and B respectively, then A' and B' must be distant less than σ and so lie in an arc α' of g of diameter less than $\varepsilon/3$. α' corresponds under the homeomorphism to an arc α of S_n^*. A and B lie in α, and the diameter of α is clearly less than $\varepsilon/3 + 2 \cdot \sigma/3 < \varepsilon$. This proves the result.

The following equicontinuity proof is modelled on one used by COURANT (see [5], pp. 542-544).

Let $\varepsilon > 0$. Clearly we may suppose ε less than the minimum of the distances between the points p, q, and r. If n is sufficiently large, then, ε is less than the minimum of the distances between the points p_n, q_n, and r_n. We fix an n_0 so that this is true for all $n \geqq n_0$. As each mapping is continuous on the boundary of the unit circle, it will suffice to prove the equicontinuity of the part of the sequence with $n \geqq n_0$. By the equi-uniform local connectedness property, there exists a $\delta > 0$ such that for any n any two points on S_n^* distant less than δ lie in an arc of S_n^* of diameter less than ε. Fix such a δ and put

$$\eta = e^{-4\pi k/\delta^2}.$$

We may assume δ taken small enough so that $\eta < 1/2$. Let O and Q be any two points on the circumference C^*, distant less than η. We shall prove that the images under $\boldsymbol{x}_n$ of O and Q, for any $n \geqq n_0$, are distant less than ε. Let P be the midpoint of the shorter arc joining O to Q. Describe with P as center two circles with radii η and $\sqrt{\eta}$ respectively. The region in which the annulus between them intersects the unit circle C will be called R. Fix any $n \geqq n_0$. Then

$$\iint_R (\boldsymbol{x}_{nu}^2 + \boldsymbol{x}_{nv}^2)\, du\, dv \leqq 2K .$$

n will remain fixed for the remainder of this argument; we drop it temporarily for convenience in notation. We now transform the integral by introducing polar coordinates (ϱ, θ) with origin at P. Within R this transformation and its inverse are analytic. R is obviously of class $\mathfrak{K}$ in the sense of MORREY ([7], p. 195). Accordingly, in virtue of the EVANS change of variable theorem (see footnote (1), p. 46), it is clear that the function $\boldsymbol{x}(u, v)$ is transformed into a function of ϱ and θ which is absolutely continuous in the sense of TONELLI, whose partial derivatives with respect to ϱ and θ are given almost everywhere by the formulas of the elementary calculus, and satisfying

$$\iint_R \left(x_\varrho^2 + \frac{1}{\varrho^2}\, x_\theta^2\right) \varrho\, d\varrho\, d\theta \leqq 2K .$$

This implies

$$\int_\eta^{\sqrt{\eta}} d\varrho\, \frac{1}{\varrho} \int_\varrho x_\theta^2\, d\theta \leqq 2K .$$

From this we see that there is a set of ϱ's of positive measure such that

$$\int_\varrho x_\theta^2\, d\theta \leqq \frac{4K}{\log (1/\eta)} .$$

For otherwise we would have

$$\int_\varrho x_\theta^2\, d\theta > \frac{4K}{\log (1/\eta)}$$

almost everywhere, so that

$$\int_\eta^{\sqrt{\eta}} d\varrho \cdot \frac{1}{\varrho} \int_\varrho x_\theta^2\, d\theta > \frac{4K}{\log (1/\eta)} \cdot \int_\eta^{\sqrt{\eta}} \frac{d\varrho}{\varrho} = 2K,$$

a contradiction. Since $\boldsymbol{x}$ is A.C.T. in ϱ and θ, then $\boldsymbol{x}$ is absolutely continuous as a function of θ for almost all ϱ, and accordingly A.C. in θ for some ϱ lying in the set of positive measure for which the inequality just proved holds. Choose such a ϱ and call it ϱ_0. The circle with center P and radius ϱ_0 intersects C^* in two points A and B. The images under $\boldsymbol{x}$ of A and B are distant less than δ. For

$$|\boldsymbol{x}(A) - \boldsymbol{x}(B)| = \left| \int_{\varrho=\varrho_0} \boldsymbol{x}_\theta \, d\theta \right| \leqq \left[\pi \int_{\varrho=\varrho_0} \boldsymbol{x}_\theta^2 \, d\theta\right]^{1/2} \leqq \left[\frac{4K\pi}{\log(1/\eta)}\right]^{1/2} = \delta .$$

Hence the images of A and B lie in an arc of S_n^* (we resume now the use of the subscript n) of diameter less than ε. This must be the arc containing the image of P, as since $\eta < 1/2$ the other arc joining the images must contain at least two of the three points p_n, q_n, r_n, and accordingly have diameter greater than ε. Hence the image of the arc $\widehat{APB}$ has diameter less than ε. But $\widehat{APB}$ contains the points O and Q, so that

$$|\boldsymbol{x}_n(O) - \boldsymbol{x}_n(Q)| < \varepsilon .$$

Accordingly the minimizing sequence is equicontinuous on the boundary.

3.4. – Weak compactness of the minimizing sequence.

On taking account of the uniform boundedness of the DIRICHLET integral and the equicontinuity on the boundary, it is easy to show that the integral

$$\iint \boldsymbol{x}_n^2(u, v) \, du \, dv$$

is bounded independent of n. Hence obviously so is the quantity

$$\overline{D}_2(\boldsymbol{x}_n, C) = D_2(\boldsymbol{x}_n, C) + \iint \boldsymbol{x}_n^2(u, v) \, du \, dv .$$

This is the square of the norm in the HILBERT space called $\mathfrak{P}_2$ by CALKIN and MORREY ([**1**] and [**7**]).

What is important here is that functions $\boldsymbol{x}$ A.C.T. with $\overline{D}_2(\boldsymbol{x}, C)$ bounded are functions of class $\mathfrak{P}_2$. Thus the functions of the minimizing sequence are of class $\mathfrak{P}_2$, and furthermore have $\overline{D}_2(\boldsymbol{x}_n, C)$ uniformly bounded. It follows from a theorem of MORREY ([7], Theorem 8.4) that a subsequence converges weakly in $\mathfrak{P}_2$ to some function $\boldsymbol{x}_0$ of class $\mathfrak{P}_2$. This limiting function need not be continuous. The next section will be devoted to proving that $\boldsymbol{x}_0$

is however equal almost everywhere to a function which is continuous on the closed circle C. First, however, we shall use some of the results so far obtained to extract a particular subsequence.

At the end of Section 3.2 we found a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ for which, over each segment of the net $\mathfrak{N}$, x_{n_k} is absolutely continuous and the integral of $(x^2_{n_k u} + x^2_{n_k v})$ uniformly bounded. Given a segment of the net, it is easy to see from this, using the SCHWARTZ inequality, that the sequence $\{x_{n_k}\}$ is equicontinuous on that segment. Accordingly a subsequence of $\{x_{n_k}\}$ converges uniformly there. Applying the diagonal process to the countable collection of segments in $\mathfrak{N}$, we obtain a subsequence of $\{x_{n_k}\}$ which converges uniformly on each segment of the net. Because of the equicontinuity on the boundary there is a further subsequence converging uniformly also on the boundary. Finally, a subsequence of this converges weakly in $\mathfrak{P}_2$ to the function x_0. We change notation and denote this sequence by $\{x_n\}$.

In summary, the sequence $\{x_n\}$ possesses the following properties:

a) $\{x_n\}$ converges weakly in $\mathfrak{P}_2$ to a function x_0 of class $\mathfrak{P}_2$;

b) $\{x_n\}$ converges uniformly on each segment of the net $\mathfrak{N}$;

c) on the boundary of the unit circle, $\{x_n\}$ converges uniformly to a representation of the JORDAN curve g;

d) the representations x_n are generalized conformal;

e) the DIRICHLET integrals $D_2(x_n, C)$ are uniformly bounded;

f) on each segment σ of the net $\mathfrak{N}$ each x_n is absolutely continuous and the integrals $\int (x^2_{nu} + x^2_{nv})\, ds$ uniformly bounded;

g) $\{x_n\}$ is a minimizing sequence.

3.5. – Equivalence of the limiting function to a continuous function.

The object of this section is to demonstrate the existence of a function $x_0'(u, v)$ which

a) is continuous on the closed unit circle C,

b) coincides on the boundary C^* of the unit circle with the limit taken on there by the there uniformly convergent sequence $\{x_n\}$,

c) coincides on each segment of the net $\mathfrak{N}$ with the limit taken on there by the there uniformly convergent sequence $\{x_n\}$, and

d) coincides almost everywhere on C with the function $x_0(u, v)$ (of the last section).

We shall prove in Section 3.6 and Section 3.7 that x'_0 is a represensation of the surface S_0 which minimizes $\mathcal{J}(S)$.

We turn to our demonstration. A theorem of C. B. Morrey ([**9**], p. 42, Theorem 6.2) states that

If a vector function x *of class* $\mathfrak{P}_2$, *whose boundary values are continuous along* C^*, *satisfies, for some* $A \geqq 1$,

$$\text{(D)} \qquad D_2(x, R) \leqq A \cdot D_2[H(x, R), R]$$

for every subregion R *of* C *which is of class* $\mathfrak{K}$, *then there exists a function* x' *which is continuous on the closed circle* C, *takes on the boundary values of* x, *and is equal to* x *almost everywhere on* C.

Let us modify the function x_0 by replacing its values on C^* by the limiting values taken on there by the there uniformly convergent sequence $\{x_n\}$. This will not affect the results of the last section. Then, according to the result quoted above, the existence of a function x'_0 satisfying a), b), and d) above will be established as soon as we have proved the condition (D) for some A. Property c) will then also follow, as follows: since the sequence $\{x_0\}$ converges weakly in $\mathfrak{P}_2$ to x'_0, it converges strongly in L_2 to x'_0 ([7], Lemma 8.3). Employing procedures similar to those used in constructing the net $\mathfrak{N}$, we find a (countable) everywhere dense set of segments parallel to the v axis and a single subsequence of $\{x_n\}$ which, along each segment of the set, converges strongly in L_2 to x'_0 and also uniformly to a function continuous along the segment. It follows that the uniform limit must coincide everywhere along each segment with x'_0. Now let $[v]$ be a segment of the net $\mathfrak{N}$ parallel to the u axis. It follows that at an everywhere dense set of points on $[v]$ the limit of $\{x_n\}$ regarded as a uniformly convergent sequence on $[v]$ coincides with x'_0. Both x'_0 and the uniform limit of $\{x_n\}$ being continuous, they coincide everywhere on $[v]$ and so c) is proved.

Thus what is required is merely a proof that x_0 satisfies a condition (D) We shall in fact prove that x_0 satisfies the condition

$$\text{(D}'\text{)} \qquad D_2(x_0, R) \leqq \frac{M}{m} \cdot D_2[H(x_0, R), R]$$

for every subregion R of C which is of class $\mathfrak{K}$.

In view of Lemma I of this paper it is sufficient to prove this for regions R bounded by a finite number of pieces of segments from the net $\mathfrak{N}$. Suppose that R is such a region. Now since $\{x_n\}$ converges weakly in $\mathfrak{P}_2$ the sequences $\{x_{nu}\}$ and $\{x_{nv}\}$ converge weakly in L_2. Hence as is well known

$$D_2(x_0, R) \leq \liminf_n D_2(x_n, R)\,.$$

Also, as the conditions of Lemma II are evidently satisfied, we have

$$D_2[H(\boldsymbol{x}_0, R), R] = \lim_{n\to\infty} [D_2 H(\boldsymbol{x}_n, R), R].$$

Hence it is sufficient to prove that

$$\liminf_n D_2(\boldsymbol{x}_n, R) \leqq \frac{M}{m} \lim_{n\to\infty} [D_2 H(\boldsymbol{x}_n, R), R].$$

Suppose this is not the case. Then there is an $\varepsilon_0 > 0$ and an n_0 such that

$$m D_2(\boldsymbol{x}_n, R) > M D_2[H(\boldsymbol{x}_n, R), R] + \varepsilon_0$$

for all $n > n_0$. Let now $\boldsymbol{w}_n$ be the vector function gotten by replacing $\boldsymbol{x}_n$ on R by $H(\boldsymbol{x}_n, R)$. Let S_n' be the surface represented by $\boldsymbol{w}_n$, $(n = 1, 2, \ldots)$. Then for $n > n_0$ we have

$$\mathcal{J}(S_n') = \iint f(\boldsymbol{w}_n, \boldsymbol{W}_n)\, du\, dv =$$

$$= \iint_R f(\boldsymbol{w}_n, \boldsymbol{W}_n)\, du\, dv + \iint_{C-R} f(\boldsymbol{x}_n, \boldsymbol{X}_n)\, du\, dv \leqq M \iint_R |\boldsymbol{W}_n|\, du\, dv + \iint_{C-R} f(\boldsymbol{x}_n, \boldsymbol{X}_n)\, du\, dv \leqq$$

$$\leqq M D_2(\boldsymbol{w}_n, R) + \iint_{C-R} f(\boldsymbol{x}_n, \boldsymbol{X}_n)\, du\, dv < m D_2(\boldsymbol{x}_n, R) + \iint_{C-R} f(\boldsymbol{x}_n, \boldsymbol{X}_n)\, du\, dv - \varepsilon_0 \leqq$$

$$\leqq \iint_R f(\boldsymbol{x}_n, \boldsymbol{X}_n)\, du\, dv + \iint_{C-R} f(\boldsymbol{x}_n, \boldsymbol{X}_n)\, du\, dv - \varepsilon_0 = \mathcal{J}(S_n) - \varepsilon_0,$$

so that

$$\liminf_n \mathcal{J}(S_n') < \lim_n \mathcal{J}(S_n).$$

This is impossible as $\{S_n\}$ is a minimizing sequence. This completes the proof of the results of this section. Thus the results summarized in Section 3.4 can be written in the sharper form — and from now on we drop the prime from $\boldsymbol{x}_0'$ —:

a) $\{\boldsymbol{x}_n\}$ converges weakly in $\mathfrak{P}_2$ to a continuous function $\boldsymbol{x}_0$ of class $\mathfrak{P}_2$ (and therefore A.C.T.);

b) $\{\boldsymbol{x}_n\}$ converges uniformly on each segment of the net $\mathfrak{N}$ to $\boldsymbol{x}_0$;

c) on the boundary of the unit circle, $\{\boldsymbol{x}_n\}$ converges uniformly to $\boldsymbol{x}_0$, and x_0 restricted to the boundary is a representation of the JORDAN curve g;

d)
e)
f)
g) } as before in Section 3.4.

3.6. – A special pair of function.

With a view to their application in Section 3.7, Section 3.6 is concerned with a pair of convex functions $f(X)$ and $\Phi(y)$.

We fix an x_0 and put $f(X) = f(x_0, X)$. Then $f(X)$ is convex, even (i.e. $f(-X) = f(X)$), positively homogeneous, of class C^2, and positively regular. Put

$$\Phi(y) = \max_{f(X)=1} X \cdot y .$$

Then Φ enjoys the following eight properties:

1) Φ is convex;
2) Φ is even;
3) Φ is positively homogeneous;
4) Φ is of class C' if $y \neq 0$;
5) $\Phi[\nabla f(X)] = 1$;
6) $f(X) = \max\limits_{\Phi(y)=1} X \cdot y$;
7) $f[\nabla \Phi(y)] = 1$;
8) On $\Phi(y) = 1, \quad m \leqq |y| \leqq M$.

These properties are well known or easily derived; their proof will be omitted here. We point out that it is in the proof of 4) that positive regularity plays a rôle.

3.7. – The modification theorem.

We state an extremely important theorem concerning the minimizing sequence $\{x_n\}$.

Modification Theorem: *Let* $\varepsilon > 0$ *and* N *a positive integer. There* *exists an* $n > N$ *and an* A.C.T. *function* $\boldsymbol{y}_n$ *such that*

1) $|\boldsymbol{y}_n(u, v) - \boldsymbol{x}_0(u, v)| < \varepsilon$ *throughout, and*

2) $\iint f(\boldsymbol{y}_n, \boldsymbol{Y}_n)\, du\, dv \leqq \iint f(\boldsymbol{x}_n, \boldsymbol{X}_n)\, du\, dv + \varepsilon$.

Proof: There exists an $\eta < \varepsilon/2$ such that if $|x_1 - x_2| < \eta$ then

$$\frac{1}{1+\delta} f(\boldsymbol{x}_1, \boldsymbol{X}) \leqq f(_2, \boldsymbol{x}\, \boldsymbol{X}) \leqq (1+\delta) f(\boldsymbol{x}_1, \boldsymbol{X}) ,$$

for all $\boldsymbol{X}$, where $\delta = \varepsilon/(3KM)$, K being, it will be recalled, the bound on $D_2(\boldsymbol{x}_n, C)$. There is a finite subnet of $\mathfrak{N}$ so fine that the image under $\boldsymbol{x}_0$ of any of its rectangles (under «rectangles» we include figures containing pieces of the boundary C^* of the unit circle on their boundaries) has diameter less than $(\eta/2)(m/M)^{3/2}$. Let (where R is a rectangle)

$$\boldsymbol{x}_0^0 = \frac{1}{m(R)} \iint_R \boldsymbol{x}_0(u, v)\, du\, dv .$$

Then $\boldsymbol{x}_0(R)$ lies in a sphere of radius $\frac{\eta}{2}\left(\frac{m}{M}\right)^{3/2}$ with center $\boldsymbol{x}_0^0$. There is an $n > N$ so large that $|\boldsymbol{x}_n(u, v) - \boldsymbol{x}_0(u, v)| < \frac{\eta}{2}\left(\frac{m}{M}\right)^{3/2}$ on R^* for each R; accordingly $\boldsymbol{x}_n(R^*)$ lies in a sphere of radius $\eta\left(\frac{m}{M}\right)^{3/2}$ with center $\boldsymbol{x}_0^0$.

Now we fix attention on a fixed R and define geometrically a new function $\boldsymbol{y}_n(u, v)$ on R which coincides with $\boldsymbol{x}_n(u, v)$ on R^*. We may assume without loss of generality that $\boldsymbol{x}_0^0$ is the origin. Let $\Phi(\boldsymbol{y})$ be the function of Section 3.6 corresponding to $f(\boldsymbol{X}) \equiv f(\boldsymbol{x}_0^0, \boldsymbol{X})$. The two surfaces σ: $\Phi(\boldsymbol{y}) = M^{-3/2} m^{1/2} \eta$ and (the larger) $\sum$: $\Phi(\boldsymbol{y}) = M^{-1}\eta$ are nested between the spheres $|\boldsymbol{y}| = \eta\left(\frac{m}{M}\right)^{3/2}$ and $|\boldsymbol{y}| = \eta$. The modification consists in projecting radially those points $\boldsymbol{x}_n(u, v)$ lying outside the smaller surface σ onto σ, without disturbing the remainder. This is effectuated for those points outside by the transformation

$$\boldsymbol{y}_n(u, v) = \frac{M^{-2}\sqrt{Mm}\,\eta}{\Phi[\boldsymbol{x}_n(u, v)]}\, \boldsymbol{x}_n(u, v) .$$

Now the points $\boldsymbol{y}_n(u, v)$ all lie in or on the smaller surface and so a fortiori differ throughout from $\boldsymbol{x}_0(u, v)$ by less than η; consequently conclusion 1) of the theorem is satisfied, clearly each y_n is A.C.T. In order to prove 2) we shall prove first, that at any point affected by the transformation

$$f(\boldsymbol{Y}_n) \leqq \frac{1}{\varrho^2} f(\boldsymbol{X}_n),$$

where $\varrho = \boldsymbol{x}_n/\boldsymbol{y}_n$. This is a simple matter; all that is necessary is to compute the derivatives

$$\boldsymbol{y}_{nu} = \frac{1}{\varrho}\left\{\boldsymbol{x}_{nu} - [\nabla\Phi(\boldsymbol{y}_n)\cdot\boldsymbol{x}_{nu}]\frac{\boldsymbol{y}_n}{\Phi(\boldsymbol{y}_n)}\right\}.$$

and

$$\boldsymbol{y}_{nv} = \frac{1}{\varrho}\left\{\boldsymbol{x}_{nv} - [\nabla\Phi\boldsymbol{y}_n)\cdot\boldsymbol{x}_{nv}]\frac{\boldsymbol{y}_n}{\Phi(\boldsymbol{y}_n)}\right\},$$

and put these into the formula

$$\boldsymbol{Y}_n = \boldsymbol{y}_{nu} \times \boldsymbol{y}_{nv}$$

to get

$$\boldsymbol{Y}_n = \frac{1}{\varrho^2}\frac{\boldsymbol{X}_n\cdot\boldsymbol{y}_n}{\Phi(\boldsymbol{y}_n)}\nabla\Phi(\boldsymbol{y}_n).$$

Then

$$f(\boldsymbol{Y}_n) = \frac{1}{\varrho^2}\frac{|\boldsymbol{X}_n\cdot\boldsymbol{y}_n|}{\Phi(\boldsymbol{y}_n)} f[\nabla\Phi(\boldsymbol{y}_n)] = \frac{1}{\varrho^2}\frac{|\boldsymbol{X}_n\cdot\boldsymbol{y}_n|}{\Phi(\boldsymbol{y}_n)} \leqq \frac{1}{\varrho^2} f(\boldsymbol{X}_n),$$

as required.

Suppose now that $\boldsymbol{x}_n(u, v)$ lies within Σ. Then $\boldsymbol{x}_n(u, v)$ and $\boldsymbol{y}_n(u, v)$ both lie within η of $\boldsymbol{x}_0^0$ and

$$\begin{aligned} f(\boldsymbol{y}_n, \boldsymbol{Y}_n) &\leqq (1+\delta) f(\boldsymbol{x}_0^0, \boldsymbol{Y}_n) \\ &\leqq (1+\delta) f(\boldsymbol{x}_0^0, \boldsymbol{X}_n) \\ &\leqq (1+\delta)^2 f(\boldsymbol{x}_n, \boldsymbol{X}_n) \\ &\leqq (1+3\delta) f(\boldsymbol{x}_n, \boldsymbol{X}_n). \end{aligned}$$

On the other hand, if $\boldsymbol{x}_n(u, v)$ lies outside Σ, then $\varrho > \sqrt{\frac{M}{m}}$ and

$$\begin{aligned} f(\boldsymbol{y}_n, \boldsymbol{Y}_n) &\leqq (1+\delta) f(\boldsymbol{x}_0^0, \boldsymbol{Y}_n) \\ &\leqq (1+\delta)\frac{m}{M} f(\boldsymbol{x}_0^0, \boldsymbol{X}_n) \\ &\leqq (1+\delta) m|\boldsymbol{X}_n| \\ &\leqq (1+\delta) f(\boldsymbol{x}_n, \boldsymbol{X}_n) \\ &\leqq (1+3\delta) f(\boldsymbol{x}_n, \boldsymbol{X}_n). \end{aligned}$$

Since otherwise $y_n(u, v) = x_n(u, v)$, we have

$$f(y_n, Y_n) \leqq (1 + 3\delta) f(x_n, X_n)$$

at every point of R where X_n exists, and indeed at almost every point of the unit circle C. Hence

$$\iint f(y_n, Y_n)\, du\, dv \leqq \iint f(x_n, X_n)\, du\, dv + 3\delta \iint f(x_n, X_n)\, du\, dv$$
$$\leqq \iint f(x_n, X_n)\, du\, dv + 3\delta K M$$
$$\leqq \iint f(x_n, X_n)\, du\, dv + \varepsilon\, .$$

This completes the proof of the modification theorem.

3.8. – Proof of the principal theorem.

For a detailed statement of this result the reader is referred to the beginning of Section 3.

Theorem: *There is a surface minimizing $\mathcal{J}(S)$ among all Fréchet surfaces bounded by the Jordan curve g.*

Proof: There is, according to the modification theorem, an infinite subsequence $n_1, n_2, \ldots, n_p, \ldots$ of the natural numbers and an infinite sequence of continuous functions y_{n_p}, $(p = 1, 2, \ldots)$ modified from the x_{n_p}, such that

1) $$| y_{n_p}(u, v) - x_0(u, v) | < \frac{1}{p} \quad \text{for all } (u, v) \in C,$$

and

2) $$\iint f(y_{n_p}, Y_{n_p})\, du\, dv \leqq \iint f(x_{n_p}, X_{n_p})\, du\, dv + \frac{1}{p}\, .$$

Hence $\{y_{n_p}\}$ is a minimizing sequence, and $\{y_{n_p}\}$ converges uniformly to x_0. Hence according to a known lower semicontinuity theorem ([4], Theorem III),

$$\iint f(x_0, X_0)\, du\, dv \leqq \liminf_p \iint f(y_{n_p}, Y_{n_p})\, du\, dv$$
$$\leqq \liminf_p \iint f(x_{n_p}, X_{n_p})\, du\, dv$$
$$= \liminf_n \iint f(x_n, X_n)\, du\, dv\, .$$

As $\{x_n\}$ is a minimizing sequence, the surface S_0 represented by x_0 is the surface required.

This completes the proof.

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