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On the differentiability of absolutely continuous functions. (**)

CESARI [1] has shown that a continuous function $f(x, y)$ which is absolutely continuous in the sense of TONELLI and whose derivatives are in L^p, $p > 2$, has a total differential in the sense of STOLZ almost everywhere. This is no longer true for $p = 2$, but the result can be improved in the following manner. Let $\varphi(t)$ be a non negative convex increasing function, defined for $0 \leqq t < +\infty$ such that $\varphi(0) = 0$. Following ORLICZ, let us denote by L_φ the class of all functions f for which there exists a constant λ, depending on f, such that

$$\int \varphi\left(\frac{|f|}{\lambda}\right) d\sigma < \infty ;$$

then if f is continuous and absolutely continuous in the sense of TONELLI and $(f_x^2 + f_y^2)^{1/2}$ belongs to L_φ locally, $\varphi(t)$ being a function such that

$$\int_1^\infty \frac{t}{\varphi(t)} dt < \infty ,$$

f has a total differential almost everywhere. This result is in a sense the best possible. In fact given any $\varphi(t)$ such that

$$\int_1^\infty \frac{t}{\varphi(t)} dt = \infty ,$$

there exists a function f which is continuous and absolutely continuous, such that $(f_x^2 + f_y^2)^{1/2}$ belongs to L_φ locally, and which fails to have a total differential almost everywhere.

The general result concerning functions of n variables can be stated as follows: if f is continuous and absolutely continuous in the sense of TONELLI

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and $|\operatorname{grad} f|$ belongs locally to L_{φ}, where φ is a function such that

$$\int_1^{\infty}\left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-1}} dt < \infty,$$

f has a total differential in the sense of STOLZ almost everywhere. Functions of this class satisfy the inequality (5) and therefore belong to the class of φ-Lipschitzian functions suggested by T. RADÓ. As pointed out by RADÓ, on account of a result of STEPANOFF [2], φ-Lipschitzian functions have a total differential almost everywhere; but we shall prove the existence of the differential directly to avoid reference to STEPANOFF's result which is far from being elementary.

Let $f(P)$, $P = (x_1, x_2, \ldots, x_n)$, be continuous and absolutely continuous in the n-dimensional unit cube K_0, $0 \leqslant x_i \leqslant 1$. It is known that for almost every point Q and almost every straight line going through Q, $f(P)$ is an absolutely continuous function of the distance between P and Q. Moreover if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the direction cosines of the line and $f_1, f_2, \ldots, f_n$, denote the partial derivatives of f, then $\frac{df}{ds} = \sum \alpha_i f_i$ for almost every s.

Let now Q and Q' be two points in K_0 such that f is absolutely continuous on almost every line through Q or Q', d be the distance between Q and Q' and D the disc of radius $\frac{d}{2\sqrt{n}}$ with center at the mid-point of the segment QQ' and contained in the hyperplane perpendicular to QQ'. Assume that D is entirely in K_0 and denote by γ and γ' the cones projecting D from Q and Q' respectively. If P is a point on D such that f is absolutely continuous on the segments PQ and PQ', then

$$|f(Q) - f(Q')| \leqslant |f(P) - f(Q)| + |f(P) - f(Q')|$$

and

$$|f(P) - f(Q)| = \int_{QP}\left(\frac{df}{ds}\right) ds \leqslant \int_{QP}\left|\frac{df}{ds}\right| ds\,,$$

where the integral is taken over the segment PQ. But almost everywhere in s we have

$$\frac{df}{ds} = \sum_1^n \alpha_i f_i \quad \text{and} \quad \left|\frac{df}{ds}\right| \leqslant |\operatorname{grad} f|\,,$$

and therefore

$$| f(P) - f(Q) | \leqslant \int_{PQ} | \operatorname{grad} f | \, ds ,$$

and similarly we obtain

$$| f(P) - f(Q') | \leqslant \int_{PQ'} | \operatorname{grad} f | \, ds ,$$

and collecting both inequalities together we find that

$$(1) \qquad | f(Q) - f(Q') | \leqslant \int_{PQ} | \operatorname{grad} f | \, ds + \int_{PQ'} | \operatorname{grad} f | \, ds .$$

Denote now by $d\Omega_Q$ the element of solid angle projecting from Q the element of area on D. Then $d\Omega_Q = d\Omega_{Q'}$ and

$$\int_D d\Omega_Q = \Omega ,$$

where Ω is a fixed number independent of the distance between Q and Q'. Let us integrate (1) with respect to $d\Omega_Q$ over D

$$\int_D | f(Q) - f(Q') | \, d\Omega_Q = \Omega \, | f(Q) - f(Q') | \leqslant$$

$$\leqslant \int_D d\Omega_Q \int_{PQ} | \operatorname{grad} f | \, ds + \int_D d\Omega_{Q'} \int_{PQ'} | \operatorname{grad} f | \, ds .$$

Now if dv denotes the element of volume in K_0 and ϱ and ϱ' the distances to Q and Q' respectively it is not difficult to see that the foregoing repeated integrals are equal to volume integrals, namely

$$\int_D d\Omega_Q \int_{QP} | \operatorname{grad} f | \, ds = \int_\gamma \frac{1}{\varrho^{n-1}} | \operatorname{grad} f | \, dv$$

and

$$\int_D d\Omega_{Q'} \int_{Q'P} | \operatorname{grad} f | \, ds = \int_{\gamma'} \frac{1}{\varrho'^{n-1}} | \operatorname{grad} f | \, dv ,$$

where γ and γ' are the cones introduced above. Therefore the preceding inequality can be written

$$\Omega \, | f(Q) - f(Q') | \leqslant \int_\gamma \frac{1}{\varrho^{n-1}} | \operatorname{grad} f | \, dv + \int_{\gamma'} \frac{1}{\varrho'^{n-1}} | \operatorname{grad} f | \, dv .$$

Let now Q be fixed and K a cube with center at Q and entirely contained in K_0. If Q' is any point of K, the cores γ and γ' constructed on the segment QQ' as above, are entirely contained in K; in fact, if 2δ denotes the length of the edge of K, the distance between the mid-point of QQ' and the boundary of K is never less than $\delta/2$, and the length of QQ' never exceeds $\sqrt{n}\,\delta$, so that the radius of D never exceeds $\delta/2$ and therefore D and a fortiori also γ and γ' are entirely contained in K. On account of this we have

$$\Omega\,|\,f(Q)-f(Q')\,|\leqslant\int_{\gamma}\frac{1}{\varrho^{n-1}}\,|\operatorname{grad} f\,|\,dv+\int_{\gamma'}\frac{1}{\varrho'^{n-1}}\,|\operatorname{grad} f\,|\,dv\leqslant$$

$$\leqslant\int_{K}\left(\frac{1}{\varrho^{n-1}}+\frac{1}{\varrho'^{n-1}}\right)|\operatorname{grad} f\,|\,dv\,.$$

Now if R is a point in K and $|\,P-R\,|$ is the distance between P and R it follows that

$$|\,f(Q)-f(Q')\,|\leqslant\frac{2}{\Omega}\sup_{R\in K}\int_{K}\frac{|\operatorname{grad} f(P)\,|}{|\,R-P\,|^{n-1}}\,dv\,. \tag{2}$$

Our next step will be to obtain inequality (5). For this purpose suppose that $|\operatorname{grad} f\,|$ belongs to L_{φ} where φ is an increasing convex function such that $\varphi(0)=0$ and

$$A=\int_{0}^{\infty}\left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-1}}dt<\infty\,. \tag{3}$$

Denote by E_m the set of points of K where

$$2^{m}\leqslant|\operatorname{grad} f\,|<2^{m+1}\,.$$

Then

$$\int_{K}\frac{|\operatorname{grad} f\,|}{|\,R-P\,|^{n-1}}\,dv\leqslant\sum_{-\infty}^{+\infty}2^{m+1}\int_{E_m}\frac{dv}{|\,R-P\,|^{n-1}}\,.$$

Now it is not difficult to see that the integral

$$\int_{E_m}\frac{dv}{|\,R-P\,|^{n-1}}$$

is less than or equal to the same integral extended over the sphere with center at R and of the same measure as E_m. Therefore if w denotes the measure of the sphere of radius 1, the radius of that sphere is

$$\left(\frac{|E_m|}{\omega}\right)^{1/n}$$

and therefore

$$\int_{E_m} \frac{dv}{|R-P|^{n-1}} \leqslant \int_0^{\left(\frac{|E_m|}{\omega}\right)^{1/n}} \frac{1}{\varrho^{n-1}}\, d(\varrho^n \omega) = n\omega^{\frac{n-1}{n}} |E_m|^{1/n}.$$

Replacing above we get

$$\text{(4)} \qquad \int_K \frac{|\operatorname{grad} f|}{|R-P|^{n-1}}\, dv \leqslant \sum_{-\infty}^{+\infty} 2^{m+1} n\omega^{\frac{n-1}{n}} |E^m|^{1/n} =$$

$$= 2n\omega^{\frac{n-1}{n}} \sum_{-\infty}^{+\infty} 2^m |E_m|^{1/n} \frac{\varphi(2^m)^{1/n}}{\varphi(2^m)^{1/n}}$$

and applying HÖLDER's inequality to the last sum, we find that

$$\sum_{-\infty}^{+\infty} 2^m |E_m|^{1/n} \frac{\varphi(2^m)^{1/n}}{\varphi(2^m)^{1/n}} \leqslant \left[\sum_{-\infty}^{+\infty} |E_m|\, \varphi(2^m)\right]^{1/n} \left[\sum_{-\infty}^{+\infty} \left[\frac{2^{mn}}{\varphi(2^m)}\right]^{\frac{1}{n-1}}\right]^{\frac{n-1}{n}}.$$

Now, on account of the definition of E_m we have

$$\sum_{-\infty}^{+\infty} |E_m|\, \varphi(2^m) \leqslant \int_K \varphi(|\operatorname{grad} f|)\, dv\ ;$$

on the other hand since $\varphi(t)$ is an increasing function

$$\left[\frac{2^{nm}}{\varphi(2^m)}\right]^{\frac{1}{n-1}} \leqslant \int_{m-1}^{m} \left[\frac{2^{n(s+1)}}{\varphi(2^s)}\right]^{\frac{1}{n-1}} ds$$

and therefore

$$\sum_{-\infty}^{+\infty} \left[\frac{2^{nm}}{\varphi(2^m)}\right]^{\frac{1}{n-1}} \leqslant \int_{-\infty}^{+\infty} \left[\frac{2^{n(s+1)}}{\varphi(2^s)}\right]^{\frac{1}{n-1}} ds$$

and introducing the variable $t = 2^s$ in the integral we obtain

$$\sum_{-\infty}^{+\infty} \left[\frac{2^{nm}}{\varphi(2^m)}\right]^{\frac{1}{n-1}} \leqslant c \int_0^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-1}} dt = cA\ ,$$

where A is a constant depending on n.

Replacing, the inequality (4) becomes

$$\int_K \frac{|\operatorname{grad} f(P)|}{|R-P|^{n-1}}\, dv \leqslant cA^{\frac{n-1}{n}} \left[\int_K \varphi(|\operatorname{grad} f|)\, dv\right]^{1/n}$$

and the inequality (2) reduces to

$$|f(Q)-f(Q')| \leqslant cA^{\frac{n-1}{n}} \left[\int_K \varphi(|\operatorname{grad} f|)\, dv\right]^{1/n}, \tag{5}$$

where the letters c denote constants depending on n.

The inequality (5) holds for any two points Q and Q' provided Q' belongs to K. This follows from the fact that it certainly holds for almost every Q and almost every Q' and that both sides are continuous functions of Q, Q' and the length of the edge of K. Furthermore, the convexity of φ has not been used in the argument, so that (5) holds also for non-convex functions φ provided the remaining assumptions are satisfied.

The proof of the differentiability of f almost everywhere depends now only on one additional fact, namely that for almost every Q and for cubes K with center at Q we have

$$\lim_{|K|\to 0} \frac{1}{|K|} \int_K \varphi[|\operatorname{grad} f(P) - \operatorname{grad} f(Q)|]\, dv = 0 . \tag{6}$$

Let us sketch the proof of this. Without loss of generality we may assume that also $\varphi[2\,|\operatorname{grad} f|]$ is integrable; then for any vector $\boldsymbol{a}_r$ with rational components and any rational number s the function $\varphi[|\operatorname{grad} f - \boldsymbol{a}_r| + s]$ is integrable and outside a set of measure zero for every Q and any $\boldsymbol{a}_r$ and s we have

$$\lim_{|K|\to 0} \frac{1}{|K|} \int_K \varphi[|\operatorname{grad} f - \boldsymbol{a}_r| + s]\, dv = \varphi[|\operatorname{grad} f(Q) - \boldsymbol{a}_r| + s] .$$

For the same Q and any vector $\boldsymbol{a}$ we have

$$(\operatorname{grad} f - \boldsymbol{a}) = (\operatorname{grad} f - \boldsymbol{a}_r) + (\boldsymbol{a}_r - \boldsymbol{a})$$

and, if $s > |\boldsymbol{a}_r - \boldsymbol{a}|$,

$$|\operatorname{grad} f - \boldsymbol{a}_r| - s \leqslant |\operatorname{grad} f - \boldsymbol{a}| \leqslant |\operatorname{grad} f - \boldsymbol{a}_r| + s ,$$

$$\varphi[|\operatorname{grad} f - \boldsymbol{a}_r| - s] \leqslant \varphi[|\operatorname{grad} f - \boldsymbol{a}|] \leqslant \varphi[|\operatorname{grad} f - \boldsymbol{a}_r| + s] .$$

Averaging the last inequality over K, letting $|K| \to 0$ and then making

$\boldsymbol{a}_r \to \boldsymbol{a}$ and $s \to 0$ we get

$$\lim_{|K|\to 0} \frac{1}{|K|} \int_K \varphi[|\operatorname{grad} f - \boldsymbol{a}|]\, dv = \varphi[|\operatorname{grad} f(Q) - \boldsymbol{a}|] .$$

Replacing $\boldsymbol{a}$ by $\operatorname{grad} f(Q)$ the desired result follows.

Going back to the function f let Q be a point where (6) holds and let $g(P)$ be $g(P) = f(P) - f(Q) - \sum_1^n \frac{\partial f}{\partial x_i}(Q)(x_i - \bar{x}_i)$ where the x_i denote the coordinates of P and the $\bar{x}_i$ those of Q. If K is a cube with center at Q and Q' is a point of the boundary of K, inequality (5) gives

$$|g(Q') - g(Q)| = |g(Q')| \leqslant cA^{\frac{n-1}{n}} \left[\int_K \varphi[|\operatorname{grad} g|]\, dv\right]^{1/n}$$

and since $\operatorname{grad} g(P) = \operatorname{grad} f(P) - \operatorname{grad} f(Q)$ and $|K| \leqslant 2^n |Q - Q'|^n$ we have

$$|g(Q')| \leqslant 2cA^{\frac{n-1}{n}} |Q - Q'| \left[\frac{1}{|K|} \int_K \varphi[|\operatorname{grad} f(P) - \operatorname{grad} f(Q)|]\, dv\right]^{1/n} .$$

When $|K| \to 0$ the quantity in the bracket tends to zero so that setting $Q' = P$ we get

$$\frac{|g(P)|}{|P - Q|} = \frac{\left|f(P) - f(Q) - \sum_1^n \frac{\partial f}{\partial x_i}(Q)(x_i - \bar{x}_i)\right|}{|P - Q|} \to 0 ,$$

as $P \to Q$. In other words $f(P)$ has a total differential at the point Q and the proof of the result is complete.

We shall prove now that given an increasing convex function $\varphi(t)$ such that $\varphi(0) = 0$ and

$$\int_1^\infty \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-1}} dt = \infty ,$$

there exists a continuous function f which is absolutely continuous, such that $|\operatorname{grad} f| \in L_\varphi$ and which fails to have a total differential almost everywhere. This can be viewed as being a consequence of the fact that under those assumptions there exists a function $F(P) \geqslant 0$ depending only on the distance ϱ between P and the origin of coordinates which is a continuously differentiable decreasing function of ϱ for $\varrho > 0$ and such that $F(P) \to \infty$ as $\varrho \to 0$ and $\int \varphi[|\operatorname{grad} F(P)|]\, dv \leqslant 1$ Once the existence of F is established the function f can be constructed as follows:

Denote by P_h^k the points interior to the unit cube K_0 whose coordinates are integral multiples of $\frac{1}{2^k}$ and let E_k be the union of the spheres of radius $\frac{1}{4^k}$ with centers at the P_h^k. Since there are less than 2^{kn} points P_h^k in K_0 we have $|E_k| < \omega \frac{1}{2^{kn}}$, where ω denotes the volume of the sphere of radius 1. Furthermore call

$$F_k(\varrho) = \begin{cases} 0 \quad \text{if} \quad \varrho \geqslant \frac{1}{4^k}, \\ \inf\left[4^{kn}\left(\frac{2}{3}\right)^k ; \quad F(\varrho) - F\left(\frac{1}{4^k}\right)\right] \quad \text{if} \quad 0 \leqslant \varrho < \frac{1}{4^k}. \end{cases}$$

On account of the properties of $F(\varrho)$ it is apparent that $F_k(\varrho)$ satisfies a uniform LIPSCHITZ condition, that $F_k(0) = 4^{kn}\left(\frac{2}{3}\right)^k$ and

$$\int \varphi[|\operatorname{grad} F_k|]\, dv \leqslant 1 .$$

Now define

$$G_k(P) = \sum_h F_k(|P - P_h^k|)$$

and

$$f(P) = \sum_{k=1}^{\infty} \frac{1}{4^{kn}} G_k(P) .$$

Since $G_k(P)$ is continuous and $G_k(P) \leqslant 4^{kn}\left(\frac{2}{3}\right)^k$, $f(P)$ is continuous; on the other hand

$$\int_{K_0} \varphi(|\operatorname{grad} G_k|)\, dv \leqslant 2^{kn}$$

and applying JENSEN's inequality we have

$$\int_K \varphi[|\operatorname{grad} f|]\, dv \leqslant \int_{K_0} \varphi\left[\sum_1^{\infty} \frac{1}{4^{kn}} |\operatorname{grad} G_k|\right] dv \leqslant$$

$$\leqslant \int_{K_0} \varphi \left[\frac{\sum_1^{\infty} \frac{1}{4^{kn}} |\operatorname{grad} G_k|}{\sum_1^{\infty} \frac{1}{4^{kn}}}\right] dv \leqslant \frac{\sum_1^{\infty} \frac{1}{4^{kn}} \int_{K_0} [|\operatorname{grad} G_k|]\, dv}{\sum_1^{\infty} \frac{1}{4^{kn}}} < \infty$$

so that $\varphi[|\operatorname{grad} f|]$ is integrable and $|\operatorname{grad} f|$ belongs to L_φ.

Let now $D_l = \bigcup_{k=l}^{\infty} E_k$; then $D_l \supset D_{l+1}$ and $\lim_{l \to \infty} | D_l | = 0$ so that the set $D = \bigcap_1^{\infty} D_k$ is of measure zero. If Q is a point not in D, Q does not belong to D_l for some l, and since $G_k(P)$ vanishes outside E_k it vanishes outside D_l for $k \geqslant l$, so that

$$f(Q) = \sum_{k=1}^{l-1} \frac{1}{4^{kn}} G_k(Q)$$

and therefore

$$\frac{f(P) - f(Q)}{| P - Q |} = \sum_{k=1}^{l-1} \frac{1}{4^{kn}} \frac{G_k(P) - G_k(Q)}{| P - Q |} + \sum_{l}^{\infty} \frac{1}{4^{kn}} \frac{G_k(P)}{| P - Q |} .$$

Now, it is possible to select a sequence $P^m_{h(m)}$ among the points P^k_h such that $P^m_{h(m)} \to Q$ as m tends to infinity, more precisely, such that $| P^m_{h(m)} - Q | \leqslant \leqslant \sqrt{n} \frac{1}{2^m}$. Since $G_m(P^m_h) = 4^{km} \left(\frac{2}{3}\right)^k$,

$$\frac{G_m[P^m_{h(m)}]}{| P^m_{h(m)} - Q |} \geqslant \frac{1}{\sqrt{n}} 2^m 4^{km} \left(\frac{2}{3}\right)^m$$

and, replacing above we get for $m \geqslant l$,

$$\frac{f[P^m_{h(m)}] - f(Q)}{| P^m_{h(m)} - Q |} \geqslant \sum_{k=1}^{l-1} \frac{1}{4^{kn}} \frac{G_k[P^m_{h(m)}] - G_k(Q)}{| P^m_{h(m)} - Q |} + \frac{1}{\sqrt{n}} \left(\frac{4}{3}\right)^m .$$

Each of the $G_k(P)$ satisfies a uniform Lipschitz condition so that the summation sign on the right is bounded and thus

$$\lim_{m \to \infty} \frac{f[P^m_{h(m)}] - f(Q)}{| P^m_{h(m)} - Q |} = \infty .$$

Therefore f is not differentiable at Q, that is f fails to have a total differential outside D.

Now it remains only to construct the function $F(\varrho)$. We shall restrict ourselves to consider the case where $\varphi(t)/t$, which on account of the convexity of φ is non-decreasing, actually tends to infinity as $t \to \infty$. When $\varphi(t)/t$ is bounded, the gradient of every absolutely continuous function belongs to L_φ.

Let

$$\Phi(t) = \int_1^t \left[\frac{s}{\varphi(s)}\right]^{\frac{1}{n-1}} ds$$

and consider the function

$$s = \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-1}} \frac{1}{\Phi(t)} = \frac{\Phi'(t)}{\Phi(t)} .$$

This function is continuous, strictly decreasing, it tends to zero as $t \to \infty$ and to infinity as $t \to 1$; therefore it has an inverse $t = h(s)$ defined for all values of $s > 0$. Moreover we shall have

$$\int_0^1 h(s)\,ds = \infty , \qquad \int_0^1 \varphi[h(s)]s^{n-1}\,ds < \infty . \tag{7}$$

To show this we shall first remark that since $\Phi'(t)$ is decreasing we have

$$\frac{\Phi(t)}{t} \backsim \frac{\Phi(t)}{t-1} \geqslant \Phi'(t) . \tag{8}$$

Then

$$\int_\varepsilon^1 h(s)\,ds = \int_\varepsilon^1 t\,ds = st]_\varepsilon^1 + \int_{t(1)}^{t(\varepsilon)} s\,dt .$$

Now as $s \to 0$, t tends to infinity so that the integrated term

$$s_t = \frac{\Phi'(t)}{\Phi(t)}\, t$$

remains bounded on account of (8). On the other hand the integral

$$\int^\infty s\,dt = \int^\infty \frac{\Phi'(t)}{\Phi(t)}\,dt$$

diverges because $\Phi(t) \to \infty$ as $t \to \infty$ so that

$$\int_0^1 h(s)\,ds = \infty .$$

Consider now

$$\int_0^1 \varphi[h(s)]s^{n-1}\,ds = \int_0^1 \varphi(t)\,\frac{t}{\varphi(t)\Phi(t)^{n-1}}\,ds = \int_0^1 \frac{t}{\Phi(t)^{n-1}}\,ds .$$

Again on account of (8) we can integrate by parts the last integral and its convergence is reduced to that of

$$\int^\infty \left[\frac{\Phi'(t)}{\Phi(t)^n} - \frac{(n-1)\Phi'(t)^2 t}{\Phi(t)^{n+1}}\right] dt .$$

But this integral is convergent as it is easily seen on account of (8).

Thus (7) is established.

Finally suppose that

$$\int_0^\alpha \varphi[h(s)]s^{n-1}\,\mathrm{d}s \leqslant 1$$

and define

$$F(\varrho) = \begin{cases} 0 \quad \text{for} \quad \varrho \geqslant \alpha, \\ \int_\varrho^\alpha [h(s) - h(\alpha)]\,\mathrm{d}s \quad \text{for} \quad 0 < \varrho \leqslant \alpha. \end{cases}$$

Now owing to (7) it is readily seen that $F(\varrho)$ has the desired properties.

This completes the argument.

References.

[1] L. Cesari, *Sulle funzioni assolutamente continue in due variabili*, Ann. Scuola Norm. Super. Pisa (2) **10**, 91-101 (1941).

[2] W. Stepanoff, *Sur les conditions de l'existence de la différentielle totale*, Rec. Math. Soc. Math. Moscou **32**, 511-526 (1925).