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Definitions of LEBESGUE area for surfaces in metric spaces. (**)

Introduction.

The problem of extending the notion of LEBESGUE area to the case of surfaces in an arbitrary metric space arose in [10]. It is the purpose of this paper to give a definition of area which is applicable to surfaces in any metric space and which is equivalent to the usual definition if the surface is in Euclidean space.

By the very nature of the definition of LEBESGUE area, a reasonable procedure for obtaining a generalization is first to define such an area for a BANACH space and then to obtain the general definition by mapping the metric space into a BANACH space. It will be noted that once the area of a triangle in a BANACH space is suitably defined, then the usual definition goes through verbatim. Certainly, any general definition should agree with this natural definition for surfaces in any BANACH space, not only three dimensional Euclidean space. The author has been able to show that his general definition agrees with this natural definition in case the BANACH space is finite dimensional.

It is clear that a LEBESGUE area exists for any suitable definition of the area of a triangle in a MINKOWSKI plane. Probably the most reasonable definition is to let π equal the area of the unit « circle » in any plane, but the writer was unable to show that this area, used by BUSEMANN [2], could be employed in conjunction with the LEBESGUE definition. Consequently use is made of an area which is equivalent to equating to four the area of a smallest parallelogram circumscribing the unit « circle ». However, BUSEMANN has since shown that the former definition can be used [3].

The results of this paper will be used to obtain certain intrinsic properties of LEBESGUE area.

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CHAPTER I.

Area of plane figures in a Banach space.

§ 1. – Isometric maps of separable metric spaces into m.

We shall concern ourselves in this section with certain isometric maps of separable metric spaces into the space m of bounded sequences [1].

The following theorems are proved in [1].

Theorem 1.1: *If the conjugate Banach space $\overline{E}$ is separable, then so in E.*

Theorem 1.2: *Suppose D is a separable metric space, (x, y) is the distance between $x, y \in D$, and $\{ x_1, x_2, \ldots \}$ is an everywhere dense sequence of points in D. Define the transformation T on D by $T(x) = \{ (x, x_i) - (x_i, x_1) \}$. Then T is an isometric map of D into m. The image of D is contained in a separable linear subspace of m.*

Lemma 1.1: *If E is a separable subset of a Banach space B, then there is a separable linear subspace of B which contains E.*

Theorem 1.3 (HAHN-BANACH): *If f is a linear functional defined in a subspace G of a vector space E, then there exists a linear functional F defined on E such that $F(x) = f(x)$ for $x \in G$ and $\| F \| = \| f \|_G$.*

Theorem 1.4: *For every x_0 in a Banach space B, there exists a linear functional f defined on B such that $f(x_0) = \| x_0 \|$ and $\| f \| = 1$.*

Definition 1.1: A linear functional of norm one will be termed a *normal linear functional.*

Theorem 1.5: *Let B be a separable Banach space. Then there exists a linear isometric transformation of B into a subset of m.*

Proof: Let $\{ x_1, x_2, \ldots \}$ be an everywhere dense sequence of points in B and let f_j be a normal linear functional such that $f_j(x_j) = \| x_j \|$. Define the trasformation T on B by $T(x) = \{ f_j(x) \}$.

Since each f_j is of norm one, $| f_j(x) | \leqq \| x \|$ for each j, so that $T(x) \in m$; since each f_j is linear, T is linear.

Now let x be any point in B and let $\{ x_{j_k} \}$ be a subsequence of $\{ x_j \}$ such that $\lim_{k\to\infty} x_{j_k} = x$. Then, for each k,

$$f_{j_k}(x) = f_{j_k}(x - x_{j_k}) + f_{j_k}(x_{j_k}) .$$

But now

$$| f_{j_k}(x - x_{j_k}) | \leqq \|x - x_{j_k}\| , \quad \lim_{k\to\infty} \|x - x_{j_k}\| = 0 ,$$

$$\lim_{k\to\infty} \Big| \, | f_{j_k}(x_{j_k}) | - \|x\| \, \Big| = \lim_{k\to\infty} \Big| \, \|x_{j_k}\| - \|x\| \, \Big| \leqq \lim_{k\to\infty} \|x_{j_k} - x\| = 0 .$$

Therefore

$$\lim_{k\to\infty} | f_{j_k}(x) | = \|x\| .$$

Hence

$$\sup_j | f_j(x) | \geqq \|x\| .$$

But since each f_j is of norm one, the equality must hold. From the linearity of T and B and the definition of distance in B, the theorem follows.

Definition 1.2: A sequence $\{ f_j \}$ of normal linear functionals in B is *mapping* if

$$\sup_j | f_j(x) | = \|x\| \quad \text{for all } x \text{ in } B.$$

Theorem 1.6: *Let B be a Banach space and suppose that $\{ f_j \}$ is a mapping sequence. Then the transformation T defined by $T(x) = \{ f_j(x) \}$ for x in B is a linear isometric transformation of B into m.*

Proof: The proof is essentially the same as that for theorem 1.5.

§ 2. – Lemmas on convex sets in the plane.

Definition 2.1: S is a convex set if, whenever P and Q belong to S, then the whole segment $\overline{PQ}$ is contained in S.

Lemma 2.1: *If S is a bounded, symmetric, closed convex set which has an interior point, then there is a parallelogram P containing S which has the smallest Euclidean area among all such parallelograms. The midpoint of each side of P is on S* [6].

We shall call such a parallelogram of minimum area a *smallest parallelogram* for S.

Lemma 2.2: *Let*

$$S = \underset{(x,y)}{E} [| \alpha_i x + \beta_i y | \leqq 1, \quad i = 1, ..., N] ,$$

and let $E(P)$ be the Euclidean area of a smallest parallelogram P of S. Then

$$E(P) = 4/\Delta\,, \qquad \Delta = \max_{i,j\leqslant n} \Delta_{ij}\,,$$

$$\Delta_{ij} = \begin{vmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{vmatrix}.$$

Remarks: Since i and j may occur in either order, $\Delta \geqq 0$, where the equality holds if and only if the sequences $\{\alpha_1, \ldots, \alpha_N\}$ and $\{\beta_1, \ldots, \beta_N\}$ are proportional, in which case P is the whole plane or an infinite strip. If we rule out this trivial case, the lemma follows without difficulty from the preceding lemma.

Lemma 2.3: *Let $\alpha = \{\alpha_i\}$ and $\beta = \{\beta_i\}$ be linearly independent points in m and*

$$S = \mathop{E}_{(x,y)} [|\alpha_i x + \beta_i y| \leqq 1\,, \quad i = 1, 2, \ldots]\,.$$

Then, if P is a smallest parallelogram of S,

$$E(P) = 4/\Delta\,, \qquad \textit{where} \qquad \Delta = \sup_{i,j} \Delta_{ij}\,,$$

$$\Delta_{ij} = \begin{vmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{vmatrix}.$$

Proof: For each N, let

$$S_N = \mathop{E}_{(x,y)} [|\alpha_i x + \beta_i y| \leqq 1\,, \quad i = 1, 2, \ldots, N]\,.$$

Let P_N be the smallest parallelogram for S_N, and let $\Delta_N = \max_{i,j\leqslant N} \Delta_{ij}$. Clearly $\Delta_N \to \Delta > 0$. Since each P_N contains S, $E(P) \leqq E(P_N)$ for each N, so that $E(P) \leqq 4/\Delta$.

Let $f(x, y) = \sup_i |\alpha_i x + \beta_i y|$. Then

$$S = \mathop{E}_{(x,y)} [f(x, y) \leqq 1]\,.$$

Now choose $\varepsilon > 0$, and define

$$f^*(x, y) = (1 - \varepsilon) f(x, y)\,,$$

$$S^* = \mathop{E}_{(x,y)} [f^*(x, y) \leqq 1]\,,$$

$$\Sigma = \mathop{E}_{(x,y)} [f^*(x, y) = 1]\,,$$

and let P^* be the parallelogram obtained from P by expanding it in the ratio $1 : (1 - \varepsilon)$. Clearly Σ is the boundary of S^*, P^* is a smallest parallelogram for S^*, and

$$E(P^*) = (1 - \varepsilon)^{-2} E(P)\,.$$

For each (x_0, y_0) on Σ, there is an i such that

$$|\alpha_i x_0 + \beta_i y_0| > f^*(x_0, y_0) = 1 .$$

Hence $|\alpha_i x + \beta_i y| > 1$ for all (x, y) sufficiently near (x_0, y_0). Thus, by the HEINE-BOREL theorem, there is a finite set $i_1, \ldots, i_k$ such that each (x, y) on Σ satisfies $|\alpha_{i_k} x + \beta_{i_k} y| > 1$ for some k. Therefore, for all sufficiently large N, S_N is interior to S^* and

$$4/\Delta_N = E(P_N) \leqq E(P^*) = (1-\varepsilon)^{-2} E(P) ,$$

or

$$E(P) \geqq \frac{4(1-\varepsilon)^2}{\Delta_N} .$$

The lemma fellows.

§ 3. – Linear subspaces of a Banach space.

Definition 3.1: Let x_0, x_1 and x_2 be linearly independent points in a BANACH space B. Then π is a plane in B if

$$x \in \pi \longleftrightarrow x = \sum_{i=0}^{2} a_i x_i \quad \text{and} \quad \sum_{i=0}^{2} a_i = 1 .$$

Definition 3.2: $\mathfrak{N}$ is a linear manifold in B if

$$x_1, \ldots, x_N \in \mathfrak{N} \rightarrow \sum_{i=1}^{N} a_i x_i \in \mathfrak{N} ,$$

for any real numbers $a_1, \ldots, a_N$.

The following lemmas are well known.

Lemma 3.1: *Let $\mathfrak{N}$ be a closed linear manifold in B such that there is an $x_0 \notin \mathfrak{N}$ and such that B is the smallest linear closed manifold containing x_0 and $\mathfrak{N}$. Then each point x in B has a unique representation on the form*

$$x = \lambda x_0 + x^* ,$$

where λ is a real number and $x^* \in \mathfrak{N}$.

Lemma 3.2: *Let $\mathfrak{N}$ be as in the preceding lemma. Then there exists a linear functional f such that*

$$x \in \mathfrak{N} \longleftrightarrow f(x) = 0 .$$

Definition 3.3: By a *hyperplane* H in B we mean the set of all points

x representable in the form $x = x_0 + x^*$ where x^* is in a closed linear manifold $\mathfrak{N}$ of the type in lemma 3.1.

Lemma 3.3: *For any hyperplane H, there exists a linear functional f and a constant c such that*

$$x \in H \longleftrightarrow f(x) = c .$$

Definition 3.4: N linear functionals $f_1, \ldots, f_N$ are *linearly independent* if

$$\sum_{i=1}^{N} c_i f_i = 0 \rightarrow c_i = 0 , \quad i = 1, \ldots, N .$$

Definition 3.5: K is a *subhyperplane* if

$$x \in K \longleftrightarrow [f_1(x) = c_1] \quad \text{and} \quad [f_2(x) = c_2] ,$$

where f_1 and f_2 are linearly independent linear functionals.

Definition 3.6: Two hyperplanes H and H' are *parallel* if there is a linear functional f and constants c and c', $c \neq c'$, such that H and H' are the loci of $f(x) = c$ and $f(x) = c'$. A similar definition applies to sub-hyperplanes.

Lemma 3.4: *Let π be a plane and H be a hyperplane. Then either $\pi \subset H$, $\pi \subset H'$, where H' is parallel to H, or π intersects H in a line.*

Proof: Suppose π consists of all points x representable in the form

$$x = x_0 + \lambda x_1 + \mu x_2 ,$$

where x_1 and x_2 are linearly independent, and suppose H is the locus of $f(x) = c$ where f is a linear functional. If H intersects π in some point $x_0 + \lambda_0 x_1 + \mu_0 x_2$, then

$$f(x_0 + \lambda_0 x_1 + \mu_0 x_2) = f(x_0) + \lambda_0 f(x_1) + \mu_0 f(x_2) = c .$$

Now, if $f(x_1) = f(x_2) = 0$, it is clear that $\pi \subset H$. Otherwise $x_0 + \lambda x_1 + \mu x_2 \in H$ if and only if

$$(\lambda - \lambda_0) f(x_1) + (\mu - \mu_0) f(x_2) = 0 .$$

Next suppose that H does not intersect π. Then $f(x_1) = 0$, $f(x_2) = 0$, and $\pi \subset H'$, determined by $f(x) = f(x_0)$.

Lemma 3.5: *Let π be a plane and N be a sub-hyperplane. Then either $\pi \subset N$, π does not intersect N, π intersects N in a line, or π intersects N in just one point.*

The proof of this lemma is similar to the preceding one.

Definition 3.7: If a plane π intersects a sub-hyperplane N in just one point, then N is *transversal* to π.

We state the following further obvious facts:

Lemma 3.6: *(a) Two hyperplanes are parallel if and only if they do not intersect.*

(b) If the sub-hyperplane N is transversal to the plane π, any parallel sub-hyperplane N' is also transversal to any plane π' obtained from π by a translation; moreover there is a unique sub-hyperplane N' parallel to N through any point of B.

(c) If π does not intersect the sub-hyperplane N, some sub-hyperplane N' parallel to N includes π or intersects it in a line.

Definition 3.8: If N is transversal to π, P is any point of B, N' is the unique sub-hyperplane parallel to N through P and P' is the intersection of N' with π, P' is called the *projection* of P on π parallel to N; if S is any set of points in B, the projection of S on π parallel to N consists of the totality of the projections on π parallel to N of the points of S.

Lemma 3.7: *If N is transversal to π and ψ is any other plane in B, the projection of ψ on π parallel to N is a point, a line, or the whole of π, the latter being the case if and only if N is also transversal to ψ.*

§ 4. – Plane measure in a Banach space B.

Let Σ_B be the set of all normal linear functionals of B and F any subset of Σ_B. If r, s and t are three points in B, define

$$A_B(F; r, s, t) = \frac{1}{2} \sup \begin{vmatrix} f(r) & g(r) & 1 \\ f(s) & g(s) & 1 \\ f(t) & g(t) & 1 \end{vmatrix}$$

for all f, g in F.

Theorem 4.1:

$$A_B(F; O, a(s-r)+b(t-r), \quad c(s-r)+d(t-r)) = |ad-bc| A_B(F; r, s, t) \geqq 0.$$

The proof follows immediately from properties of determinants and linear functionals.

Definition 4.1: $A_B(\Sigma_B; r, s, t)$ is the *area* of the triangle Δ in B with vertices r, s and t. We may write $A_B(F, \Delta)$ for $A_B(F; r, s, t)$ and $A_B(\Delta)$ for $A_B(\Sigma_B, \Delta)$.

By theorem 4.1, it follows that if we introduce an affine coördinate system

T on π, the area of any triangle in π is merely some constant multiple of the Euclidean area of the corresponding triangle in E_2, the Euclidean plane. The constant depends upon π and T. Hence there is a unique LEBESGUE measure of sets on π which agrees with the area defined above for triangles. Since the unit circle in π clearly corresponds under T to a bounded, closed, convex set S symmetric to the origin, which is an interior point, the existence of a smallest parallelogram for the unit circle in π follows from lemma 2.1.

We shall denote by π_{fg} the parallelogram (or infinite set) cut out on π by the hyperplanes $f(x) = +1$, $g(x) = +1$, where f and g are normal linear functionals.

Lemma 4.1: *If P is a smallest parallelogram for the unit circle in the plane π, then there exist normal linear functionals f and g such that $P = \pi_{fg}$. Furthermore, there exist points e_1 and e_2 in π such that e_1 and e_2 are linearly independent, $\|e_1\| = \|e_2\| = f(e_1) = g(e_2) = 1$, and $f(e_2) = g(e_1) = 0$.*

Proof: We can certainly define normal linear functionals f^* and g^* in the plane π such that P is bounded by the lines

$$f^*(x) = \pm 1, \qquad g^*(x) = \pm 1.$$

Now use the HAHN-BANACH theorem to extend f^* and g^* to the whole space. The remainder of the lemma follows from lemma 2.1.

Lemma 4.2: *Let e_1 and e_2 be linearly independent points in π. Associate coordinates (x, y) with $xe_1 + ye_2$. Then*

$$E(\pi_{f_1 f_2}) = 4/\Delta_f, \qquad \Delta_f = \begin{vmatrix} f_1(e_1) & f_2(e_1) \\ f_1(e_2) & f_2(e_2) \end{vmatrix},$$

where $E(\pi_{f_1 f_2})$ is the Euclidean area of the parallelogram in E_2 corresponding to $\pi_{f_1 f_2}$.

Proof: The lines bounding $\pi_{f_1 f_2}$ are

$$x f_1(e_1) + y f_1(e_2) = \pm 1, \qquad x f_2(e_1) + y f_2(e_2) = \pm 1.$$

The lemma follows by analytic geometry.

Theorem 4.2: *The area of a smallest parallelogram (for the unit circle) in any plane π is 4.*

Proof: We may as well assume that the plane π passes through the origin. Let $P = \pi_{f_1 f_2}$ be a smallest parallelogram with e_1 and e_2 linearly independent points in π such that $f_1(e_1) = f_2(e_2) = \|e_1\| = \|e_2\| = 1$ and $f_1(e_2) = f_2(e_1) = 0$.

Now, since the area of P equals 8 times the area of the triangle with vertices O, e_1 and e_2, it follows that

$$\text{area } P = 8A_B(O, e_1, e_2) \geqq 4 \begin{vmatrix} f_1(e_1) & f_2(e_1) \\ f_1(e_2) & f_2(e_2) \end{vmatrix} = 4 .$$

Next, if g_1 and g_2 are any two normal linear functionals, $\pi_{g_1g_2}$ certainly contains the unit circle, and so, associating π with E_2 as in the last lemma,

$$E(\pi_{g_1g_2}) \geqq E(\pi_{f_1,f_2})$$

since $\pi_{f_1f_2}$ is a smallest parallelogram by hypothesis. Hence, by lemma 4.2,

$$\Delta_f \geqq \Delta_g , \quad \text{where} \quad \Delta_g = \begin{vmatrix} g_1(e_1) & g_2(e_1) \\ g_1(e_2) & g_2(e_2) \end{vmatrix} .$$

Finally,

$$\sup_{g_1, g_2 \in \Sigma_B} \Delta_g \leqq 1 ,$$

from which the theorem follows.

From this theorem, it can be seen that for any triangle Δ and any affine coordinate system of π,

$$A_B(\Delta) = \frac{4E(\Delta)}{E(P)},$$

where $E(\Delta)$ and $E(P)$ are the Euclidean areas of the sets corresponding to Δ and a smallest parallelogram, respectively.

Theorem 4.3: *Defining the area of a smallest parallelogram for the unit circle to be four is equivalent to defining area by definition* 4.1.

This follows immediately from the preceding theorem.

Theorem 4.4: *Area as defined in this section agrees with Eulcidean area for a Euclidean plane.*

Proof: The Euclidean area of the smallest parallelogram circumscribing the unit circle is four.

Theorem 4.5: *If $A = \{A_i\}$, $B = \{B_i\}$, and $C = \{C_i\}$ are any three points in m, then the area of the triangle with vertices A, B, and C is*

$$\frac{1}{2} \sup_{i,j} \begin{vmatrix} A_i & A_j & 1 \\ B_i & B_j & 1 \\ C_i & C_j & 1 \end{vmatrix} .$$

Proof: Set up a coordinate system in the plane so that the point $\lambda(B - A) + \mu(C - A)$ will have coordinates (λ, μ). The convex set in E_2

corresponding to the unit circle about A is given by

$$S = E[\,|\,\lambda(B_i - A_i) + \mu(C_i - A_i)\,| \leqq 1\,, \quad i = 1, 2, \ldots]\,.$$

Now use lemmas 2.3, 4.2 and theorem 4.3.

Theorem 4.6: *In any Banach space B which has a mapping sequence $\{f_i\}$,*

$$A_B(x, y, z) = \frac{1}{2} \sup_{i,j} \begin{vmatrix} f_i(x) & f_j(x) & 1 \\ f_i(y) & f_j(y) & 1 \\ f_i(z) & f_j(z) & 1 \end{vmatrix}.$$

Proof: First map B into m linearly and isometrically by means of the mapping sequence. The unit circle and smallest parallelogram are carried into a unit circle and smallest parallelogram. Now use theorem 4.2 and theorem 4.5.

Definition 4.3: A sub-hyperplane N in B is *area-perpendicular* $(a\text{–}p)$ to a plane π in B if N is transversal to π and if any triangle T in B projects parallel to N onto a triangle T' in π such that area $T' \leqq$ area T.

Theorem 4.7: *If f and g are normal linear functionals such that α_{fg} is a smallest parallelogram for the unit circle in π, then the sub-hyperplane N determined by $f(x) = 0$, $g(x) = 0$ is a-p to π.*

Proof: From the proofs of lemmas 3.4 and 3.6 and theorem 4.2, it follows that f and g are linearly independent and that N intersects the plane π' parallel to π through the origin only at the origin. Thus N is transversal to π. If the projection of a triangle in B on π parallel to N is a point or a segment, the theorem is evident. Otherwise N is also transversal to the plane of the triangle (lemma 3.7).

Now consider a smallest parallelogram $\overline{P}$ in the plane ψ through the origin parallel to any given plane in B to which N is transversal. The area of the parallelogram P cut out on ψ by $f(x) = \pm 1$, $g(x) = \pm 1$ will be greater than or equal to the area of $\overline{P}$, since $\overline{P}$ is a smallest parallelogram. But if T is any triangle in ψ and T' its projection in π, then

$$\frac{\text{area } T}{\text{area } T'} = \frac{\text{area } P}{\text{area } \overline{P}} \geqq 1$$

since P projects into a smallest parallelogram P^* in π and P^* and P both have area 4.

Theorem 4.8: *The area of a parallelogram spanned by vectors of length l_1 and l_2 does not exceed $2l_1l_2$.*

The theorem follows directly from definition 4.1.

Theorem 4.9: *Suppose that there is a linear transformation from a plane π in a Banach space B onto a plane $\bar{\pi}$ in a Banach space $\bar{B}$ (possibly B) such that*

$$e_0 + ae_1 + be_2 \longleftrightarrow \bar{e}_0 + a\bar{e}_1 + b\bar{e}_2$$

where e_0, e_1 and e_2 are in π, and $\bar{e}_0$, $\bar{e}_1$, and $\bar{e}_2$ are in $\bar{\pi}$. If $\|a\bar{e}_1 + b\bar{e}_2\| \leqq$ $\leqq k\,\|ae_1 + be_2\|$ for all a, b then

$$A_{\bar{B}}(\bar{x}, \bar{y}, \bar{z}) \leqq k^2 A_B(x, y, z)$$

where x, y and z are in π, $\bar{x}$, $\bar{y}$ and $\bar{z}$ are the corresponding points in $\bar{\pi}$.

Proof: The unit circle in $\bar{\pi}$ (with center at $\bar{e}_0$) corresponds to a set in π which contains the circle in π (with center at e_0) of radius $1/k$. Thus a smallest parallelogram in $\bar{\pi}$ corresponds to a parallelogram P in π which contains this circle. Hence

$$\text{area } P \geqq 4/k^2\,,$$

$$\frac{A_{\bar{B}}(\bar{x}, \bar{y}, \bar{z})}{A_B(x, y, z)} = \frac{\text{area } P}{\text{area } \bar{P}} \leqq k^2\,.$$

CHAPTER II.

Lebesgue area in a Banach space.

§ 5. – Preliminary definitions.

By *curve* or *surface* we shall always mean FRÉCHET curve or FRÉCHET surface. If a continuous function x is a representation of a surface S, and if the range of x, $R(x)$, is in a metric space E, then we shall say that x and S are *in* E. By S: $[x, R]$, we shall understand that x is a continuous function whose domain, $D(x)$, includes R and that S is the surface determined by $x|\,R$.

If x and y are two continuous functions and S and T are two surfaces in a metric space E, then $D(x, y)$ and $D(S, T)$ are the FRÉCHET distances between x and y and between S and T, respectively.

It is convenient to define a PEANO or GEÖCZE area in terms of an essential multiplicity function. For the two dimensional case, RADÓ and REICHEL–

DERFER [15] have defined one such function and CESARI [4] has defined another, which he calls the *characteristic* function. FEDERER [7] has discussed a more general case where the dimension is not limited to two. For the two dimensional case to be considered, the functions of CESARI and FEDERER are equivalent [7] and differs from the other at most on a countable set [14].

Let R be a JORDAN domain, E_2 the Euclidean plane, and C the set of all continuous functions on R to E_2 which is metrized by the function d such that

$$d(f, g) = \sup_{x \in R} \| f(x) - g(x) \| , \qquad f, g \in C .$$

Then let $M(f, y)$ be the essential multiplicity with which f assumes the value y in the sense of FEDERER. We could equally well use either of the other functions.

Let L_2 be the two dimensional LEBESGUE measure on E_2.

Define the GEÖCZE area, G, on C by

$$G(f) = \int_{E_2} M(f, y)\, \mathrm{d}L_2 y , \qquad f \in C .$$

We shall make use of the standard theorems relating to LEBESGUE and GEÖCZE area in the plane [4, 5, 7, 13].

If A is a set, then $\bar{A}$, A^*, $\delta(A)$ and A^0 will denote the closure, boundary, diameter and interior, respectively, of A. If A is contained in a plane π, then $| A |$ is the LEBESGUE measure of A in π.

Definition 5.1: f is of *bounded variation in the sense of* CESARI, B.V.C., if $G(f) < + \infty$.

Definition 5.2: f is *absolutely continuous in the sense of* CESARI, A.C.C., if

(*a*) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sum_{i=1}^{n} G(f \mid \pi_i) < \varepsilon$, whenever $\pi_1, \ldots, \pi_n$ are simple polygonal regions without common interior points such that $\sum_{i=1}^{n} | \pi_i | < \delta$;

(*b*) if A is any simple polygonal region in R, then $G(f \mid A) = \sum_{i=1}^{n} G(f \mid \pi_i)$ for any subdivision of A into simple polygonal regions $\pi_1, \ldots, \pi_n$.

Now let us suppose that a Cartesian coordinate system is introduced on E_2.

Definition 5.3: Let f be B.V.C. . Then, if the limit exists, define

$$J(f; u, v) = \lim_{\delta(q) \to 0} \frac{G(f \mid q)}{| q |} ,$$

where q is any oriented square in R (sides parallel to the coordinate axes) containing (u, v) in its interior. J is the *generalized Jacobian* of CESARI [5].

Definition 5.4: Let x be a function defined on the interval $[a, b]$ with range in an arbitrary metric space D. Then the *variation* of x, $V_a^b(x)$, is defined by

$$V_a^b(x) = \sup \sum_{i=1}^{n} (x(t_i), x(t_{i-1}))$$

for all subdivision $a = t_0 < t_1 < \ldots < t_n = b$.

Definition 5.5: If x is as in Definition 5.4, then x is *absolutely continuous*, A.C., if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_i (x(t_i''), x(t_i')) < \varepsilon ,$$

whenever $\sum_i | t_i'' - t_i' | < \delta$, the intervals $[t_i'', t_i']$ have no interior points in common and are contained in $[a, b]$.

With these definitions of bounded variation and absolute continuity of a function of one real variable in any metric space D, we extend verbatim the definitions of bounded variation and absolute continuity in the sense of TONELLI, B.V.T. and A.C.T., to apply to functions of two variables with range in D.

We shall make use of the PETTIS integral [11] for integrating a function with range in a BANACH space.

Lemma 5.1: *If x is a continuous function defined on a region R with range in a Banach space B, and if E is a measurable subset of R, then there exists a point x_E in B such that*

$$f(x_E) = \iint_E f(x(s, t))\, ds\, dt$$

for every linear functional f in B. By definition

$$\iint_E x(s, t)\, ds\, dt = x_E .$$

Definition 5.6: Let x be a continuous function defined on a JORDAN region R with range in a BANACH space B, and let D be any domain whose closure is contained in R^0. Let δ be the shortest distance between D^* and R^*. Then define x_h on D by

$$x_h(u, v) = \frac{1}{4h^2} \int_{u-h}^{u+h} \int_{v-h}^{v+h} x(s, t)\, ds\, dt , \qquad 0 < h < \delta , \qquad (u, v) \in D .$$

As a consequence of the preceding lemma, we have

$$f(x_h(u, v)) = \frac{1}{4h^2}\int_{u-h}^{u+h}\int_{v-h}^{v+h} f(x(s, t))\, ds\, dt .$$

Definition 5.7: Let $f \in C$ be defined by $f\colon (u, v) \to (x(u, v), y(u, v))$. Then f is of class L [10] if

(a) x_u, x_v, y_u, y_v are defined almost everywhere in R^0,

(b) $(x_u y_v - x_v y_u)$ is summable over R^0,

(c) $\lim\limits_{h\to 0} \iint\limits_{\mathcal{R}} \big|\, | x_u y_v - x_v y_u | - | x_{hu} y_{hv} - x_{hv} y_{hu} | \,\big|\, du\, dv = 0$

for every oriented rectangle $\mathcal{R} \subset R^0$.

§ 6. – Peano area in a Banach space.

The exposition of PEANO and LEBESGUE area in a BANACH space will be complicated by the introduction of pseudo areas. At times it seems to be necessary to approximate to the LEBESGUE area of a surface by means of pseudo LEBESGUE areas of the same surface rather than by area of approximating surfaces. The study of these pseudo LEBESGUE areas requires the use of corresponding PEANO areas.

Let x be a continuous function with domain a JORDAN region R in the (u, v) plane and range in a BANACH space B. If f and g are linear functionals, let x^{fg} be the continuous transformation from R to E_2 defined by

$$x^{fg}\colon\quad (u, v) \to (s, t) = (f(x(u, v)), g(x(u, v))) , \qquad (u, v) \in R,$$

where s and t are cartesian coordinate in E_2.

If Σ_B is the set of all normal linear functionals on B and $F \subset \Sigma_B$, let

$$G_B^{fg}(x, D) = \iint\limits_{E_2} M(x^{fg} \mid \bar{D}; s, t)\, ds\, dt$$

and

$$G_B(F, x, D) = \sup_{f,g \in F} G^{fg}(x, D) ,$$

where D is any domain contained in R^0.

Definition 6.1: Let $P_B(F, x) = \sup\limits_{\sigma} \sum\limits_{D \in \sigma} G_B(F, x, D)$ where σ is any finite set of disjoint domain in R^0. Let $P_B(x) = P_B(\Sigma_B, x)$. $P_B(x)$ is the *Peano area* of x in B.

Theorem 6.1: *Let B' be any closed linear subspace of B containing x. Then $P_{B'}(x) = P_B(x)$.*

Proof: If $f \in \Sigma_B$, let $f' = f/\|f\|_{B'}$. Then $f' \mid B'$ is a normal linear functional on B'. If $f, g \in \Sigma_B$, let

$$s' = s\ \|f\|_{B'}, \qquad t' = t\ \|g\|_{B'}.$$

Then

$$M(x^{fg} \mid \overline{D}; s, t) = M(x^{f'g'} \mid \overline{D}; s', t')$$

for any domain $D \subset R^0$. Hence

$$\iint_{E_2} M(x^{fg} \mid \overline{D}; s, t)\, ds\, dt = \frac{1}{\|f\|_{B'} \cdot \|g\|_{B'}} \iint_{E_2} M(x^{f'g'} \mid \overline{D}; s', t')\, ds'\, dt'$$

in case the integrands are summable. In any event

$$G_B^{fg}(x, D) = G_{B'}^{f'g'}(x, D),$$

and so

$$P_B(x) \leqq P_{B'}(x).$$

On the other hand, if f' and $g' \in \Sigma_{B'}$, then they may be extended by the Hahn-Banach theorem to f and $g \in \Sigma_B$. Obviously

$$G_B^{fg}(x, D) = G_{B'}^{f'g'}(x, D)$$

and so

$$P_{B'}(x) \leqq P_B(x).$$

Theorem 6.2: *For each x, there exists a denumerable subset $\overline{F} \subset F$ such that $P_B(\overline{F}, x) = P_B(F, x)$.*

Proof: Let $D_1^{(n)}, \ldots, D_{j_n}^{(n)}$ be a system of disjoint domains in R^0 such that

$$P_B(F, x) < \sum_{p=1}^{j_n} G_B(F, x, D_p^{(n)}) + 1/n, \qquad n = 1, 2, \ldots$$

and let $f_r^{(n)}$, $r = 1, \ldots, l_n$ be elements in F such that

$$G_B(F, x, D_p^{(n)}) < \max_{b,c} G_B^{f_b^{(n)} f_c^{(n)}}(x, D_p^{(n)}) + 1/nj_n.$$

Now combine the countable number of countable sequences $\{f_r^{(n)}\}$, $n=1, 2, \ldots$, $r = 1, \ldots, l_n$, into a single sequence $\{f_k\}$. Then $\overline{F} = \{f_k\}$ satisfies the condition of the theorem.

If F is denumerable, then we shall suppose that the elements of F are arranged in a single sequence and that F_n consists of the first n elements of F. Otherwise, F_n consists of the first n elements of $\overline{F}$.

Theorem 6.3: $P_B(F, x) = \lim_{n \to \infty} P_B(F_n, x)$.

In case F is a finite set, then all of the theorems that we require are proved for P_B exactly as the corresponding theorems are proved for the lower area of RADÓ. If F is not finite, these theorems can be proved with the aid of Theorem 6.3. In particular, if $D(x, y) = 0$, then $P_B(F, x) = P_B(F, y)$, and so we can define the PEANO area of a FRÉCHET surface S, $P_B(F, S)$, to be $P_B(F, x)$, where x is any representation of S.

Definition 6.2: If $F \subset \Sigma_B$ and $t \in B$, let $\|t\|_F = \sup_{f \in F} | f(t) |$. We shall call $\|t\|_F$ the F-form of t. Evidently, if F is a mapping sequence then $\|t\|_F = \|t\|$, for each $t \in B$. Define $d_F(x, y)$, $D_F(x, y)$, and $D_F(S, T)$ in terms of the F-norm as $d(x, y)$, $D(x, y)$, and $D(S, T)$ were defined in terms of the norm, where x and y are continuous functions, and S and T are surfaces in B. If $D_F(S_n, S) \to 0$, then we shall write $S_n \underset{F}{\to} S$.

Definition 6.3: Let

$$J'(x^{fg}; u, v) = \begin{cases} J(x^{fg}; u, v) & \text{if defined,} \\ 0 & \text{otherwise,} \end{cases}$$

$$W(F; x; u, v) = \sup_{f,g \in F} J'(x^{fg}; v, u) .$$

Let $\mathcal{J}(x^{fg}; u, v)$ be the value of the ordinary Jacobian of x^{fg} at (u, v). Then define $\mathcal{J}'(x^{fg}; u, v)$ and $\mathcal{W}(F; x; u, v)$ by means of $\mathcal{J}(x^{fg}; u, v)$ as $J'(x^{fg}; u, v)$ and $W(F; x; u, v)$ were defined by means of $J(x^{fg}; u, v)$.

Definition 6.4: x is of class $C_B(F)$ on R if

(*a*) x^{fg} is A.C.C. for $f, g \in F$,

(*b*) $W(F; x)$ is summable on R^0.

For brevity, we write C_B for $C_B(\Sigma_B)$ and $W(x)$ for $W(\Sigma_B, x)$.

Theorem 6.4: *If* S: $[x, R]$ and $P_B(F, S) < +\infty$, *then* $J(x^{fg}; u, v)$ *exists almost everywhere in* R^0 *for* $f, g \in F$, *and*

$$\iint_{R^0} W(F; x; u, v)\, du\, dv \leqq P_B(F, S) .$$

Theorem 6.5: *If* $x \in C_B(F)$ *on* R, *then* $P_B(F, x) < +\infty$, $J(x^{fg}; u, v)$ *exists almost everywhere in* R^0 *for* $f, g \in F$, *and*

$$\iint_{R^0} W(F; x; u, v)\, du\, dv = P_B(F, x) .$$

Definition 6.5: x is of class C' on R if

(*a*) $f(x)$ is of class C' on R for any $f \in \Sigma_B$,

(*b*) $\mathcal{W}(x)$ is summable on R^0.

Definition 6.6: A continuous function x is quasilinear on R, if R is a simple polygonal region and if there exists a subdivision of R into a finite number of triangles on each of which x is linear.

Definition 6.7: Suppose x is quasilinear on R, being linear on each triangle $\Delta_1, \ldots, \Delta_n$. We define

$$E_B(F, x) = \sum_{i=1}^{n} A_B(F, x(\Delta_i)) .$$

It is clear that $E_B(F, x)$ is independent of the division of R into Δ_i as long as x is linear on each Δ_i.

Definition 6.8: A surface P is a polyhedron if it admits a quasilinear representation x. Let $E_B(F, P) = E_B(F, x)$.

Definition 6.9: x is Lipschitzian on R if for some $M > 0$, $\|x\ (U, V) - x(u, v)\| < M\ \|(U, V) - (u, v)\|$ for every (U, V), $(u, v) \in R$.

Theorem 6.6: *If x is of class C', quasilinear or Lipschitzian on R, then*

$$P_B(F, x) = \iint_{R^0} \mathcal{W}(F; x; u, v)\, du\, dv .$$

If x is quasilinear, then $P_B(F, x) = E_B(F, x)$.

It follows that $E_B(F, P)$ is a lower semi-continuous function in the class of polyhedra. If $F = \Sigma_B$, then we write $E_B(P)$ for $E_B(F, P)$ and call $E_B(P)$ the *elementary area* of P.

§ 7. – Lebesgue area in a Banach space.

Lemma 7.1: *If S is a surface, then there exists a sequence $\{P_n\}$ of polhyedra such that $P_n \to S$.*

Definition 7.1: For each $F \subset \Sigma_B$, we define an F-Lebesgue area of S,

$$L_B(F, S) = \inf \{ \liminf_{n \to \infty} E_B(F, P_n) \}$$

for all sequences $\{P_n\}$ of polyhedra such that $P_n \underset{F}{\to} S$. We define the Lebesgue *area* of S by $L_B(S) = L_B(\Sigma_B, S)$.

Theorem 7.1: *If S is any surface, then there exists a sequence $\{P_n\}$ of polyhedra such that $P_n \underset{F}{\rightarrow} S$ and $E_B(F, P_n) \rightarrow L_B(F, S)$.*

By definition, $L_B(F, S)$ is independent of any representation. If x is a representation of S, then we put $L_B(F, x) = L_B(F, S)$.

If S is a surface in B and $\{P_n\}$ is a sequence of polyhedra in B such that $P_n \rightarrow S$ and $E_B(P_n) \rightarrow L_B(S)$, then there obviously exists a separable closed linear subspace B' of B such that S and P_n are all in B', $n = 1, 2, \ldots$. Since $E_{B'}(P_n) = E_B(P_n)$ and $P_n \rightarrow S$ considered as surfaces in B', it follows that $L_{B'}(S) \leqq \liminf_{n\to\infty} E_{B'}(P_n) = \liminf_{n\to\infty} E_B(P_n) = L_B(S)$. On the other hand, $L_B(S) \leqq L_{B'}(S)$ directly from the definition. Hence, since $P_{B'}(S) = P_B(S)$, there is no loss in generality in always supposing that B is separable.

Theorem 7.2: *If F contains a mapping sequence of B, then $L_B(F, S) = L_B(S)$.*

Proof: By Theorem 4.6, $E_B(F, P) = E_B(P)$ for any polyhedron P in B. Furthermore, by definition 1.2, the F-form equals the norm.

Many of the standard theorems of LEBESGUE area are now proved in the usual way. In particular, if x is of class C', then $L_B(F, x) = \iint_{R^0} \mathcal{W}(F; x; u, v)\, du\, dv$.

Definition 7.2: x is of class $L_B(F)$ on R if for $f, g \in F$,

(*a*) $\mathcal{J}(x^{fg}; u, v)$ exists almost everywhere in R^0,

(*b*) $\mathcal{W}(F, x)$ is summable over R^0,

(*c*) $\lim_{h\to 0} \iint_T | \mathcal{J}(x^{fg}; u, v) - \mathcal{J}(x_h^{fg}; u, v) |\, du\, dv = 0$

for every oriented rectangle $T \subset R^0$. We write L_B for $L_B(\Sigma_B)$.

Theorem 7.3: *If $x \in L_B(F)$ on R and F is finite, then*

$$P_B(F, x) = L_B(F, x) = \iint_{R^0} \mathcal{W}(F; x; u, v)\, du\, dv\,.$$

Definition 7.3: x is of class $\overline{L}_B$ on R if x satisfies (*a*) and (*b*) of Definition 7.2 and if for each oriented rectangle $T \subset R^0$,

$$\lim_{h\to 0} \iint_T | \mathcal{W}(x; u, v) - \mathcal{W}(x_h; u, v) |\, du\, dv = 0\,.$$

The reader will observe that if x is Lipschitzian on R, then x is also of class $\overline{L}_B$ on R.

Theorem 7.4: *If* $x \in \bar{L}_B$ *on* R, *then* $L_B(x) = P_B(x) = \iint_{R_0} \mathcal{W}(x;\, u,\, v)\, du\, dv$.

Theorem 7.5: *Let* $x \in L_B$. *If there exists a denumerable set* $F \subset \Sigma_B$ *such that* $L_B(x) = \lim_{n \to \infty} L_B(F_n, x)$, *then* $x \in \bar{L}_B$ *on* R.

Theorem 7.6: *If* B *is finite dimensional and* F *is a mapping sequence for* B, *then* $L_B(S) = \lim_{n \to \infty} L_B(F_n, S)$ *for every surface* S *in* B.

Proof: Let σ be the unit sphere in B and σ^* its boundary. Choose ε, $0 < \varepsilon < 1/2$. Since σ^* is compact, there exist points $x_1, \ldots, x_k$ of σ^* such that $\| x - x_i \| < \varepsilon$ for every $x \in \sigma^*$ and some i. Now choose $f_{j_i} \in F$ such that $| f_{j_i}(x_i) | > 1 - \varepsilon$. Then $| f_{j_i}(x) | = | f_{j_i}(x - x_i) + f_{j_i}(x) | \geqq 1 - 2\varepsilon$. Hence for each $x \in B$ there is an i such that $\| x \| \leqq (1 - 2\varepsilon)^{-1} | f_{j_i}(x) |$. This, and Theorem 4.9, imply that if P is any polyhedron in B, $E_B(P) \leqq (1 - 2\varepsilon)^{-2} E_B(F_n, P)$ for n larger than any of the j_i, $i = 1, \ldots, k$. The theorem follows.

Chapter III.

Lebesgue area in m.

§ 8. – General theorems.

Lemma 8.1: *Let* $\{ x^i \}$ *be a uniformly bounded equicontinuous sequence of functions defined on a Jordan region* R. *Then for each* $\varepsilon > 0$, *there exists a finite set* $x^{i_1}, \ldots, x^{i_k}$ *such that for each* i *and all* (u, v) *in* R, *there exists a* j *for which*

$$| x^i(u, v) - x^{i_j}(u, v) | < \varepsilon .$$

Proof: The sequence is conditionally compact in the space of continuous functions defined on R.

Lemma 8.2: *A necessary and sufficient condition that a function* $x = \{ x^i \}$ *be continuous in* m *is that the sequence* $\{ x^i \}$ *be uniformly bounded and equicontinuous.*

The proof follows immediately from the definition of m.

If $y = \{ y^i \}$ is any point in m, let $g_i \in \Sigma_m$ be defined by $g_i(y) = y^i$. Let $G_n = \{ g_1, \ldots, g_n \}$ and $G = \{ g_i,\ i = 1, 2, \ldots \}$. By Definition 1.2, G is a mapping sequence so, by Theorem 7.2, $L_m(x) = L_m(G, x)$. In computing

$L_m(x)$, the use of this equality is preferable to the use of the original definition $L_m(x) = L_m(\Sigma_m, x)$.

Definition 8.1: If T maps G into G', a subset of G, and x is any continuous function on R to m, then define the function Tox on R to m by

$$g(Tox) = \begin{cases} g(x) & \text{if } g \in G', \\ 0 & \text{otherwise}. \end{cases}$$

Define the function Tx on R to m by

$$g(Tx) = Tg(x) \quad \text{for each} \quad g \in G.$$

Obviously Tox and Tx are continuous.

Assume, in the following lemmas, that T and G' are as in this definition amd that x and y are continuous functions.

Lemma 8.3: $D(Tx, Ty) = D(Tox, Toy) \leqq D(x, y)$.

Definition 8.2: If S is a surface, then let ToS and TS be the surfaces defined by Tox and Tx where x is any representation of S.

We notice that if P is a polyhedron, then TP and ToP are polyhedra and that $E_m(TP) = E_m(ToP) = E_m(G', P) \leqq E_m(P)$.

Lemma 8.4: $L_m(TS) = L_m(ToS) = L_m(G', S) \leqq L_m(S)$ *for any surface* S *in* m.

Proof: The first equality follows easily from the preceding paragraph.

Let $P_n \underset{G'}{\rightrightarrows} S$, $E_m(G', P_n) \to L_m(G', S)$, each P_n being a polyhedron. Then $ToP_n \underset{G'}{\rightrightarrows} ToS$ and $E_m(ToP_n) = E_m(G', P_n)$. Hence

$$L_m(ToS) \leqq \liminf_{n\to\infty} E_m(G', P_n) = L_m(G', S).$$

Next, let $\pi_n \underset{G'}{\rightrightarrows} ToS$, $E_m(\pi_n) \to L_m(ToS)$, each π_n being a polyhedron. Then $To\pi_n \underset{G'}{\rightrightarrows} ToS$ and $E_m(To\pi_n) \to L_m(ToS)$. Now let x be a representation of S on Q: $0 \leqq u, v \leqq 1$, and let z_n be a quasilinear representation of $T\pi o_n$ on Q such that $z_n \underset{G'}{\rightrightarrows} Tox$, this being possible exactly as in E_3. Furthermore, there is no loss in generality in supposing that the maximum diameter of any triangle of linearity of z_n is less than $1/n$. Now define the quasilinear function w_n on Q such that (i) $Tow_n = z_n$, (ii) if $g \notin G'$, then $g(w_n)$ agrees with $g(x)$ on vertices of the triangles of linearity of z_n, and (iii) w_n is linear on these triangles of linearity.

It is clear that $w_n \rightrightarrows x$ and that $E_m(G', w_n) = E_m(z_n)$. Hence $L_m(G', S) = L_m(G', x) \leqq \liminf_{n\to\infty} E_m(G', w_n) = \liminf_{n\to\infty} E_m(z_n) = \liminf_{n\to\infty} E_m(To\pi_n) = L_m(ToS)$.

Theorem 8.1: *$L_m(S) = \lim_{n\to\infty} L_m(G_n, S)$ for any surface S in m.*

Proof: We need only show that $L_m(S) \leqq \liminf_{n\to\infty} L_m(G_n, S)$. Suppose $L_m(S) < +\infty$. Choose $\varepsilon > 0$ and take $\delta > 0$ such that $L_m(\mathcal{S}) > L_m(S) - \varepsilon$ for all surfaces $\mathcal{S}$ with $D(S, \mathcal{S}) < \delta$.

By Lemma 8.1, there exists an n_ε and a map T_ε: $G \to G_{n_\varepsilon}$ such that $D(S, T_\varepsilon S) < \delta$. Hence $L_m(G_{n_\varepsilon}, S) = L_m(T_\varepsilon S) > L_m(S) - \varepsilon$. If $L_m(S) = +\infty$, the obvious modification completes the proof.

Theorem 8.2: *If x is of class L_m on R, then $L_m(x) = \iint_{R^0} \mathcal{W}(x; u, v)\, du\, dv$.*

Theorem 8.3: *Let x_k, $k = 0, 1, 2, \ldots$ be defined on R and suppose that $x_k^i \rightrightarrows x_0$ for each $i = 1, 2, \ldots$. Then $L_m(x_0) \leqq \underline{\lim}\,\inf_{k\to\infty} L_m(x_k)$.*

Proof: By Theorem 8.1, $L_m(x_k) = \lim_{k\to\infty} L_m(G_n, x_k)$ for each k. Hence

$$L_m(x_0) = \lim_{n\to\infty} \{ L_m(G_n, x_0) \} \leqq \lim_{n\to\infty} \{ \liminf_{k\to\infty} L_m(G_n, x_k) \} \leqq$$

$$\leqq \lim_{n\to\infty} \{ \liminf_{k\to\infty} L_m(x_k) \} = \liminf_{k\to\infty} L_m(x_k) .$$

§ 9. – Kolmogoroff's principle.

The development of this section was suggested by « The Kolmogoroff principle for the Lebesgue area » by R. G. Helsel and E. J. Mickle.

If x, P, and $U(P)$ are points in m, then denote the n^{th} component by x_n, P_n, and $U(P)_n$, respectively.

Theorem 9.1 (Extension theorem): *Let U be a transformation with domain $E \subset m$ and range $\overline{E} \subset m$ such that $\| U(P) - U(Q) \| \leqslant k \, \| P - Q \|$ for P and Q any two points in E. Then U can be extended to a Lipschitzian transformation U^* defined for all of m, with range included in m, such that $\| U^*(x) - U^*(y) \| \leqq k \, \| x - y \|$, where x and y are any two points in m.*

If we define $T_n(x) = \inf_{P\in E} (U(P)_n + k \, \|P - x\|)$, then we may define U^* by $U^*(x) = \{ T_n(x) \}$ for all x in m.

Proof: We must show (i) the range of U^* is contained in m, (ii) $P \in E \to$ $\to U^*(P) = U(P)$, (iii) $\| U^*(x) - U^*(y) \| \leqq k \, \| x - y \|$.

To prove (i), let x be any point in m and P be any point in E. Then

$$|T_n(x) - T_n(P)| = |\inf_{Q\in E}(U(Q)_n + k\|Q-x\|) - \inf_{Q\in E}(U(Q)_n + k\|Q-P\|)| \leqq$$

$$\leqq \sup_{Q\in E}|(U(Q)_n + k\|Q-x\|) - (U(Q)_n + k\|Q-P\|)| \leqq k\|P-x\| .$$

Therefore

$$|T_n(x)| \leqq |T_n(P)| + k\|P-x\| \leqq \|U(P)\| + k\|P-x\|$$

for all n, and hence $U^*(x) \in m$.

To prove (ii), notice that, if $P \in E$,

$$(U(Q)_n + k\|Q-P\|) - (U(P)_n + k\|P-P\|) \geqq$$

$$\geqq k\|Q-P\| - \|U(Q)_n - U(P)_n\| \geqq 0 ,$$

and so

$$\inf_{Q\in E}(U(Q)_n + k\|Q-P\|) \geqq U(P)_n$$

or

$$U^*(P) = U(P) .$$

To prove (iii), use the same proof as in (i), but replace P by any point y in m.

Theorem 9.2: *If* $\|a\bar{x} + b\bar{y}\| \leqq k\|ax + by\|$ *for all* a, b *where* $x = \{x_n\}$, $y = \{y_n\}$, $\bar{x} = \{\bar{x}_n\}$, *and* $\bar{y} = \{\bar{y}_n\}$, *then*

$$\sup_{m,n}\begin{vmatrix}\bar{x}_m & \bar{x}_n \\ \bar{y}_m & \bar{y}_n\end{vmatrix} \leqq k^2 \sup_{m,n}\begin{vmatrix}x_m & x_n \\ y_m & y_n\end{vmatrix} .$$

This is an immediate consequence of Theorems 4.5 and 4.9.

Theorem 9.3: *If y is the image of a quasilinear function x under a Lipschitzian transformation of constant k, then* $L_m(y) \leqq k^2 L_m(x)$.

Proof: For almost all (u, v) in R, for all (u, v) interior to some triangle of linearity of x,

$$\|y(u+a, v+b) - y(u, v)\| \leqq k\|x(u+a, v+b) - x(u, v)\| = k\|ax_u + bx_v\|$$

for all sufficiently small a and b, where x_u and x_v are constant vectors. Hence

$$|y^i(u+a, v+b) - y^i(u, v)| \leqq k\|ax_u + bx_v\|$$

for all i under the same conditions. How, by RADEMACHER's theorem of Lipschitzian functions [12], y^i has a total differential for almost all (u, v).

Hence, for almost all (u, v) in R, and all a and b we have

$$| a y_u^i + b y_v^i | \leqq k \, \| a x_u + b x_v \|$$

which implies

$$\| a y_u + b y_v \| \leqq k \, \| a x_u + b x_v \|$$

where $y_u = \{ y_u^i \}$ and $y_v = \{ y_v^i \}$.

An application of the preceding theorem now gives

$$\sup_{i,k} \begin{vmatrix} y_u^i & y_u^k \\ y_v^i & y_v^k \end{vmatrix} \leqq k^2 \sup_{i,k} \begin{vmatrix} x_u^i & x_u^k \\ x_v^i & x_v^k \end{vmatrix} .$$

Noting that y is of class L_m, we see that

$$L_m(y) = \iint_R \mathcal{W}(y;\, u,\, v)\, du\, dv \leqq k^2 \iint_R \mathcal{W}(x;\, u,\, v)\, du\, dv = k^2 L_m(x) .$$

Theorem 9.4 (KOLMOGOROFF's principle): *If x and y are continuous functions such that $\| y(u, v) - y(U, V) \| \leqq k \, \| x(u, v) - x(U, V) \|$ for all (u, v), (U, V) in $R = D(x) = D(y)$, then $L_m(y) = k^2 L_m(x)$.*

Proof: First we observe that there is no loss in generality in supposing that R is Q. Then we notice that if $x(u, v) = x(U, V)$, then $y(u, v) = y(U, V)$. Thus we may define a transfomation T from $R(x)$ to $R(y)$ as follows: Let $P \in R(x)$ and suppose $x(u, v) = P$. We define $T(P) = y(u, v)$. Clearly T is single valued and satisfies a LIPSCHITZ condition with constant k. Then we may extend T (using the extension theorem) to the whole of m to be Lipschitzian with the same constant k.

Let $\{ z_n \}$ be a sequence of quasilinear functions defined on Q such that $z_n \rightrightarrows x$ and $E_m(z_n) \to L_m(x)$. Then $\{ \bar{z}_n \} = \{ T(z_n) \}$ is a sequence of Lipschitzian functions of the type of the preceding theorem. We have $\bar{z}_n \rightrightarrows y$ since the transformation is Lipschitzian, and so

$$L_m(y) \leqq \liminf_{n \to \infty} L_m(\bar{z}_n) \leqq k^2 \liminf_{n \to \infty} L_m(z_n) = k^2 L_m(x) .$$

Corollary: *If x and y are isometric, then $L_m(x) = L_m(y)$, where, by isometric, we mean that $\| x(u, v) - x(U, V) \| = \| y(u, v) - y(U, V) \|$ for all (u, v), (U, V) in $R = D(x) = D(y)$.*

CHAPTER IV.

Cyclic additivity.

§ 10. – Preliminary remarks.

We have already seen that for our problem there is no loss in generality in supposing that a BANACH space B containing a surface S is separable. Hence, by Theorems 1.5, 6.2, and 7.2, there exist countable sequences F' and F'' such that $L_B(S) = L_B(F', S)$ and $P_B(S) = P_B(F'', S)$. If we combine these two sequences into another sequence F, then obviously both of the above equalities will hold with F replacing F' and F'' The use of the sequence F enables us, in a simple manner, to obtain a particularly useful surface $\mathcal{S}$ in m which is isometric to S. It would be desirable to show that $L_B(S) = L_m(\mathcal{S})$, since it would follow that the LEBESGUE area of any two isometric surfaces imbedded in BANACH spaces would be the same. The writer cannot show that the equality holds for all surfaces S and all BANACH spaces B.

Let the transformation $U: B \to m$ be defined by $U(y) = \{ f_i(y) \}$ where $F = \{ f_i \}$. If x is any continuous function in B, then let x_* be defined by $x_*(u, v) = U(x(u, v))$.

Theorem 10.1: $P_B(x) = P_m(G, x_*)$.

Proof: Let D be any domain interior to R^0, where R is the JORDAN region on which x is defined. Then

$$G^{f_i f_j}(x, D) = \iint_{E_2} M(x^{f_i f_j} \mid \overline{D}; s, t)\, ds\, dt = G_m^{g_i g_j}(x_*, D)$$

for all i, j.

As a result of this theorem, we see that

$$P_B(x) = P_m(G, x_*) \leq P_m(x_*) \leqq L_m(x_*) \leqq L_B(x) .$$

Hence, if $x \in \overline{L}_B$, and x' is any isometric function in m, then

$$P_B(x) = L_m(x') = L_B(x) .$$

Definition 10.1: Let x be continuous on R. Then x is *light* (*monotone*) if the inverse image of each point of the range of x is totally disconnected (a continuum).

If we identify points at F-norm zero, B remains a metric space. If x is light in this space, then x is *F-light* in B.

Definition 10.2: A surface S is *nondegenerate* (*F-nondegenerate*) if it admits a light (F-light) representation. Such a representation may be called nondegenerate (F-nondegenerate).

Lemma 10.1: *If $D(x, y) = 0$, then there exist monotone-light factorizations*

$$x(u, v) = l(m_1(u, v)), \quad m_1(D(x)) = \mathfrak{M}, \quad (u, v) \in D(x),$$

$$y(s, t) = l(m_2(s, t)), \quad m_2(D(y)) = \mathfrak{M}, \quad (s, t) \in D(y),$$

where m_1 and m_2 are monotone and l is light [13].

$\mathfrak{M}$, or any set homeomorphic to it, may be referred to as the *middle space* of the surface determined by x.

We observe that if x is light, then we can take m_1 to be the identity map of $D(x)$. Then $m_2(D(y)) = D(x)$ and $y(s, t) = x(m_2(s, t))$ for $(s, t) \in D(y)$.

Now let $T : B \to E_n$, $T(\zeta) = \{ f_1(\zeta), \ldots, f_n(\zeta) \}$ for $\zeta \in B$. Then $1/n \| T(\zeta) \|_{E_n} \leqq \| \zeta \| \leqq \| T(\zeta) \|_{E_n}$. Hence, by Theorem 4.9, $1/n^2 E_{E_n}(T(\Delta)) \leqq \leqq E_B(F_n, \Delta) \leqq E_B(T(\Delta))$, where Δ is any triangle in B. From this it is easy to see that $L_{E_n}(\bar{x})$ and $L_B(F_n, x)$ are simultaneously finite, where $\bar{x} = T(x)$ and x is in B. Also, if x is F_n-light then $\bar{x}$ is light.

Theorem 10.2: Any F_n-nondegenerate surface S in B for which $L_B(F_n, S) < +\infty$ admits a representation of class $L_B(F_n)$ on the unit circle C.

Proof: Let x be a nondegenerate representation of S and $\bar{x}$ be defined in terms of x as above. Then $\bar{x}$ is nondegenerate. Furthermore, $L_{E_n}(\bar{x}) < +\infty$, so there exists a function $\bar{y}$ of class L_{E_n} on C such that $D(\bar{x}, \bar{y}) = 0$. Also, since $\bar{x}$ is light, there exists a monotone map m: $C \to D(x)$ such that $\bar{y}(s, t) = = \bar{x}(m(s, t))$. Now define $y(s, t) = x(m(s, t))$ for $(s, t) \in C$. Since m is continuous, y is continuous. Next, $f_i(y(s, t)) = f_i(x(m(s, t))) = \bar{x}_i(m(s, t)) = \bar{y}_i((s, t)$, where $\bar{x}_i$ and $\bar{y}_i$ are the i^{th} coordinates of $\bar{x}$ and $\bar{y}$ respectively, and so y is of class $L_B(F_n)$.

§ 11. – Cyclic addititivity.

Let r be any positive number. Then define

$$\lambda(\varrho) = \begin{cases} 0 & 0 \leqq \varrho \leqq r, \\ \dfrac{\varrho - r}{\varrho} & \varrho \geqq r. \end{cases}$$

If α is a fixed point of B and x is continuous in B, let

$$x'(u, v) = \alpha + \lambda(\| x(u, v) - \alpha \|)(x(u, v) - \alpha), \qquad (u, v) \in R(x).$$

Clearly x' is continuous.

Lemma 11.1: *If* $D(x, y) = 0$, *then* $D(x', y') = 0$.

As a consequence of this lemma, we can define the surface S' obtained from S as follows: If x is a representation of S, then let S' be the surface determined by x'. Obviously, $D(S', S) \leqq r$.

As a result of a straightforward computation, we obtain

Lemma 11.2: $\|x'(u, v) - x'(U, V)\| \leqq \|x(u, v) - x(U, V)\|$.

Theorem 11.1: *If P is a polyhedron, then* $L_B(P') \leqq E_B(P)$.

Proof: Let x be a quasilinear representation of P. Then $(x_*)' = (x')_*$ where x_* is defined as in § 10. Next, since x and x' are Lipschitzian, $L_B(x) = L_m(x_*)$, and $L_B(x') = L_m(x'_*)$ where $x'_* = (x_*)'$. Finally, using KOLMOGOROFF's principle,

$$L_B(P') = L_B(x') = L_m(x'_*) \leqq L_m(x_*) = L_B(x) = E_B(P) .$$

The following theorem is now proved as in [13].

Theorem 11.2: $L_B(S) = 0$ *if and only if the middle space reduces to a dendrite. If* $L_B(S) > 0$, *and* $S_1, \ldots$ *are the surfaces in the cyclic decomposition of* S, *then* $L_B(S) = \sum_n L_B(S_n)$.

The corresponding theorem for P_B is also true.

Theorem 11.3: *If* $L_B(F_n, S) < +\infty$, *then* $L_B(F_n, S) = P_B(F_n, S)$.

Theorem 11.4: *If* $L_m(S) < +\infty$, *then* $L_m(S) = P_m(S)$.

Proof: For each n, $L_m(G_n, S) < +\infty$, and so

$$L_m(S) = \lim_{n\to\infty} L_m(G_n, S) = \lim_{n\to\infty} P_m(G_n, S) \leqq P_m(S) \leqq L_m(S) .$$

CHAPTER V.

Lebesgue area in metric spaces.

§ 12. – Definitions and theorems.

In this section we shall always suppose that x is a continuous function defined on a JORDAN region into a metric space D.

Theorem 12.1: *There exists an isometric map T from $R(x)$ into m.*

Proof: We may regard $R(x)$ as a subspace of D. $R(x)$ is obviously separable.

Definition 12.1: Let T be an isometric transformation of $R(x)$ into m. Define

$$L(x) = L_m(y)$$

where $y = T(x)$. By KOLMOGOROFF's principle, $L(x)$ is independent of T.

Theorem 12.2: *If $\{x_n\}$ is a sequence of continuous functions in D such that $x_n \rightrightarrows x$, then $L(x) \leqq \liminf_{n\to\infty} L(x_n)$.*

Proof: Map D into m by means of an everywhere dense sequence of points in the subspace consisting of $R(x)$ and $R(x_n)$. This can be done since each range is separable. The mapping is clearly isometric as regards points of $R(x)$ and $R(x_n)$. If the image of x is y and the image of x_n is y_n, then $y_n \rightrightarrows y$ by the above remarks. But $L(x) = L_m(y)$, $L(x_n) = L_m(y_n)$ by Definition 12.1, and $L_m(y) \leqq \liminf_{n\to\infty} L_m(y_n)$.

Since it is clear that $L(x) = L(y)$ if x and y are two representations of a surface S in D, we can make the following definition:

Definition 12.2: If S is a surface in a metric space D, and x is a representation of S, then we define the LEBESGUE area of S, $L(S)$, by $L(S) = L(x)$.

Definition 12.3: Let x_1 and x_2 be continuous functions defined on JORDAN regions R_1 and R_2 with ranges in metric spaces D_1 and D_2, respectively. Then define

$$\overline{D}(x_1, x_2) = \inf \{ \max_{\substack{(u,v)\in R_1 \\ (U,V)\in R_1}} | (x_1(u, v), x_1(U, V)) - (x_2(\tau(u, v)), x_2(\tau(U, V))) | \}$$

for all homeomorphisms τ: $R_1 \to R_2$.

Clearly $\overline{D}$ metrizes the space of all continuous functions defined on JORDAN regions with range in metric spaces.

Definition 12.4: The relation between x and y defined by $\overline{D}(x, y) = 0$ is an equivalence relation. An *E-Fréchet-surface* is a class of such equivalent functions.

Definition 12.5: If S_1 and S_2 are E-FRÉCHET surfaces, then $\overline{D}(S_1, S_2) = \overline{D}(x_1, x_2)$ where x_1 and x_2 are representations of S_1 and S_2, respectively.

Definition 12.6: If $P \in R$, let $x^P(u, v) = (x(u, v), x(P))$. If $\mathcal{P} = \{P_n\}$ is an everywhere dense sequence in R, let $x^{\mathcal{P}}(u, v) = \{x^{P_n}(u, v)\}$. Define $\mathcal{W}_{\mathcal{P}}(x; u, v) = \mathcal{W}(G; x^{\mathcal{P}}; u, v)$ and $\mathcal{W}(x; u, v) = \sup_{\mathcal{P}} \mathcal{W}_{\mathcal{P}}(x; u, v)$.

Theorem 12.3: *Let x_k, $k = 0, 1, \ldots$ be defined on R into metric spaces D_k. Suppose that for each $P \in R$, $x_k^P(u, v) \rightrightarrows x_0^P(u, v)$. Then $L(x_0) \leqq \liminf_{k\to\infty} L(x_k)$.*

Proof: Let $\mathfrak{P} = \{ P_n \}$ be an everywhere dense sequence in R and define $X_k(u, v) = \{ x_k^{P_n}(u, v) \}$. Now, using Theorem 8.3 and Definition 12.1, $L(x_0) = L_m(X_0) \leqq \liminf_{k \to \infty} L_m(X_k) = \liminf_{k \to \infty} L(x_k)$.

Corollary: *If* $\overline{D}(x_k, x_0) \to 0$, *then* $L(x_0) \leqq \liminf_{k \to \infty} L(x_k)$.

Definition 12.7: x is of class L on R if, for some $\mathfrak{P}$, $x^{\mathfrak{P}}$ is of class L_m on R.

Definition 12.8: Define

$$D_u x(u_0, v_0) = \lim_{u \to u_0} \frac{(x(u, v_0), x(u_0, v_0))}{| u - u_0 |},$$

$$D_v x(u_0, v_0) = \lim_{v \to v_0} \frac{(x(u_0, v), x(u_0, v_0))}{| v - v_0 |},$$

if these limits exist.

Lemma 12.1: *If x is continuous on R and* B.V.T. *on each oriented rectangle $T \subset R^0$, then $D_u x$ and $D_v x$ exist almost everywhere in R^0 and are summable.*

The proof of this lemma is similar to the case in which x is a real valued function [9].

Theorem 12.4: *If x is* A.C.T. *on each oriented rectangle $T \subset R^0$ with $| D_u x |^p$ and $| D_v x |^q$ summable over R^0, $1/p + 1/q \leqq 1$, $p \geqq 1$, $q \geqq 1$, then x is of class L.*

Proof: By the triangle inequality, $| x^P(u_1, v_1) - x^P(u_2, v_2) | \leqq (x(u_1, v_1), x(u_2, v_2))$ for every (u_1, v_1) and $(u_2, v_2) \in R$. Thus each x^P is continuous and is A.C. in u for the same values of v for which this is true of x. Thus each x^P is A.C.T. Also

$$\left| \frac{x^P(u, v_0) - x^P(u_0, v_0)}{u - u_0} \right| \leqq \frac{(x(u, v_0), x(u_0, v_0))}{| u - u_0 |}$$

and so

$$| x^P(u_0, v_0) | \leqq D_u x(u_0, v_0) .$$

Similarly

$$| x^P(u_0, v_0) | \leqq D_v x(u_0, v_0) .$$

From these results, the hypothesis, and a well known result on transformations of class L [9], we can conclude that the transformations $(u, v) \to (x^P(u, v), x^Q(u, v))$ are of class L, whether or not P and Q are in $\mathfrak{P}$.

Finally, for almost all $(u, v) \in R$, $\mathcal{W}_{\mathfrak{P}}(x; u, v) \leqq 2(D_u x(u, v))(D_v x(u, v))$, which is summable by the Hölder inequality.

Theorem 12.5: *If x is of class L on R, and if $\mathfrak{P}$ is the everywhere dense sequence of Definition 12.7, then $L(x) = \iint_{\dot{R}_0} \mathcal{W}_{\mathfrak{P}}(x; u, v)\, du\, dv$.*

This theorem is an immediate consequence of Definition 12.1 and Theorem 8.2, where use is made of an everywhere dense mapping into m by means of the points corresponding to $\mathfrak{P}$.

Theorem 12.6: *If the flat transformation $(u, v) \to (x^P(u, v), x^Q(u, v))$ is of class L for all P, Q in R, and if $\mathcal{W}(x; u, v)$ is summable over R^0, then*

$$L(x) = \iint_{\dot{R}^0} \mathcal{W}(x; u, v)\, du\, dv .$$

Proof: Obviously x is of class L on R, and if $\mathfrak{P}$ is any everywhere dense sequence in R, then

$$L(x) = \iint_{\dot{R}^0} \mathcal{W}_{\mathfrak{P}}(x; u, v)\, du\, dv \leqq \iint_{\dot{R}^0} \mathcal{W}(x; u, v)\, du\, dv .$$

Now let M be the Banach space of bounded functions on R with $\|f\| = \sup_{P \in R} |f(P)|$ for all f in M. Then the transformation $T(P) = (P, Q) - (\bar{P}, Q)$, where $\bar{P}$ is a fixed point and Q is any point of R, is isometric from $x(\dot{R})$ into M.

Next, if y is a continuous function in m and Y is an isometric function in M, then clearly $L_M(Y) = L_m(y)$.

Finally, if $X = T(x)$, then $L(x) = L_M(X)$ by the above remarks, and so

$$L(x) = L_M(X) \geqq P_M(X) \geqq \iint_{\dot{R}^0} \mathcal{W}(X; u, v)\, du\, dv = \iint_{\dot{R}^0} \mathcal{W}(x; u, v)\, du\, dv .$$

Theorem 12.7: *If S is in a finite dimensional Banach space B, then $L(S) = L_B(S)$.*

Proof: Let F be the sequence and $\mathcal{S}$ the surface of § 10. Then $L_B(S) = \lim_{n \to \infty} L_B(F_n, S) = \lim_{n \to \infty} L_m(G_n, \mathcal{S}) = L_m(\mathcal{S}) = L(S)$.

Theorem 12.8: *If S is in a in Banach space B, then $P_B(S) \leqq L(S) \leqq L_B(S)$. If S admits a representation of class $\bar{L}_B$, then the equality holds througho ut. If for some $F \subset \Sigma_B$, $L_B(S) = \lim_{n \to \infty} L_B(F_n, S)$, then $L(S) = L_B(S)$. If $L(S) = 0$, then $L(S) = L_B(S)$.*

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