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On generalized Lipschitzian transformations. (**)

1. – This Note is a revised and improved version of an unpublished paper in which we generalized the following result of Cesari [3]. (Numbers in square brackets refer to the bibliography at the end of this Note.) Let T be a continuous mapping from the unit square $Q: 0 \leqslant u, v \leqslant 1$ into the Euclidean xy-plane, given by functions $x = x(u, v)$, $y = y(u, v)$, $(u, v) \in Q$, for which the following conditions hold.

(i) Each of the functions $x(u, v)$, $y(u, v)$ is ACT in Q (absolutely continuous in Q in the sense of Tonelli [5, III. 2.64]).

(ii) There exists a positive constant α such that each of the first partial derivatives x_u, x_v, y_u, y_v belongs to the Lebesgue class $L^{2+\alpha}$ on Q [5, I. 3.10].

Then the mapping T is ACB in Q (absolutely continuous in Q in the Banach sense [5, IV. 5.4]) — that is, for every positive number ε there exists a positive number δ such that for every system σ of non-overlapping oriented squares s in Q the sum of whose areas is less than δ one has the relation $\sum_{s \in \sigma} | Ts | < \varepsilon$. (A square, or a rectangle, is said to be oriented if its sides are parallel to the respective coordinate axes. A system of squares, or of rectangles, is said to be nonoverlapping if no pair in the system has any common interior points. If E is a measurable set in the Euclidean plane then $|E|$ denotes its two dimensional Lebesgue measure.)

2. – We sought to generalize this result of Cesari in two directions: first, by considering the case in which T is a continuous mapping from Euclidean n-space into Euclidean N-space, and second, by weakening the requirement that the first partial derivatives of the mapping functions belong to the Lebesgue class $L^{2+\alpha}$. Now Cesari has given an example in [3] to show that in his condition (ii) described above the Lebesgue class $L^{2+\alpha}$ cannot be replaced by the Lebesgue class L^2 if his theorem is to be valid. Yet there remained

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the possibility that the LEBESGUE class $L^{2+\alpha}$ might be replaced by some appropriately defined class L^{2+0}. In the course of conversations on this subject, A. P. CALDERÓN called our attention to a class of functions more general than those occurring in the CESARI theorem for which he established in [1] a generalization of another result of CESARI in [2] concerning the existence of a complete differential. The functions considered by CALDERÓN, to be termed C-functions in the sequel, are defined in section 3. Our study of these results of CALDERÓN led to the consideration of an even more general class of functions, to be termed the class of generalized Lipschitzian functions in what follows (see section 6), which, in a certain sense, seems to play the role of a class L^{2+0} (cf. sections 12, 13). In particular, we found that these generalized Lipschitzian functions lead to adequate extensions of the result of CESARI described above (see section 20). The purpose of this Note is to describe the generalized Lipschitzian functions and some of their properties, as well as certain questions to which they give rise.

3. – A C-function of CALDERÓN is a function $f(u, v)$ which is defined, real-valued, and continuous on the unit square Q: $0 \leqslant u, v \leqslant 1$ of the uv-plane and satisfies the following conditions.

(i) The function f is ACT in Q (see section 1 (i)).

(ii) There exists a real-valued function $\varphi(t)$, $0 \leqslant t < \infty$, with the following properties: (a) the function φ is non-decreasing on $0 \leqslant t < \infty$; (b) the function $t/\varphi(t)$ has a finite (improper) integral on $0 \leqslant t < \infty$; (c) the function $\varphi(|\operatorname{grad} f|)$ is summable in Q.

4. – In [1] CALDERÓN establishes the following remarkable inequality for any C-function f. For each oriented square s in Q, let $\omega(s, f)$ denote the oscillation of f on s. Then

$$\omega(s, f) \leqslant K \left\{ \iint_s \varphi(|\operatorname{grad} f|)\, du\, dv \right\}^{1/2}, \qquad s \subset Q,$$

where K is a finite constant independent of the square s in Q.

5. – Actually CALDERÓN makes further restrictions on the function φ (eg., that $\varphi(0) = 0$ and φ is convex) for other purposes in his paper. Inspection of his proof for the above inequality shows that the only assumptions needed on φ are those listed in section 3 (ii) above. Moreover, CALDERÓN states and proves his results in Euclidean n-space. Our results, to be described presently, may also be established in n-space, but the case $n = 2$ reveals all the essentials of the question, so we restrict ourselves to the plane in order to simplify the presentation.

6. – The inequality of CALDERÓN described in section 4 suggests the consideration of the class of functions f satisfying the following conditions:

(i) The function f is real-valued and continuous on the unit square $Q: 0 \leqslant u,\ v \leqslant 1$.

(ii) There exists a real-valued, non-negative, summable function Φ defined on Q such that

$$\omega(s, f) \leqslant \left\{ \iint_s \Phi \, du dv \right\}^{1/2},$$

for every oriented square s in Q.

If Φ is bounded on Q, it is readily seen that the preceding inequality is equivalent to the assertion that f satisfies a LIPSCHITZ condition on Q. Thus conditions (i) and (ii) may be regarded as defining a generalized LIPSCHITZ condition; accordingly, a function f satisfying conditions (i) and (ii) is termed generalized Lipschitzian in Q — or, more explicitly, generalized Lipschitzian in Q with function Φ. The inequality of CALDERÓN states that every C-function is generalized Lipschitzian in Q. Problems relating to HAUSDORFF measure suggest an extension of the preceding definition to continuous mappings, as follows. Given a metric space M, let $T: Q \to M$ be a continuous mapping from the unit square Q into M. For subsets E of Q, put $\omega(E, T)$ equal to the least upper bound of the distance in M between the images Tp and Tq of two points p and q in E; in other words, $\omega(E, T)$ is the diameter of TE. Then the mapping T is termed generalized Lipschitzian in Q — or, more explicitly, generalized Lipschitzian in Q with function Φ — if Φ is a real-valued, non-negative, summable function on Q such that

$$\omega(s, T) \leqslant \left\{ \iint_s \Phi \, du \, dv \right\}^{1/2},$$

for every oriented square s in Q. If T is generalized Lipschitzian in Q with function Φ and Ψ is any real-valued summable function on Q such that $\Phi \leqslant \Psi$ on Q, then T is clearly generalized Lipschitzian in Q with function Ψ.

7. – Given a continuous mapping $T: Q \to M$ as in section 6, and a set E in Q, $\mathrm{H}^2(TE)$ will denote the two-dimensional HAUSDORFF measure of the image TE of E in M under T. Explicitly, for a subset e of M, $\mathrm{H}^2(e)$ is defined as follows. Given a positive number ε, let $e_1, \ldots, e_n, \ldots$ be a countable family of subsets of M whose union covers e, and such that the diameter $d(e_n)$ of each e_n is less than ε. The greatest lower bound of $(\pi/4) \sum_n d(e_n)^2$ for all such coverings of e is denoted by $\mathrm{H}^2_\varepsilon(e)$. (This quantity may be equal to $+\infty$.) If ε decreases, then $\mathrm{H}^2(e)$ does not decrease. Hence as ε converges to zero,

$H_\varepsilon^2(e)$ converges to a limit (possibly $+\infty$) which is, by definition, the two-dimensional HAUSDORFF measure $H^2(e)$ of the set e in M.

Let σ be the generic notation for a (finite or infinite) system of non-overlapping oriented squares s in Q (see section 1). The mapping T in termed BVH² in Q (of bounded variation relative to two-dimensional HAUSDORFF measure) if there exists a finite constant K such that $\sum H^2(Ts)$, $s \in \sigma$, is less than K for every system σ. The mapping T is termed ACH² in Q (absolutely continuous relative to two-dimensional HAUSDORFF measure) if $H^2(TQ) < +\infty$ and for every positive number ε there exists a positive number δ such that $\sum H^2(Ts)$, $s \in \sigma$, is less than ε for every system σ such that the sum of the areas of the squares s in σ is less than δ. It is easy to show that the property ACH² in Q implies the property BVH² in Q (cf. [5, III.1.3]).

8. – Lemma. If the mapping $T\colon Q \to M$ is generalized Lipschitzian in Q, then T is ACH² in Q.

The proof is immediate. Let S be any oriented square in Q. For each natural number n let σ_n be a subdivision of S into n^2 congruent non-overlapping oriented squares s. If ε is any positive number it follows from the uniform continuity of T on Q that there is an integer N such that $\omega(s, T) < \varepsilon$ for $s \in \sigma_n$, $n > N$. Since clearly $TS \subset \bigcup Ts$, $s \in \sigma_n$, one finds that, for $n > N$.

$$H_\varepsilon^2(TS) \leqslant \pi/4 \sum_{s \in \sigma_n} \omega(s, T)^2 \leqslant \pi/4 \sum_{s \in \sigma_n} \iint_s \Phi \, du \, dv = \pi/4 \iint_S \Phi \, du \, dv ,$$

if T is generalized Lipschitzian in Q with function Φ. Since the right-most member of these inequalities is independent of ε, it follows that

$$H^2(TS) \leqslant \pi/4 \iint_S \Phi \, du \, dv ,$$

and the fact that T is ACH² is now obvious.

9. – Given a continuous mapping $T\colon Q \to M$ as in section 6, suppose there is a positive number δ and a non-negative summable function φ defined on Q such that

$$\omega(s, T) \leqslant \left\{ \iint_s \varphi \, du \, dv \right\}^{1/2}$$

for every oriented square s in Q having edge less than δ. Let N be the first integer greater than $1/\delta$, and consider any oriented square S in Q. If p and q are arbitrary points in S, one can find a sequence containing $n \leqslant N$ non-overlapping oriented squares s_j in S each with edge less than δ and such that s_1 contains p, s_n contains q, and s_j and s_{j+1} have common boundary points for

$1 \leqslant j \leqslant n-1$. Then the distance between Tp and Tq is clearly dominated by

$$\sum_{j=1}^{n} \omega(s_j, T) \leqslant \sum_{j=1}^{n} \left\{ \iint_{s_j} \varphi \, du \, dv \right\}^{1/2} \leqslant N^{1/2} \left\{ \iint_{S} \varphi \, du \, dv \right\}^{1/2},$$

and, consequently,

$$\omega(S, T) \leqslant N^{1/2} \left\{ \iint_{S} \varphi \, du \, dv \right\}^{1/2}.$$

In other words, T is generalized Lipschitzian in Q with function $N\varphi$.

10. - Given a continuous mapping $T\colon Q \to M$ which is generalized Lipschitzian in Q with function Φ (see section 6). Let r be an oriented rectangle in Q whose length does not exceed N times its width. If p and q are arbitrary points in r, one can find a sequence containing $n \leqslant N$ non-overlapping oriented squares s_j in r such that s_1 contains p, s_n contains q, and s_j and s_{j+1} have common boundary for $1 \leqslant j \leqslant n-1$. The reasoning used above now shows that

$$\omega(r, T) \leqslant N^{1/2} \left\{ \iint_{r} \Phi \, du \, dv \right\}^{1/2}.$$

11. - Let $T\colon Q \to M$ be a continuous mapping from the unit square Q into a metric space M (see section 6). Extend the definition of T by reflections to obtain a continuous periodic mapping $T^*\colon E_2 \to M$ from the Euclidean two-space E_2 into M. Explicitly, the extension T^* may be described as follows: Denote by Q^* the square $-1 \leqslant u, v \leqslant +1$. Then for $(u, v) \in Q^*$,

$$T^*(u, v) = \begin{cases} T(u, v) & \text{if } (u, v) \in Q, \\ T(-u, v) & \text{if } (-u, v) \in Q, \\ T(-u, -v) & \text{if } (-u, -v) \in Q, \\ T(u, -v) & \text{if } (u, -v) \in Q. \end{cases}$$

For an arbitrary point (u, v) in E_2 there exist unique even integers i, j such that $(u-i, v-j)$ belongs to Q^*, and $T^*(u, v) = T^*(u-i, v-j)$. Now suppose that $T\colon Q \to M$ is generalized Lipschitzian in Q with function Φ. Extend the definition of Φ by the same method as that used to extend the definition of T, to obtain a non-negative function Φ^* defined in E_2 and summable on any compact set in E_2. Is the extended mapping $T^*\colon E_2 \to M$ now generalized Lipschitzian with the extended function Φ^*? — that is, if s is any oriented square in E_2, does it follow that

$$\omega(s, T^*) \leqslant \left\{ \iint_{s} \Phi^* \, du \, dv \right\}^{1/2} ?$$

The answer to this question is not known to the writers, but there is the following result.

Lemma. Under the assumptions and definitions stated above, the extended mapping $T^*: E_2 \to M$ is generalized Lipschitzian with function $2\Phi^*$.

Proof. A square in E_2 of the form $i \leqslant u \leqslant i+1$, $j \leqslant v \leqslant j+1$ where i and j are integers is termed an elementary square. If an oriented square S in E_2 is wholly contained in some elementary square, then it is clear that

$$\omega(S, T^*) \leqslant \left\{ \iint_S \Phi^* \, du \, dv \right\}^{1/2} \leqslant 2^{1/2} \left\{ \iint_S \Phi^* \, du \, dv \right\}^{1/2}.$$

If an oriented square S contains an elementary square s, then one has

$$\omega(S, T^*) = \omega(s, T^*) \leqslant \left\{ \iint_s \Phi^* \, du \, dv \right\}^{1/2} \leqslant 2^{1/2} \left\{ \iint_S \Phi^* \, du \, dv \right\}^{1/2}.$$

There remains only the case when the oriented square S in E_2 neither is contained in nor contains an elementary square; then S contains interior points of not fewer than two and not more than six elementary squares. If r denotes the largest rectangle in S which is contained in just one of the elementary squares which S meets, it is seen that the length of r does not exceed 2 times its width, and $\omega(T^*, S) = \omega(T^*, r)$. From the preceding section it follows that

$$\omega(T^*, S) = \omega(T^*, r) \leqslant 2^{1/2} \left\{ \iint_r \Phi^* \, du \, dv \right\}^{1/2} \leqslant 2^{1/2} \left\{ \iint_S \Phi^* \, du \, dv \right\}^{1/2}.$$

Thus the lemma is established.

12. – The remarks made in the preceding sections concerning continuous mappings from the unit square Q into some metric space M clearly apply equally well to a real-valued and continuous function f defined on Q, since one may consider the special continuous mapping $x = f(u, v)$, $(u, v) \in Q$, from Q into Euclidean one-space (see section 6). Naturally there arises the question of the scope of the class of generalized Lipschitzian functions. The following lemma yields some information in this respect.

Lemma. Let the continuous function $f(u, v)$, $(u, v) \in Q$, be generalized Lipschitzian in the unit square Q. Then the following hold:

(i) the function f is ACT in Q (see section 1);

(ii) the squares of the first partial derivatives f_u, f_v are summable in Q;

(iii) the function f has a complete differential almost everywhere in Q.

Proof. In view of the lemma in section 11, no generality is lost if one assumes that the definition of f is extended to the entire plane and that the extended f is generalized Lipschitzian in E_2. Let r: $a \leqslant u \leqslant b$, $c \leqslant v \leqslant d$ be any closed oriented rectangle in E_2. For $c \leqslant v \leqslant d$, denote by $V_u(a, b, v, f)$ the total variation of $f(u, v)$ as a function of u on $a \leqslant u \leqslant b$, where the value $+\infty$ is to be used if $f(u, v)$ is not of bounded variation on $a \leqslant u \leqslant b$ for any given v. If $V_u(a, b, v, f)$, $c \leqslant v \leqslant d$, is summable, put

$$W_u(r, f) = \int_c^d V_u(a, b, v, f)\, dv:$$

otherwise, set $W_u(r, f) = +\infty$. Define $W_v(r, f)$ in a similar way by interchanging the roles of u and v. In [**5**, III.2.55, III.2.64] it is shown that the function f is ACT in Q if and only if both of the rectangle functions $W_u(r, f)$ and $W_v(r, f)$, $r \subset Q$, are absolutely continuous in Q [**5**, III.1.2]. It is presently shown that $W_u(r, f)$, $r \subset Q$, is absolutely continuous in Q. Of course, the proof for $W_v(r, f)$, $r \subset Q$, is entirely analogous, and so part (i) of the lemma is established.

For any positive number h, define (cf. [**5**, III.2.65])

$$f_h(u, v) = \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} f(u+\xi,\, v+\eta)\, d\xi\, d\eta .$$

The integral mean f_h has continuous partial derivatives of the first order [**5**, III.2.66]. In particular,

$$\frac{\partial f_h}{\partial u} = \frac{1}{4h^2} \int_{-h}^{h} [f(u+h,\ v+\eta) - f(u-h,\ v+\eta)\, d\eta . \tag{1}$$

Since f is by assumption generalized Lipschitzian with some function Φ one has, for $-h \leqslant \eta \leqslant h$,

$$|f(u+h,\ v+\eta) - f(u-h,\ v+\eta)| \leqslant \left\{ \int_{-h}^{h} \int_{-h}^{h} \Phi(u+\alpha,\ v+\beta)\, d\alpha\, d\beta \right\}^{1/2} .$$

The bound on the right does not depend upon η in $-h \leqslant \eta \leqslant h$; consequently one obtains from (1) the estimate

$$\left| \frac{\partial f_h}{\partial u} \right| \leqslant \left\{ \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} \Phi(u+\alpha,\ v+\beta)\, d\alpha\, d\beta \right\}^{1/2} . \tag{2}$$

For any oriented rectangle r: $a \leqslant u \leqslant b$, $c \leqslant v \leqslant d$, in E_2, let $W_u(r, f_h)$ be defined in the same manner as $W_u(r, f)$. Since f_h has continuous first partial derivatives, it follows that

$$W_u(r, f_h) = \iint_r \left| \frac{\partial f_h}{\partial u} \right| du\, dv . \tag{3}$$

By the HÖLDER inequality and the theorem of FUBINI one obtains from (2) and (3) the relations

$$W_u(r, f_h) \leqslant \left\{ \iint_r \left| \frac{\partial f_h}{\partial u} \right|^2 du dv \right\}^{1/2} |r|^{1/2} \leqslant$$

$$\leqslant \left\{ \iint_r \left[\frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} \Phi(u+\alpha,\ v+\beta)\, d\alpha\, d\beta \right] du\, dv \right\}^{1/2} |r|^{1/2} \leqslant$$

$$\leqslant \left\{ \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} \left[\iint_r \Phi(u+\alpha,\ v+\beta)\, du\, dv \right] d\alpha d\beta \right\}^{1/2} |r|^{1/2},$$

where $|r|$ denotes the area of the rectangle r. Let r_h denote the oriented rectangle $a-h \leqslant u \leqslant b+h$, $c-h \leqslant v \leqslant d+h$. Then

$$W_u(r, f_h) \leqslant \left\{ \iint_{r_h} \Phi(u, v)\, du\, dv \right\}^{1/2} |r|^{1/2}. \tag{4}$$

Since f_h converges to f uniformly on r and r_h converges to r as h tends to zero it follows from [**5**, III. 2.53] that

$$W_u(r, f) \leqslant \liminf_{h\to 0} W_u(r, f_h), \qquad \lim_{h\to 0} \iint_{r_h} \Phi\, du dv = \iint_r \Phi\, du\, dv. \tag{5}$$

Relations (4) and (5) imply that

$$W_u(r, f) \leqslant \left\{ \iint_r \Phi\, du\, dv \right\}^{1/2} |r|^{1/2}. \tag{6}$$

Now consider any finite system of oriented rectangles $r_1, \ldots, r_n$ in Q whose interiors are pairwise disjoint. In view of relation (6), the HÖLDER inequality yields

$$\sum_{j=1}^{n} W_u(r_j, f) \leqslant \sum_{j=1}^{n} \left\{ \iint_{r_j} \Phi\, du dv \right\}^{1/2} |r_j|^{1/2} \leqslant$$

$$\leqslant \left\{ \sum_{j=1}^{n} \iint_{r_j} \Phi\, du\, dv \right\}^{1/2} \left\{ \sum_{j=1}^{n} |r_j| \right\}^{1/2} \leqslant \left\{ \iint_Q \Phi du dv \right\}^{1/2} \left\{ \sum_{j=1}^{n} |r_j| \right\}^{1/2}.$$

This reveals that $W_u(r, f)$, $r \subset Q$, is absolutely continuous [**5**, III.1.2]. Thus assertion (i) of the lemma is established.

Next, let (u, v) be any point in Q where

$$\frac{1}{4\varrho^2} \int_{-\varrho}^{\varrho} \int_{-\varrho}^{\varrho} \Phi(u+\alpha,\ v+\beta)\, d\alpha\, d\beta \xrightarrow[\varrho\to 0+]{} \Phi(u, v). \tag{7}$$

For real numbers h, k such that $\varrho^2 = h^2 + k^2 > 0$, let s_ϱ be the closed oriented square with center (u, v) and side length 2ϱ. Then

$$|f(u+h, v+k) - f(u, v)| \leqslant \omega(f, s_\varrho) \leqslant \left\{ \iint_{s_\varrho} \Phi \, du dv \right\}^{1/2}$$

and, consequently,

$$\frac{|f(u+h, v+k - f(u, v)|}{\varrho} \leqslant 2 \left\{ \frac{1}{4\varrho^2} \iint_{s_\varrho} \Phi \, du \, dv \right\}^{1/2}. \tag{8}$$

From (7) and (8) it follows that

$$\limsup_{\varrho \to 0+} \frac{|f(u+h, v+k) - f(u, v)|}{\varrho} \leqslant 2\Phi(u, v)^{1/2}. \tag{9}$$

Now relation (7), and consequently relation (9), hold almost everywhere in the square Q. Since $\Phi(u, v)^{1/2}$, $(u, v) \in Q$, is summable, it follows by a well known result of RADEMACHER [4] that f has a total differential almost everywhere in Q. This establishes part (iii) of the lemma. Moreover, the first partials of f exist almost everywhere in Q and are measurable. From relation (9) one obtains $|f_u(u, v)| \leqslant 2\Phi(u, v)^{1/2}$, $|f_v(u, v)| \leqslant 2\Phi(u, v)^{1/2}$, almost everywhere in Q. Thus f_u^2 and f_v^2 are summable in Q, and part (ii) of the lemma is verified. This completes the proof.

13. – The preceding result may be interpreted as showing that the generalized Lipschitzian functions represent a LEBESGUE class L^{2+0} in a certain sense. It would be of interest to determine further properties (beyond those established in the lemma) which may completely characterize generalized Lipschitzian functions. In particular, the C-functions of CALDERÓN (see section 3) may be profitably studied from this point of view. One may ask, for instance: is every generalized Lipschitzian function also a C-function?

14. – One observes that the definition of a generalized Lipschitzian function or transformation given in section 6 is not independent of the choice of the coordinate system, since it requires a certain inequality to hold only for oriented squares — that is, for squares whose sides are parallel to the respective coordinate axes. To overcome this defect certain plausible conditions are presently introduced, and their relationships to the generalized Lipschitzian concept are explored.

15. – Let $T: Q \to M$ be a continuous mapping from the unit square Q into a metric space M, as in section 6. Then T is said to satisfy condition C_1 in Q — or, more explicitly, condition C_1 in Q with function Ψ — if Ψ is a real-valued, non-negative, summable function on Q such that

$$\omega(s^*, T) \leqslant \left\{ \iint_{s^*} \Psi \, du dv \right\}^{1/2}$$

for every square s^* — oriented or not — in Q.

It is obvious that if T satisfies condition C_1 in Q with function Ψ then T is also generalized Lipschitzian in Q with function Ψ. However, the simplest examples show that T may be generalized Lipschitzian in Q with function Φ, but not satisfy condition C_1 in Q with function Φ. Naturally the following question arises. If a mapping $T: Q \to M$ is generalized Lipschitzian in Q does it also satisfy condition C_1 in Q? If so, and if T is generalized Lipschitzian in Q with function Φ how may a function Ψ be related to Φ so that T satisfies condition C_1 in Q with function Ψ? Also, if $T: Q \to M$ satisfies condition C_1 in Q, will its extension $T^*: E_2 \to M$ described in section 11 satisfy condition C_1 in E_2?

16. – Let $T: Q \to M$ be a continuous mapping as in section 15. Then T is said to satisfy condition C_2 in Q — or, more explicitly, condition C_2 in Q with function χ — if χ is a real-valued, non negative, summable function on Q such that

$$\omega(d, T) \leqslant \left\{ \iint_{d} \chi \, du \, dv \right\}^{1/2}$$

for every circular disc d in Q. If T satisfies condition C_2 in Q, does it satisfy condition C_1 in Q? Is it generalized Lipschitzian in Q? Does a mapping T which is generalized Lipschitzian in Q satisfy condition C_2? There is the following result.

L e m m a. If $T: Q \to M$ satisfies condition C_1 in Q with function Ψ, then T satisfies condition C_2 in Q with function 4Ψ.

P r o o f. Let d be a disc in Q with center p. If q is any other point in d, the square s^* with diagonal pq is contained in d, and consequently the distance between Tp and Tq does not exceed $\left\{ \iint_{s^*} \Phi \, du \, dv \right\}^{1/2}$, which in turn is dominated by $\left\{ \iint_{d} \Psi \, du \, dv \right\}^{1/2}$. From this it follows that

$$\omega(d, T) \leqslant 2 \left\{ \iint_{d} \Psi \, du dv \right\}^{1/2},$$

and the lemma is established.

If $f(u, v)$, $(u, v) \in Q$ is any C-function (see section 3), the method of CALDERÓN may be used to show that the mapping $x = f(u, v)$, $(u, v) \in Q$, from Q into E_1 satisfies both conditions C_1 and C_2 in Q.

17. – Given a mapping $T: Q \to M$ as in section 15, one may without loss of generality assume that the definition of T has been extended by reflections to the entire Euclidean plane E_2 (see section 11). Let ϱ be a positive number and p any point in E_2. Put $M_\varrho(p)$ equal to the least upper bound of the ratio of the distance (Tp, Tq) in M between Tp and Tq to the distance (p, q) in E_2 between p and q for all points q distinct from p whose distance from p does not exceed ϱ. For fixed ϱ, $M_\varrho(p)$, $p \in E_2$, is clearly lower semi-continuous. As ϱ decreases, M_ϱ does not increase. It is readily seen that

$$\lim_{\varrho \to 0+} M_\varrho(p) = \limsup_{(p,q) \to 0+} \frac{(Tp, Tq)}{p(, q)} .$$

Now suppose there is a positive number ϱ such that $M_\varrho^2(p)$, $p \in Q$, is summable. When this condition is satisfied, the mapping T is said to satisfy condition C_3 in Q. Consider any square s^* in E_2 — oriented or not — whose diameter d does not exceed ϱ. Let p, q, r be any three points in s^*. Then clearly $(Tq, Tr) \leqslant (Tq, Tp) + (Tp, Tr) \leqslant M_\varrho(p)(q, p) + M_\varrho(p)(p, r) \leqslant 2dM_\varrho(p)$ and hence $\omega(s^*, T) \leqslant 2dM_\varrho(p)$, $p \in s^*$. Squaring this relation and integrating over the square s^*, one finds

$$\omega(s^*, T) \leqslant 8^{1/2}\left\{\iint_{s^*} M_\varrho^2 \, du \, dv \right\}^{1/2} .$$

The reasoning used in section 9 may now be employed to show that, for a suitable constant N depending only on ϱ, T satisfies condition C_1 in Q with function NM_ϱ^2. Thus the following result has been established (see sections 15, 16).

Lemma. If $T: Q \to M$ is a continuous mapping satisfying condition C_3 in Q, then T also satisfies condition C_1 and C_2 in Q.

18. – Given a non-negative summable function Φ in Q, one may without loss of generality assume that the definition of Φ has been extended by reflections to the entire Euclidean plane E_2 (see section 11). Let ϱ be a positive number and p any point in E_2. Put $N_\varrho(p, \Phi)$ equal to the least upper bound of

$$\iint_s \frac{\Phi \, du \, dv}{|s|}$$

for all oriented square s with center p and edge not exceeding 2ϱ. For fixed ϱ,

$N_\varrho(p, \Phi)$, $p \in E_2$, is clearly lower semi-continuous. As ϱ decreases, N_ϱ does not increase. It is easily seen that

$$\lim_{\varrho \to 0+} N_\varrho(p, \Phi) = \Phi(p) \quad \text{almost everywhere.}$$

If k is any positive constant, then clearly $N_\varrho(p, k\Phi) = kN_\varrho(p, \Phi)$, $p \in E_2$. The function Φ is said to satisfy condition C whenever there exists some positive constant ϱ such that $N_\varrho(p, \Phi)$, $p \in Q$, is summable.

Now suppose that a mapping T: $Q \to M$ is generalized Lipschitzian in Q with function Φ satisfying condition C. One may without loss of generality assume that the definition of T has been extended by reflections to the entire Euclidean plane E_2, and that the extended mapping is generalized Lipschitzian with function 2Φ (see section 11). Let p be an arbitrary point of E_2 and q any point distinct from p whose distance d from p does not exceed ϱ; then q lies in the oriented square s with center p and side length $2d$. Consequently

$$\frac{(Tp, Tq)}{(p, q)} \leqslant \frac{\omega(s, T)}{d} \leqslant 8^{1/2} \left\{ \iint_s \frac{\Phi \, du \, dv}{|s|} \right\}^{1/2} \leqslant 8^{1/2} N_\varrho^{1/2}(p, \Phi) .$$

Thus

$$M_\varrho^2(p) \leqslant 8 N_\varrho(p, \Phi), \qquad p \in E_2 ,$$

and $M_\varrho^2(p)$, $p \in Q$, is summable if $N_\varrho(p, \Phi)$, $p \in Q$, is summable. Combining this fact the result in section 17, one obtains the following.

Lemma. Let T: $Q \to M$ be a continuous mapping which is generalized Lipschitzian in Q with function Φ satisfying condition C. Then T also satisfies condition C_3, C_2, C_1 in Q.

19. – For the applications to be made in the next section the following fact is needed.

Lemma. Let T: $Q \to E_3$ be a continuous mapping from the unit square Q into Euclidean three-space (x_1, x_2, x_3) given by formulas T: $x_i = x_i(u, v)$, $(u, v) \in Q$, $(i = 1, 2, 3)$. Then T is a generalized Lipschitzian mapping in Q if and only if each of the functions x_i is generalized Lipschitzian in Q for $i = 1, 2, 3$.

Proof. If T is generalized Lipschitzian in Q, then each x_i is also generalized Lipschitzian in Q because, on any oriented square s in Q, one has $\omega(s, x_i) \leqslant \omega(s, T)$ for $i = 1, 2, 3$. On the other hand, if each x_i is generalized Lipschitzian in Q with function Φ_i, $(i = 1, 2, 3)$, then the reader will verify at once that T is generalized Lipschitzian in Q with function $\Phi_1 + \Phi_2 + \Phi_3$.

20. – This Note concludes with an application of the preceding results to extend the theorem of Cesari described in section 1 in both of the directions indicated in section 2. Let $x(u, v)$, $(u, v) \in Q$, be any function which is ACT in Q (see section 1) and each of whose first partial derivatives belongs to the LEBESGUE class $L^{2+\alpha}$ for some positive constant α. Define

$$\varphi(t) = \begin{cases} 1, & 0 \leqslant t \leqslant 1, \\ t^{2+\alpha}, & 1 \leqslant t < +\infty. \end{cases}$$

The reader will verify that the function x then satisfies conditions (i) and (ii) of section 3 for this choice of φ — that is, x is a C-function, and consequently is generalized Lipschitzian in Q (see section 6). In view of the lemma in section 19, it is now clear that a continuous mapping T possessing the properties required by CESARI in [3], and stated in section 1 (i) (ii), is generalized Lipschitzian in Q. Since in the plane two-dimensional HAUSDORFF measure coincides with two-dimensional LEBESGUE measure, the theorem of CESARI stated in section 1 is seen to be a very special case of the lemma in section 8. In other words, the lemma in section 8 is a generalization of the theorem of CESARI in the two directions: the Euclidean plane xy of CESARI has been replaced by an arbitrary metric space and the requirement that the first partial derivatives belong to the LEBESGUE classs $L^{2+\alpha}$ for some positive number α has been reduced to the requirement that T be a generalized Lipschitzian mapping.

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