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On a theorem of BESICOVITCH in surface area theory. (**)

Introduction.

In recent years, a number of important and interesting results were obtained by various authors concerning the relationships between the Lebesgue definition of surface area and other definitions based upon Hausdorff measure. For background, see the book [7], which will be referred to by LA in the sequel (numbers in square brackets refer to the Bibliography at the end of this paper). The present paper was motivated by the study of a very beautiful result of Besicovitch [1] to the effect that if

$$S: \qquad z = f(x, y), \quad (x, y) \in Q: \ 0 \leqq x \leqq 1, \ 0 \leqq y \leqq 1 \,,$$

is a non-parametric continuous surface, then its Lebesgue area is equal to the two-dimensional Hausdorff measure of the set of its points, provided that $f(x, y)$ is absolutely continuous in the Tonelli sense. This result of Besicovitch has been shown by Federer [4] to hold for all continuous non-parametric surfaces. The method of Federer brings to bear upon the problem on hand a series of general results on various measures which he established in [3]. In particular, he uses a certain generalized version of Favard measure as a *majorant*, in a sense, to obtain the necessary estimates in the course of his argument. The main purpose of the present paper is to show that the Lebesgue area of the given non-parametric surface may be used as such a majorant. The main reason for this is the fact that for a non-parametric surface of finite Lebesgue area there is available a particularly convenient parametric representation, namely, the so-called generalized conformal representation in the sense of C. B. Morrey (see LA for literature and for technical details). As a result, we obtain a proof of the theorem of Besicovitch, in the generalized form due to Federer, by methods which are germane to the theory of the Lebesgue area. Furthermore, we obtain a

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generalization to those parametric surfaces which admit of a *quasi-Lipschitzian representation* in the sense of 5.23 below. This generalization is stated in 5.28. As a corollary we obtain in 5.33 the theorem that certain area-definitions proposed recently by BESICOVITCH [2] and L. C. YOUNG [9] agree with the LEBESGUE area for surfaces admitting of quasi-Lipschitzian representations.

The essential contribution of the present paper is contained in part 5. Parts 1-4 are devoted to background material, including a discussion of certain portions of the argument used by BESICOVITCH [1] and FEDERER [4]. Our study of their work led us to various observations which we thought may justify the rather detailed presentation in the preliminary parts 1-4 of this paper.

1. - Preliminary definitions and remarks.

1.1. - In this section we shall give some definitions and remarks needed in the sequel.

1.2. - For two points p, q in Euclidean 3-space E_3 we shall denote by $d(p, q)$ the distance between them and for a set $E \subset E_3$ we shall denote by $d(E)$ the diameter of the set.

1.3. - For a set E in E_3 let σ_ε be a generic notation for a finite or denumerable number of sets $e_1, e_2, \dots$ which satisfy the conditions: (i) $E = e_1 + e_2 + \dots$. (ii) For each j, $d(e_j) < \varepsilon$. We set

$$H^2_\varepsilon(E) = \text{glb} \sum_j \frac{1}{4} \pi d(e_j)^2 \quad \text{for all} \quad \sigma_\varepsilon .$$

Since the set function $H^2_\varepsilon(E)$ increases when ε decreases, it has a limit (finite of infinite) for $\varepsilon \to 0$. The HAUSDORFF *two-dimensional measure* of E is defined to be

$$H^2(E) = \lim_{\varepsilon \to 0} H^2_\varepsilon(E) .$$

1.4. - For a set E in E_3 let τ_ε be a generic notation for a finite or denumerable number of open spheres $s_1, s_2, \dots$ which satisfy the conditions: (i) $E \subset s_1 + s_2 + \dots$. (ii) For each j, $d(s_j) < \varepsilon$. Then $S^2_\varepsilon(E)$ and $S^2(E)$ are defined as $H^2_\varepsilon(E)$ and $H^2(E)$ in 1.3, using of course the diameters of the open spheres s_j.

For a set E in E_3 let τ be a generic notation for a finite or denumerable number of open spheres $s_1, s_2, \dots$, which cover E, i.e., $E \subset s_1 + s_2 + \dots$ (note there is no restriction on the size of the diameters). Then we define

$$S^2_\infty(E) = \text{glb} \sum_j \frac{1}{4} \pi d(s_j)^2 \quad \text{for all} \quad \tau .$$

1.5. – For a set E in E_3 let σ be a generic notation for a finite or denumberable number of sets $e_1, e_2, \ldots$ which satisfy the condition that $E = e_1 + e_2 + \ldots$. We set

$$H^1_\infty(E) = \text{glb} \sum_j d(e_j) \quad \text{for all} \quad \sigma,$$

$$H^2_\infty(E) = \text{glb} \sum_j \frac{1}{4}\pi d(e_j)^2 \quad \text{for all} \quad \sigma.$$

Note that there is no restriction on the diameters of the covering sets e_j in defining $H^1_\infty(E)$, $H^2_\infty(E)$.

1.6. – The set functions defined in 1.3, 1.4 and 1.5 possess the following properties.

(a) If $\Phi(E)$ is any one of these functions then $\Phi(E_1) \leqslant \Phi(E_2)$ whenever $E_1 \subset E_2$ and $\Phi(\sum_j E_j) \leqslant \sum_j \Phi(E_j)$ for any sequence of sets $E_1, E_2, \ldots$.

(b) $S^2_\infty(E) \leqslant S^2_\varepsilon(E)$ for every $\varepsilon > 0$, and $H^2_\infty(E) \leqq S^2_\infty(E)$ for every set E.

(c) $H^2(E) = 0$ if and only if $S^2_\infty(E) = 0$.

(d) $H^2(E)$ is a CARATHÉODORY outer measure (see SAKS [8], pp. 39-54).

1.7. – For a set E in a metric space we shall denote by $\text{fr}(E)$ the frontier of E, by $\complement E$ the complement of E, by $c(E)$ the closure of E, and by $\text{Int}\, E$ the interior of E.

1.8. – For a set E in xyz-space we shall designate by E' the set in the xy-plane obtained by projecting E orthogonally upon the xy-plane.

1.9. – Let a be a point in E_3 and let r be a positive number. We shall denote by $K(a, r)$ the open sphere with center at a and radius r, by $C(a, r)$ the closed sphere with center at a and radius r and by $B(a, r)$ the boundary of $K(a, r)$.

2. – Lemmas on HAUSDORFF measure and density.

2.1. – The purpose of this section is to collect, for convenience of application, a series of lemmas which occur in the work of BESICOVITCH and FEDERER. The reason for a detailed presentation will be explained in 2.8.

2.2. – Lemma. In Euclidean three-space E_3 let Γ be a straight circular (solid) cylinder of height h, whose base circle has diameter d. If $d \leqslant h$ then

(see 1.4)

$$S^2_{2d}(\Gamma) \leqslant \pi d h .$$

Proof. Since $d \leqslant h$ there is an integer $n \geqslant 2$ such that

(1) $$(n-1)d \leqslant h < nd .$$

Then Γ can be covered by n straight circular cylinders $\gamma_1, \ldots, \gamma_n$, each having height less than d and a base circle of diameter d. If O_i is the center of γ_i then γ_i is contained in a sphere with center at O_i and diameter $\sqrt{2}\, d < 2d$. Hence

$$S^2_{2d}(\gamma_i) \leqslant \frac{1}{4}\pi(\sqrt{2}\, d)^2 = \frac{1}{2}\pi d^2 .$$

Thus

$$S^2_{2d}(\Gamma) \leqslant \frac{1}{2}\pi n d^2 = \frac{1}{2}\pi d(nd) \leqslant \pi d h ,$$

where the last inequality follows from (1) since $nd \leqslant h + d \leqslant 2h$.

2.3. – Lemma. In the xy-plane, let A' be a bounded set and, for z_0 and $\varrho > 0$ given, in xyz-space let A be the set of points (x, y, z) satisfying the relation $(x, y) \in A'$, $z_0 - \varrho \leqslant z \leqslant z_0 + \varrho$. Assume $d(A') \leqslant 2\varrho$. Then

$$S^2_{4d(A')}(A) \leqslant 8\pi\varrho d(A') .$$

Proof. A' can be covered by a circular disc of diameter $d = 2d(A')$. Hence A can be covered by a straight circular cylinder Γ whose base circle has a diameter $d = 2d(A')$ and height $h = 4\varrho \geqslant 2d(A') = d$. Hence, by the Lemma in 2.2,

$$S^2_{4d(A')}(A) = S^2_{2d}(A) \leqslant S^2_{2d}(\Gamma) \leqslant \pi d h = \pi 2d(A')4\varrho = 8\pi\varrho d(A') .$$

2.4. – Lemma. Let B be a bounded set in E_3, let $\varrho > 0$ be given, and let λ be a number such that (see 1.5, 1.9, 1.8)

$$H^1_\infty \{ [B \cap K(a, \varrho)]' \} < \lambda .$$

Then

$$S^2_{4\lambda}[B \cap K(a, \varrho)] \leqslant 8\pi\varrho H^1_\infty \{ [B \cap K(a, \varrho)]' \} .$$

Proof. By the definition of H^1_∞ (see 1.5) we have an intrinsic covering of $[B \cap K(a, \varrho)]'$ by sets $A'_1, \ldots, A'_j, \ldots$, such that $[B \cap K(a, \varrho)]' = \bigcup_j A'_j$ and

(1) $$\sum_j d(A'_j) < H^1_\infty \{ [B \cap K(a, \varrho)]' \} + \varepsilon < \lambda ,$$

where $\varepsilon > 0$ is arbitrarily small. Let z_0 be the z-coordinate of the center a, and let ϱ be the ϱ of $K(a, \varrho)$. For each A'_j, let A_j be the set (as in the Lemma in 2.3) of points (x, y, z) satisfying $(x, y) \in A'_j$, $z_0 - \varrho \leqslant z \leqslant z_0 + \varrho$. Clearly $B \cap K(a, \varrho) \subset \bigcup_j A_j$. Hence, by the Lemma in 2.3 (note that $d(A'_j) \leqslant 2\varrho$)

$$S^2_{4d(A'_j)}(A_j) \leqslant 8\pi\varrho d(A'_j) .$$

From (1) we have $d(A'_j) < \lambda$. Hence

$$S^2_{4\lambda}(A_j) \leqslant S^2_{4d(A'_j)}(A_j) \leqslant 8\pi\varrho d(A'_j) ,$$

and

$$S^2_{4\lambda}[B \cap K(a, \varrho)] \leqslant \sum_j S^2_{4\lambda}(A_j) \leqslant 8\pi\varrho \sum_j d(A'_j) < 8\pi\varrho[H^1_\infty \{ [B \cap K(a, \varrho)]' \} + \varepsilon] .$$

Since $\varepsilon > 0$ is arbitrary, the statement of the Lemma follows.

2.5. - Lemma. In E_3 let B be a bounded set with the following property: there exists a $\delta > 0$ such that

$$H^1_\infty \{ [B \cap K(b, r)]' \} < r/16$$

for every $K(b, r)$ with $0 < r < \delta$ (b is not required to lie in B). Then $H^2(B) = 0$.

Proof. It is sufficient (see 1.6) to show that

(1) $$S^2_\infty(B) = 0 .$$

To show (1) we set

(2) $$\mu = \delta/8 ,$$

and let ε be an arbitrary positive number. By the definition of $S^2_\mu(B)$ we can cover B with open spheres $K(b_n, r_n)$ (centers not generally in B) such that

(3) $$\sum_n \pi r_n^2 < S^2_\mu(B) + \varepsilon , \qquad d[K(b_n, r_n)] = 2r_n < \mu .$$

Then

(4) $$S^2_\mu(B) \leqslant \sum_n S^2_\mu[B \cap K(b_n, r_n)] .$$

Now $r_n < \mu/2 = \delta/16 < \delta$. Hence, by assumption,

(5) $$H^1_\infty \{ [B \cap K(b_n, r_n)]' \} < r_n/16 < \mu/32 < \delta/32 .$$

Hence ,by applying the Lemma in 2.4 with $\lambda = \delta/32$ and hence $4\lambda = \delta/8 = \mu$, we obtain

$$S^2_\mu[B \cap K(b_n, r_n)] \leqslant 8\pi r_n H^1_\infty \{ [B \cap K(b_n, r_n)]' \} ,$$

and hence, using (5),

(6) $$S^2_\mu[B \cap K(b_n, r_n)] < 8\pi r_n r_n/16 = \frac{1}{2}\pi r_n^2.$$

The relations (4), (6) and (3) yield

$$S^2_\mu(B) < \frac{1}{2}\sum_n \pi r_n^2 < \frac{1}{2}[S^2_\mu(B) + \varepsilon]\,. \tag{7}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (7) that $S^2_\mu(B) = 0$ and hence, *a fortiori*, $S^2_\infty(B) = 0$.

2.6. – Theorem. Let A be a bounded set in E_3. Then

$$\limsup_{r\to 0} \frac{H^1_\infty\{[A \cap K(a, r)]'\}}{r} \geqslant \frac{1}{32}, \tag{1}$$

for $a \in A$, except at a subset of A whose H^2 measure is equal to zero.

Proof. Let E be the exceptional set in A where (1) does not hold. For $a \in E$,

$$\limsup_{r\to 0} \frac{H^1_\infty\{[E \cap K(a, r)]'\}}{r} \leqslant \limsup_{r\to 0} \frac{H^1_\infty\{[A \cap K(a, r)]'\}}{r} < \frac{1}{32}\,.$$

Hence we have rational numbers $\delta(a)$, $\lambda(a)$, such that $\delta(a) > 0$, $0 < \lambda(a) < 1/32$, and

$$\frac{H^1_\infty\{[E \cap K(a, r)]'\}}{r} < \lambda(a) < 1/32 \quad \text{for} \quad 0 < r < \delta(a)\,, \quad a \in E\,.$$

Now for assigned rationals $0 < \delta < \infty$, $0 < \lambda < 1/32$, let $E_{\delta\lambda}$ be the subset of E where $\delta(a) = \delta$, $\lambda(a) = \lambda$. Clearly $E \subset \bigcup E_{\delta\lambda}$ and hence, (see 1.6) it is sufficient to show that $H^2(E_{\delta\lambda}) = 0$. Setting, for fixed δ, λ, $B = E_{\delta\lambda}$, we know that, since $B \subset E$,

$$\frac{H^1_\infty\{[B \cap K(a, \varrho)]'\}}{\varrho} < \lambda < 1/32 \quad \text{for} \quad 0 < \varrho < \delta\,, \quad a \in B\,.$$

Now taking any $K(b, r)$, b not necessarily in B and $0 < r < \delta/2$, we assert that

$$\frac{H^1_\infty\{[B \cap K(b, r)]'\}}{r} < \frac{1}{16}\,. \tag{2}$$

Indeed, if $B \cap K(b, r) = 0$, then (2) is obvious. If $B \cap K(b, r) \neq 0$ take $a \in B \cap K(b, r)$. Then $K(b, r) \subset K(a, 2r)$ and hence

$$\frac{H^1_\infty\{[B \cap K(b, r)]'\}}{r} \leqslant \frac{H^1_\infty\{[B \cap K(a, 2r)]'\}}{2r} < 2\lambda < 1/16\,,$$

since $0 < 2r < \delta$ and $a \in B$. Thus (2) holds for $0 < r < \delta/2$, b not necessarily in B. From the Lemma in 2.5 it follows that $H^2(B) = 0$.

2.7. – Lemma. Let A be a bounded BOREL set in xyz-space, let $\varphi(B)$ be a finite, non-negative, completely additive function of bounded BOREL sets $B \subset A$. Assume that there exists a real number λ such that $0 < \lambda < \infty$ and

$$\limsup_{r\to 0} \frac{\varphi[A \cap K(a, r)]}{\pi r^2} > \lambda$$

for $a \in A$ except on a set E with $H^2(E) = 0$. Then

$$H^2(A) \leqslant \varphi(A)/\lambda .$$

Proof. We note first that we have *a fortiori* (see 1.9)

$$\limsup_{r\to 0} \frac{\varphi[A \cap C(a, r)]}{\pi r^2} > \lambda \quad \text{for} \quad a \in A^* = A - E .$$

Now let $\varepsilon > 0$ be given and let C_ε be the class of all $C(a, r)$ such that: (i) $a \in A^*$, (ii) $10r < \varepsilon$, (iii) $\varphi[A \cap C(a, r)] > \lambda\pi r^2$. Then C_ε covers A^* in the sense of VITALI. Hence by a well-known covering theorem (see MORSE [5], § 3.10) we can select from C_ε a sequence of disjoint elements $C(a_j, r_j)$ such that: (α) $a_j \in A^*$, (β) $10r_j < \varepsilon$, (γ) $\varphi[A \cap C(a_j, r_j)] > \lambda\pi r_j^2$, ($\delta$) for every positive integer n,

$$A^* \subset \bigcup_{j=1}^{n} C(a_j, r_j) + \bigcup_{j=n+1}^{\infty} C(a_j, 5r_j) .$$

It thus follows that

$$(1) \qquad A^* \subset \bigcup_{j=1}^{n} [A^* \cap C(a_j, r_j)] + \bigcup_{j=n+1}^{\infty} [A^* \cap C(a_j, 5r_j)] .$$

Now all the sets on the right of (1) have diameters less than or equal to $10r_j < \varepsilon$. Hence

$$(2) \qquad H_\varepsilon^2(A^*) \leqslant \sum_{j=1}^{n} \frac{1}{4}\pi(2r_j)^2 + \sum_{j=n+1}^{\infty} \frac{1}{4}\pi(10r_j)^2 = \sum_{j=1}^{n} \pi r_j^2 + 25 \sum_{j=n+1}^{\infty} \pi r_j^2 \leqslant$$

$$\leqslant \frac{1}{\lambda} \sum_{j=1}^{n} \frac{1}{4} \varphi[A \cap C(a_j, r_j)] + \frac{25}{\lambda} \sum_{j=n+1}^{\infty} \varphi[A \cap C(a_j, r_j)] .$$

Now the sets $A \cap C(a_j, r_j)$ are disjoint BOREL subsets of A. Hence

$$(3) \qquad \sum_{j=1}^{\infty} \varphi[A \cap C(a_j, r_j)] \leqslant \varphi(A) ,$$

$$(4) \qquad \sum_{j=n+1}^{\infty} \varphi[A \cap C(a_j, r_j)] \to 0 \quad \text{for} \quad n \to \infty .$$

(2), (3), and (4) imply that $H_\varepsilon^2(A^*) \leqslant \varphi(A)/\lambda$ for every $\varepsilon > 0$. Hence

$H^2(A^*) \leqslant \varphi(A)/\lambda$. Therefore, since $H^2(E) = 0$,

$$H^2(A) = H^2(A^* + E) \leqslant H^2(A^*) + H^2(E) = H^2(A^*) \leqslant \varphi(A)/\lambda .$$

2.8. – On comparing the lemmas in 2.2, 2.3, 2.4, 2.5 with analogous statements in BESICOVITCH [1] and FEDERER [4], the reader will notice that our lemmas are worded in a more detailed manner. The purpose is to remedy certain minor discrepancies involving the diameters of the covering sets and spheres, in order to conform to the exact definitions of the quantities $H^2_\varepsilon(E)$, $H^2(E)$, $H^1_\infty(E)$, $S^2_\varepsilon(E)$, $S^2(E)$, $S^2_\infty(E)$ (see 1.3, 1.4, 1.5).

3. – Lemmas on total variation.

3.1. – The lemmas to be discussed in this section are related to corresponding statements in BESICOVITCH [1] and FEDERER [4] as follows. BESICOVITCH considers a surface $z = f(x, y)$, where he assumes that $f(x, y)$ is ACT (absolutely continuous in the TONELLI sense). To extend his ingenious argument to the case when $f(x, y)$ is BVT (of bounded variation in the TONELLI sense), FEDERER [4] replaces the various definite integrals bearing upon derivatives (as used by BESICOVITCH), by integrals bearing upon total variations. These latter integrals coincide with one of the alternative forms of the so-called GEÖCZE expressions which play an important role in the theory of the LEBESGUE area. Beyond this point, FEDERER [4] proceeds to use his important and interesting results on a generalized version of FAVARD measure which serves in a sense as a majorant. At this point, we deviate from FEDERER [4] by observing, essentially, that the LEBESGUE area itself (extended to BOREL sets as a completely additive set function) may be used as a majorant. Another reason for giving a detailed presentation of this portion of the argument is the presence of minor topological discrepancies in both FEDERER [4] and BESICOVITCH [1]. For instance, in connection with a certain open set O which arises in the course of the proof, it is asserted that the set of the components of the closure of O is countable, a manifest slip. Accordingly, we go into various details which seem to be necessary and which may also help the reader to better understand the remarkably ingenious geometrical argument of BESICOVITCH [1].

3.2. – Let F be a compact set in the plane and let σ be a generic notation for a countable system of mutually disjoint continua $C_1, \ldots, C_j, \ldots$ such that $\bigcup C_j \subset F$. We define

$$\varkappa(F) = \text{lub} \sum_j d(C_j) \quad \text{for all} \quad \sigma .$$

3.3. – Lemma. Let O be a bounded open set in the plane. Then

(1) $$H^1_\infty(O) \leqslant \varkappa[\mathrm{fr}(O)].$$

Proof. We can assume $O \neq 0$ [otherwise (1) is trivial]. Let $\Gamma_1, \Gamma_2, \ldots$ be the components of O (the set of these components is finite or countably infinite). Let $C_1, C_2, \ldots$ be those components of the closure $c(O)$ of O which contain at least one of $\Gamma_1, \Gamma_2, \ldots$. Then

$$O \subset \bigcup_j \Gamma_j \subset \bigcup_n C_n,$$

and hence

(2) $$H^1_\infty(O) \leqslant \sum_n d(C_n).$$

Now C_n is a bounded continuum. Hence, in particular, the unbounded component D_n of the complement $\complement C_n$ of C_n has a connected frontier $F_n \subset C_n$ (see Newman [6], p. 117, Theorem 14.3). We assert

(3) $$d(F_n) = d(C_n).$$

Let p_n, q_n be points in C_n such that $d(p_n, q_n) = d(C_n)$. Let λ be the line through p_n and q_n and let λ_0 be the closed segment of λ from p_n to q_n. Then $\lambda - \lambda_0$ has two components each of which is in D_n. Hence p_n and q_n are in F_n and $d(F_n) \geqslant d(C_n)$. Since $F_n \subset C_n$, $d(F_n) \leqslant d(C_n)$ and (3) follows.

We next assert that

(4) $$F_n \subset \mathrm{fr}(O).$$

Since $F_n \subset C_n \subset c(O)$ we have to show that $F_n \cap O = 0$. Suppose $p_n \in F_n \cap O$. Then $p_n \in \Gamma_j$ for some j and $\Gamma_j \subset C_k$ for some k. Thus $p_n \in C_k \cap C_n$ and, since C_k and C_n are components of $c(O)$, $k = n$. Thus $p_n \in \Gamma_j \subset \mathrm{Int}\, C_n$ and $(\mathrm{Int}\, C_n) \cap F_n = 0$. Thus $p_n \notin \Gamma_j \cap F_n$. This is a contradiction. Hence $F_n \cap O = 0$ and (4) follows.

By (4) and the fact that $F_n \subset C_n$ where $C_1, C_2, \ldots$ are disjoint continua [components of $c(O)$] it follows that $F_1, F_2, \ldots$ are disjoint continua in $\mathrm{fr}(O)$. Hence by definition

(5) $$\sum_n d(F_n) \leqslant \varkappa[\mathrm{fr}\,(O)].$$

(2), (3), and (5) imply (1).

3.4. – Let $E \neq 0$ be a compact set in the plane. For $h > 0$ we define $\mu(E, h)$ as follows.

(i) A subset G of E is h-admissible if $p \in G$, $q \in G$, $p \neq q$, implies that $d(p, q) \geqslant h$.

(ii) If $N(G)$ is the number of distinct points of an h-admissible set $G \subset E$ then clearly $N(G) \leqslant \nu(E, h) < \infty$, where, since E is compact, $\nu(E, h)$ depends only on E and h.

(iii) It now follows that there is an h-admissible set $G_0 \subset E$ such that $N(G_0)$ is a maximum.

We now define $\mu(E, h) = N(G_0) = \max N(G)$ for all h-admissible sets $G \subset E$. If $E = 0$ we define $\mu(E, h) = 0$.

3.5. - $\mu(E, h)$ has the following upper semi-continuity property. There is an open set $O \supset E$ such that $\mu(E^*, h) \leqslant \mu(E, h)$ for every compact set $E^* \subset O$.

Proof. Let O_n be the set of points p in the plane for which $d(p, E) < 1/n$. Take E_n compact in O_n and set $\mu = \mu(E, h)$. Suppose E_n contains points $p_1^n, \ldots, p_{\mu+1}^n$ such that $d(p_i^n, p_j^n) \geqslant h$ for $i \neq j$. For each p_i^n have $q_i^n \in E$ such that $d(p_i^n, q_i^n) < 1/n$. Hence, for $i \neq j$, $h \leqslant d(p_i^n, p_j^n) \leqslant d(q_i^n, q_j^n) + 2/n$ or $d(q_i^n, q_j^n) \geqslant h - 2/n$. For a proper subsequence, we have $q_i^n \to q_i \in E$, $i = 1, \ldots, \mu + 1$. This yields an h-admissible set $G \subset E$ with $N(G) = \mu(E, h) + 1$. This is a contradiction. Thus for some n the set O_n can be taken as O.

3.6. - Let E be a compact set in the xy-plane. Define

$$\xi(x, h, E) = \mu(E_x, h),$$

where E_x is the intersection of E with the line $x =$ constant. Then $\xi(x, h, E)$ is uniformly bounded (less than or equal to $\mu(E, h)$), is identically zero outside of some bounded interval and by the remark in 3.5 is an upper semicontinuous function of x. Hence the integral

$$\int_{-\infty}^{\infty} \xi(x, h, E)\, dx = \int \xi(x, h, E)\, dx$$

exists and is finite.

In a similar way $\eta(y, h, E)$ is defined by exchanging the roles of x and y.

3.7. - Lemma. If E is a compact set in the xy-plane then (see 3.2, 3.6)

(1) $$\varkappa(E) \leqslant \liminf_{h \to 0} \left[\int \xi(x, h, E)\, dx + \int \eta(y, h, E)\, dy \right].$$

Proof. Take any (finite of infinite) sequence of disjoint continua $C_1, \ldots, C_j, \ldots$ in E. We can assume $\varkappa(E) \neq 0$ (otherwise (1) is obvious) and hence that $d(C_j) > 0$. The projection of C_j onto the x-axis is an interval $a_j \leqslant x \leqslant b_j$ (may have $a_j = b_j$) and onto the y-axis is an interval

$c_j \leqslant y \leqslant d_j$. Let $g_j(x)$ be the characteristic function of the interval on the x-axis. Take any integer n and let $\delta(n)$ be the minimum distance of pairs in $C_1, \ldots, C_n$. Take h such that $0 < h < \delta(n)$. For given x, clearly $g_1(x) + \ldots + g_n(x)$ is the number of those amongst $C_1, \ldots, C_n$ that are intersected by the vertical line through x. Hence

$$g_1(x) + \ldots + g_n(x) \leqslant \xi(x, h, E)\,,$$

and thus on integrating

$$\sum_{j=1}^{n} (b_j - a_j) \leqslant \int \xi(x, h, E)\, \mathrm{d}x\,.$$

Similarly

$$\sum_{j=1}^{n} (d_j - c_j) \leqslant \int \eta(y, h, E)\, \mathrm{d}y\,.$$

Thus

$$\sum_{j=1}^{n} d(C_j) \leqslant \sum_{j=1}^{n} [(b_j - a_j) + (d_j - c_j)] \leqslant \int \xi(x, h, E)\, \mathrm{d}x + \int \eta(y, h, E)\, \mathrm{d}y\,.$$

Now h was subject only to the condition $0 < h < \delta(n)$. So, keeping n fixed and making $h \to 0$, we get

$$\sum_{j=1}^{n} d(C_j) \leqslant \liminf_{h \to 0} \left[\int \xi(x, h, E)\, \mathrm{d}x + \int \eta(y, h, E)\, \mathrm{d}y \right].$$

Letting $n \to \infty$ we obtain

$$\sum_{j=1}^{\infty} d(C_j) \leqslant \liminf_{h \to 0} \left[\int \xi(x, y, E)\, \mathrm{d}x + \int \eta(y, h, E)\, \mathrm{d}y \right].$$

Since the sequence $C_1, \ldots, C_j, \ldots$ was arbitrary, (1) follows.

3.8. – Throughout the remainder of this section we shall assume that $f(x, y)$ is a continuous function defined for all x and y and that the LEBESGUE area of the surface $z = f(x, y)$ is finite over every rectangle.

Whe shall denote by S the point set consisting of the points $[x, y, z = f(x, y)]$.

3.9. – Lemma. Under the assumptions of 3.8 assume $S \cap K(a, r) \neq 0$. Then $[S \cap K(a, r)]'$ is an open set in the xy-plane, and (see 1.9)

$$\mathrm{fr}\,\{\,[S \cap K(a, r)]'\,\} \subset [S \cap B(a, r)]'\,.$$

Proof. Obvious.

3.10. – Theorem. Under the assumptions of 3.8 we have

$$H^1_\infty \{ [S \cap K(a, r)]' \} \leqslant \liminf_{h \to 0} \left\{ \int \xi(x, h, [S \cap B(a, r)]')\, dx + \right.$$

$$\left. + \int \eta(y, h, [S \cap B(a, r)]')\, dy \right\} .$$

Proof. The proof of the theorem follows by using successively the lemmas in 3.3, 3.7, and 3.9 (note that clearly $E_1 \subset E_2$ implies that $\varkappa(E_1) \leqslant \varkappa(E_2)$).

3.11. – For a bounded BOREL set B in the xy-plane we define $V_x(B_y, y, f)$ and $V_y(B_x, x, f)$ as in LA III.2.45. We set

$$W_x(B, f) = \int V_x(B_y, y, f)\, dy , \qquad W_y(B, f) = \int V_y(B_x, x, f)\, dx ,$$

$$\mathfrak{R}_x(B, f) = | B |_2 + W_x(B, f) , \qquad \mathfrak{R}_y(B, f) = | B |_2 + W_y(B, f) ,$$

where $| B |_2$ denotes the two-dimensional LEBESGUE measure of B.

It should be noted that $\mathfrak{R}_x$ and $\mathfrak{R}_y$ are finite, non-negative, completely additive functions of bounded BOREL sets in the plane.

3.12. – Theorem. Under the assumptions of 3.8 we have the inequality

$$H^1_\infty \{ [S \cap K(a, \varrho)]' \} \leqslant 8 \, \frac{\mathfrak{R}_x \{ [S \cap K(a, 2\varrho)]' \} + \mathfrak{R}_y \{ [S \cap K(a, 2\varrho)]' \}}{2\varrho} ,$$

where a is not required to lie on S and $\varrho > 0$ is arbitrary.

Proof. Set (see 1.9)

$$D = S \quad [K(a, r+h) - C(a, r-h)] , \qquad 0 < h < r , \qquad B = S \cap B(a, r) .$$

We first want an estimate of $\xi(x, h, B')$. Keep x and h fixed. Assume $\xi(x, h, B') \geqslant 1$ (otherwise we have a trivial estimate). On a vertical line through x we have points $(x, y_1), \ldots, (x, y_\xi)$ in B' such that $| y_i - y_j | \geqslant h$ for $i \neq j$. Consider, for $0 \leqslant \lambda \leqslant h$,

$$\varphi_j(\lambda) = \lambda + | f(x, y_j + \lambda/2) - f(x, y_j) | + | f(x, y_j) - f(x, y_j - \lambda/2) | .$$

Then $\varphi_j(0) = 0$, $\varphi_j(h) \geqslant h$. Thus there is a smallest $\lambda > 0$, call it λ_j, such that

$$\varphi_j(\lambda_j) = h/2 \quad \text{and} \quad \varphi_j(\lambda) \leqslant h/2 \quad \text{for} \quad 0 \leqslant \lambda \leqslant \lambda_j .$$

We assert that the intervals $I_j^+ : y_j \leqslant y \leqslant y_j + \lambda_j/2$ and $I_j^- : y_j - \lambda_j/2 \leqslant y \leqslant y_j$ on the vertical line through x lie in D'. Indeed, take I_j^+ and let $0 \leqslant \lambda \leqslant \lambda_j$. Let p_j and p_j^λ be the points of S over (x, y_j) and $(x, y_j + \lambda/2)$. Then

$$d(p_j, p_j^\lambda) \leqslant \lambda/2 + | f(x, y_j + \lambda/2) - f(x, y_j) | \leqslant \varphi_j(\lambda) \leqslant h/2 .$$

Since $(x, y_j) \in B'$, $p_j \in B$ and hence $p_j^\lambda \in D$. Therefore, $(x, y_j + \lambda/2) \in D'$. Thus the 2ξ intervals I_j^+, I_j^- are non-overlapping intervals in $(D')_x$. We thus have the inequality

$$\frac{1}{2} h\xi(x, h, B') = \sum_{j=1}^{\xi} \varphi_j(\lambda_j) \leqslant |(D')_x|_1 + V_y[(D')_x, x, f],$$

where $|(D')_x|_1$ indicates the linear LEBESGUE measure of $(D')_x$. Integration yields

$$\frac{1}{2} h \int \xi(x, h, B')\,\mathrm{d}x \leqslant \mathfrak{N}_y(D', f)\,. \tag{1}$$

Now, since the projection from the surface S upon the xy-plane is one-to-one and $\mathfrak{N}_y(D', f)$ is additive, we have the following relationships:

$$D \subset S \cap [K(a, r+h) - K(a, r-h)] = S \cap K(a, r+h) - S \cap K(a, r-h)\,,$$

$$D' \subset [S \cap K(a, r+h)]' - [S \cap K(a, r-h)]'\,,$$

$$\mathfrak{N}_y(D', f) \leqslant \mathfrak{N}_y\{[S \cap K(a, r+h)]' - \mathfrak{N}_y\{[S \cap K(a, r-h)]'\}\,.$$

From (1) we thus have the inequality

$$\int \xi\{x, h, [S \cap B(a, r)]'\}\,\mathrm{d}x \leqslant 4\,\frac{\mathfrak{N}_y\{[S \cap K(a, r+h)]'\} - \mathfrak{N}_y\{[S \cap K(a, r-h)]'\}}{2h}\,. \tag{2}$$

Similarly

$$\int \eta\{y, h, [S \cap B(a, r)]'\}\mathrm{d}y \leqslant 4\,\frac{\mathfrak{N}_x\{[S \cap K(a, r+h)]'\} - \mathfrak{N}_x\{[S \cap K(a, r-h)]'\}}{2h}\,. \tag{3}$$

Now let

$$\varphi(r) = \mathfrak{N}_y\{[S \cap K(a, r)]'\}, \qquad \psi(r) = \mathfrak{N}_x\{[S \cap K(a, r)]'\}\,. \tag{4}$$

Then $\varphi(r)$ and $\psi(r)$ are non-decreasing functions of r and hence $\varphi'(r)$ and $\psi'(r)$ exist for almost every r. From the inequalities (2) and (3) we thus have

$$\liminf_{h \to 0} \left[\int \xi\{x, h, [S \cap B(a, r)]'\}\,\mathrm{d}x + \int \eta\{y, h, [S \cap B(a, r)]'\}\,\mathrm{d}y \leqslant \right. \tag{5}$$

$$\left. \leqslant 4[\varphi'(r) + \psi'(r)]\right] \qquad \text{for almost every } r\,.$$

Set

$$U(r) = H^1_\infty\{[S \cap K(a, r)]'\}\,. \tag{6}$$

From the theorem in 3.10 and the relations (5) and (6) we have

$$U(r) \leqslant 4[\varphi'(r) + \psi'(r)] \qquad \text{for almost every } r.$$

Thus, for any $\varrho > 0$,

$$(7)\quad \varrho U(\varrho) \leqslant \int_{\varrho}^{2\varrho} U(r)\,\mathrm{d}r \leqslant 4\left[\int_{\varrho}^{2\varrho} \varphi'(r)\,\mathrm{d}r + \int_{\varrho}^{2\varrho} \psi'(r)\,\mathrm{d}r\right] \leqslant$$

$$\leqslant 4\,[\varphi(2\varrho) + \psi(2\varrho) - \varphi(\varrho) - \psi(\varrho)] \leqslant 4[\varphi(2\varrho) + \psi(2\varrho)]\,.$$

The statement of the Theorem then follows from the relations (7), (6) and (4).

3.13. – For a bounded BOREL set B in the xy-plane we set (see 3.11)

$$\mathfrak{R}(B) = \mathfrak{R}_x(B) + \mathfrak{R}_y(B)\,.$$

Then $\mathfrak{R}(B)$ is a finite, non-negative, completely additive function of bounded BOREL sets. From the Theorem in 3.12 we have the following Theorem.

Theorem. Under the assumptions of 3.8 we have the inequality

$$H^1_\infty\,\{[S \cap K(a,\,\varrho)]'\} \leqslant 8\,\frac{\mathfrak{R}\,\{[S \cap K(a,\,2\varrho)]'\}}{2\varrho}\,,$$

where the point a need not lie on S and $\varrho > 0$ is arbitrary.

4. – The inequality of BESICOVITCH.

4.1. – This inequality is stated as a theorem in 4.3 below. Let us recall that BESICOVITCH proved the inequality for the special case where $f(x, y)$ is ACT. The more general version of the inequality, stated as a theorem in 4.4 below, illustrates the convenience of using the important fact that the LEBESGUE area gives rise to a completely additive function of BOREL sets.

4.2. – Lemma. Under the assumptions of 3.8, let S_R be the portion of S over the oriented rectangle R. Then

$$H^2(S_R) \leqslant 2048\,\mathfrak{R}(R)\,,$$

where $\mathfrak{R}$ is the set function defined in 3.13.

Proof. Let A be the set of points (x, y, z) such that $(x, y) \in R^0$ and $z = f(x, y)$, where R^0 indicates the interior of the rectangle R. On A we define for BOREL subsets $B \subset A$ the set function

$$\varphi(B) = \mathfrak{R}(B')\,.$$

Then $\varphi(B)$ is a finite, non-negative, completely additive function of BOREL sets on A. Take a point a on A. Then $\mathfrak{R}\{[A \cap K(a, r)]'\} = \mathfrak{R}\{[S \cap K(a, r)]'\}$ for r small enough. Hence $\varphi\{[A \cap K(a, r)]\} = \mathfrak{R}\{[S \cap K(a, r)]'\}$ for r small enough. Thus, by the Theorem in 3.13 with $\varrho = r/2$,

$$\frac{\varphi[A \cap K(a, r)]}{\pi r^2} \geqslant \frac{1}{16\pi} \frac{H^1_\infty\{[S \cap K(a, r/2)]'\}}{r/2}.$$

Hence, by the Theorem in 2.6,

$$\limsup_{r \to 0} \frac{\varphi[A \cap K(a, r)]}{\pi r^2} \geqslant \frac{1}{16 \cdot 32\pi}$$

for $a \in A$, except for a set of H^2 measure zero. Therefore, by the Lemma in 2.7,

$$H^2(A) \leqslant 16 \cdot 32\pi\varphi(A) \leqslant 16 \cdot 32\pi\varphi(S_R) = 16 \cdot 32\pi\mathfrak{R}(R) \leqslant 16 \cdot 32 \cdot 4\,\mathfrak{R}(R) = 2048\,\mathfrak{R}(R).$$

Since clearly $H^2(S_R - A) = 0$ we have $H^2(S_R) \leqslant 2048\,\mathfrak{R}(R)$.

4.3. – By LA V.3.9 we have that $\mathfrak{R}(R, f) = 2\,|\,R\,|_2 + W_x(R, f) + W_y(R, f) \leqslant \leqslant 4A(R)$, where $A(R)$ denotes the LEBESGUE area of the part of S over R. Since $4 \cdot 2048 = 8192 < 10^4 < 10^5$, we have in view of 4.2 the following inequality of BESICOVITCH (cf. 4.1).

Theorem. Under the assumptions of 3.8 let S_R be the portion of S over the oriented rectangle R. Then

$$H^2(S_R) \leqslant 10^5 A(R).$$

4.4. – Under the assumptions of 3.8, by LA V.3.28, the rectangle function $A(R)$ has a completely additive extension to BOREL sets in the plane, and this extension satisfies the relation $A(R^0) = A(R)$. Let O be a bounded open set in the plane. Then $O = R_1 + R_2 + \ldots$ where R_n is an oriented rectangle and $R_i^0 \cap R_j^0 = 0$ for $i \neq j$. Hence $A(O) \geqslant A(R_1^0) + A(R_2^0) + \ldots = A(R_1) + A(R_2) + \ldots \geqslant \geqslant A(O)$ and therefore $A(O) = A(R_1) + A(R_2) + \ldots$. Let S_O denote the portion of S over O. Then

$$H^2(S_O) \leqslant \sum_i H^2(S_{R_i}) \leqslant 10^5 \sum_i A(R_i) = 10^5 A(O).$$

We thus have the following corollary to the Theorem in 4.3.

Theorem. Under the assumptions of 3.8 let S_O denote the portion of S over a bounded open set O in the xy-plane. Then

$$H^2(S_O) \leqslant 10^5 A(O).$$

5. - Relations between HAUSDORFF measure and LEBESGUE area.

5.1. – Let

(1) $$T: \quad x = x(u, v)\,, \quad y = y(u, v)\,, \quad z = z(u, v)\,, \quad (u, v) \in Q\,,$$

be a continuous mapping from the unit square Q: $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$, in the uv-plane, into Euclidean xyz-space E_3. We shall use vector notation $\mathfrak{x}$ to denote a point (x, y, z) in E_3. In the next few sections we shall give some definitions and properties for continuous mappings T.

5.2. – A continuous mapping T as given in 5.1 is called BVT if each of the coordinate functions in (1) of 5.1 are BVT and is called ACT if each of the coordinate functions in (1) of 5.1 are ACT (see LA III.2.64).

5.3. – A component of an inverse set $T^{-1}(\mathfrak{x})$ for $\mathfrak{x} \in T(Q)$ is called a *maximal model continuum*. A maximal model continuum will be called non-degenerate if it does not reduce to a single point.

5.4. – A continuous mapping T as given in 5.1 is called monotone if $T^{-1}(\mathfrak{x})$ is a continuum for each $\mathfrak{x} \in T(Q)$.

5.5. – If $x_u, \ldots, z_v$ exist at a point then $W(u, v) = \sqrt{EG - F^2}$ will be called the Jacobian of the mapping, where $E = x_u^2 + y_u^2 + z_u^2$, $G = x_v^2 + y_v^2 + z_v^2$, $F = x_u x_v + y_u y_v + z_u z_v$.

5.6. – For each $\mathfrak{x} \in E_3$ and set $E \subset Q$ we define $N(\mathfrak{x}, T, E)$ to be the number (possibly $+\infty$) of points $(u, v) \in E$ for which $T(u, v) = \mathfrak{x}$. We shall use the well known fact that $N(\mathfrak{x}, T, B)$ is an H^2 measurable function whenever B is a BOREL set.

5.7. – Lemma. Let T be a continuous mapping as given in 5.1. Assume T is BVT (see 5.2). Then there exists a dense denumerable set of vertical line segments and a dense denumerable set of horizontal line segment

$$s_{u_i}: \ u = u_i\,, \quad 0 \leqslant v \leqslant 1\,; \qquad s_{v_j}: \ v = v_j\,, \quad 0 \leqslant u \leqslant 1\,; \qquad i, j = 1, 2, \ldots,$$

such that

$$H^2[T(s_{u_i})] = 0\,, \qquad H^2[T(s_{v_j})] = 0\,.$$

Proof. Choose the line segments so that T is of bounded variation on each of them.

5.8. – Lemma. Let T be a continuous mapping as given in 5.1. Assume T is BVT and let E_0 be the set-sum of all the non-degenerate maximal model continua determined by T (see 5.3). Then $H^2[T(E_0)] = 0$.

Proof. Let γ be a non-degenerate maximal model continuum determined by T and let s_{u_i}, s_{v_j} be the line segments obtained in the Lemma in 5.7. Then either $\gamma \cap s_{u_i} \neq 0$ for some i or $\gamma \cap s_{v_j} \neq 0$ for some j. Hence

$$T(E_0) \subset \bigcup_i T(s_{u_i}) + \bigcup_j T(s_{v_j}),$$

and

$$H^2[T(E_0)] \leqslant \sum_i H^2[T(s_{u_i})] + \sum_j H^2[T(s_{v_j})] = 0 .$$

5.9. – Lemma. Let T be a continuous mapping as given in 5.1. Assume T is BVT and monotone (see 5.4). Then for any Borel set $B \subset Q$ (see 5.6)

$$\int N(\mathfrak{x}, T, B)\, dH^2 = H^2[T(B)] . \tag{1}$$

Proof. Let E_0 be the set-sum of all the non-degenerate maximal model continua determined by T. By the Lemma in 5.8, $H^2[T(E_0)] = 0$. Thus, since T is monotone, $N(\mathfrak{x}, T, B) > 1$ only on a set of H^2 measure zero. Hence, (1) follows.

5.10. – Lemma. Let T be a continuous mapping as given in 5.1. Assume T is BVT, monotone and that $W(u, v)$ (see 5.5) is summable. Then, for any Borel set $B \subset Q$,

$$\iint_B W\, du\, dv \leqslant H^2[T(B)] .$$

Proof. We have a Borel set $B_0 \subset B$ such that $|B_0|_2 = |B|_2$ and W exists everywhere on B_0. By a theorem of Federer ([3], 6.3, p. 455) we have then

$$\iint_{B_0} W\, du\, dv = \int N(\mathfrak{x}, T, B_0)\, dH^2 . \tag{1}$$

Hence, by (1) and the Lemma in 5.9,

$$\iint_B W\, du\, dv = \iint_{B_0} W\, du\, dv = \int N(\mathfrak{x}, T, B_0)\, dH^2 = H^2[T(B_0)] \leqslant H^2[T(B)] .$$

5.11. – Let T be a continuous mapping as given in 5.1. Then T may be thought of as a representation of a Fréchet surface of the type of the two cell (see LA II.3.44). We denote by $A(T)$ the LEBESGUE area of this surface.

If R is an oriented rectangle in Q then T_R will denote the continuous mapping T operating from R. We shall also use $A(R)$ to denote $A(T_R)$.

5.12. – Lemma. Let T be a continuous mapping as given in 5.1. Let O be an open set in Q such that T is ACT on every oriented rectangle in O and the partial derivatives of the coordinate functions are summable with their squares on O. Then

$$A(T) \geqslant \iint_O W \, du \, dv .$$

Proof. On every oriented rectangle $R \subset O$ we have (see LA V.2.27)

$$A(T_R) = \iint_R W \, du \, dv .$$

Now $O = R_1 + R_2 + \ldots$, where R_n is an oriented rectangle and $R_i^0 \cap R_j^0 = 0$ for $i \neq j$. By LA V.2.15 we thus have

$$A(T) \geqslant \sum_n A(T_{R_n}) = \sum_n \iint_{R_n} W \, du \, dv = \iint_O W \, du \, dv .$$

5.13. – Let $f(x, y)$ be a continuous function defined for (x, y) in the unit square Q: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Then $z = f(x, y)$, $(x, y) \in Q$ is a non parametric surface. We shall denote by

$$\bar{f}: \quad x = x, \quad y = y, \quad z = f(x, y), \quad (x, y) \in Q,$$

the continuous mapping from Q into E_3.

It should be noted that if $A(\bar{f}) < \infty$ then $f(x, y)$ can be extended by reflections to the entire xy-plane to satisfy the conditions of 3.8.

5.14. – Let $f(x, y)$ be a continuous function defined for (x, y) in the unit square Q: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Assume $f(x, y)$ is BVT. Then (see 5.13) $A(\bar{f}) < \infty$ and $\bar{f}$ is topological. Hence there exists a generalized conformal representation (of the surface represented by $\bar{f}$)

$$(1) \qquad T: \quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in K,$$

where K is the unit square $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$ in the uv-plane. T satisfies the following conditions:

(i) T is a continuous monotone mapping.

(ii) T is ACT, the partial derivatives $x_u, \ldots, z_v$ exist almost everywhere in K and are summable with their square on K, and

$$A(T) = \iint_K W \, du \, dv .$$

(iii) For the first two coordinate functions in (1) the mapping

$$m: \quad x = x(u, v), \quad y = y(u, v), \quad (u, v) \in K,$$

is a continuous monotone mapping from K onto Q and

$$T = \bar{f}m, \quad m: K \Rightarrow Q, \quad \bar{f}: Q \to E_3.$$

5.15. - Lemma. Under the assumptions of 5.14, for every oriented rectangle $R \subset Q$

(1)
$$\iint\limits_{m^{-1}(R-R^0)} W \, du \, dv = 0.$$

Proof. By the Lemma in 5.10 the integral in (1) is less than or equal to $H^2\{T[m^{-1}(R-R^0)]\} = H^2[\bar{f}(R-R^0)]$ and clearly $H^2[\bar{f}(R-R^0)] = 0$.

5.16. - Lemma. Under the assumptions of 5.14, for every oriented rectangle $R \subset Q$

$$A(R) = \iint\limits_{m^{-1}(R)} W \, du \, dv.$$

Proof. Let m_R be a monotone retraction of Q onto R (see LA II.2.39) and let $\bar{f}_R$ be the mapping $\bar{f}$ operating from R. Then (see LA II.1.64) $\bar{f}_R$ and $\bar{f}m_R m$ are representations of the same surface and hence $A(\bar{f}_R) = A(\bar{f}m_R m)$. Now $\bar{f}m_R m$ and T are identical on the open set $m^{-1}(R^0)$ and hence, by the Lemmas in 5.12 and 5.15 we have

(1)
$$A(R) = A(\bar{f}_R) = A(\bar{f}m_R m) \geqslant \iint\limits_{m^{-1}(R^0)} W \, du \, dv = \iint\limits_{m^{-1}(R)} W \, du \, dv.$$

Now Q can be divided into 9 oriented rectangle $R_1, \ldots, R_9$ one of which is R and for each of which (1) holds. Then, by LA V.3.18, (1) and the Lemma in 5.15 we have

(2)
$$A(Q) = \sum_{i=1}^{9} A(R_i) \geqslant \sum_{i=1}^{9} \iint\limits_{m^{-1}(R_i)} W \, du \, dv = \iint\limits_{K} W \, du \, dv = A(T) = A(Q).$$

From (2) it follows that the equality sign holds in (1).

5.17. - Lemma. Under the assumptions of 5.14 for every open set $O \subset Q$ (see 4.4)

$$A(O) = \iint\limits_{m^{-1}(O)} W \, du \, dv.$$

Proof. $O = R_1 + R_2 + \dots$, where R_n is an oriented rectangle and $R_i^0 \cap R_j^0 = 0$ for $i \neq j$. Then (see 4.4) by the Lemmas in 5.16 and 5.15 we have

$$A(O) = \sum_n A(R_n) = \sum_n \iint_{m^{-1}(R_n)} W\, du\, dv = \iint_{m^{-1}(O)} W\, du\, dv .$$

5.18. – Lemma. Under the assumptions of 5.14 let E be a set of LEBESGUE plane measure zero in K. Then

$$(1) \qquad H^2[T(E)] = 0 \quad \text{and} \quad H^2[T(K - E)] = H^2[T(K)] .$$

Proof. Let E_0 be the set-sum of the non-degenerate maximal model continua determined by T. Set $E_1 = E - E_0$. For $\varepsilon > 0$ given we have an open set $O \supset E_1$ such that

$$\iint_O W\, du\, dv < \varepsilon .$$

Let O^* be the set-sum of all the maximal model continua determined by T and lying in O. Then O^* is an open set (see LA II.1.12) containing E_1, $m(O^*)$ is an open set in Q and $O^* = m^{-1}m(O^*)$. Thus by the Theorem in 4.4 and the Lemma in 5.17

$$H^2[T(E_1)] \leqslant H^2[T(O^*)] = H^2[\bar{f}m(O^*)] \leqslant 10^5 A[m(O^*)] = 10^5 \iint_{O^*} W\, du\, dv < 10^5 \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, $H^2[T(E_1)] = 0$. Since $E \subset E_1 + E_0$, by the Lemma in 5.8

$$H^2[T(E)] \leqslant H^2[T(E_1) + T(E_0)] \leqslant H^2[T(E_1)] + H^2[T(E_0)] = 0 .$$

Thus $H^2[T(E)] = 0$ and the first part (1) holds. Now

$$H^2[T(K - E)] \leqslant H^2[T(K)] \leqslant H^2[T(K - E)] + H^2[T(E)] = H^2[T(K - E)] .$$

Thus the equality sign holds in the last relationship and the second part of (1) holds.

5.19. – Theorem (BESICOVITCH [1], FEDERER [4]; cf. the introduction). Let $f(x, y)$ be a continuous function defined for (x, y) in the unit square Q: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Assume $f(x, y)$ is BVT. Then

$$A(Q) = H^2[\bar{f}(Q)] .$$

Proof. Using the notation in 5.14 let E be the set of points where W does not exist. Then E is of LEBESGUE plane measure zero. Let B be a

BOREL set of LEBESGUE plane measure zero containing E. By the theorem of FEDERER ([3], 6.3, p. 455), the Lemmas in 5.9 and 5.18 we have

$$A(Q) = A(T) = \iint_K W \,du\,dv = \iint_{K-B} W \,du\,dv =$$

$$= \int N(\mathfrak{x}, T, K-B)\,dH^2 = H^2[T(K-B)] = H^2[T(K)] = H^2[\bar{f}(Q)] .$$

5.20. – Lemma. Let Φ be a Lipschitzian transformation with constant 1 from xyz-space E_3 into $x'y'z'$-space E_3'. If E is a bounded H^2 measurable set in E_3 with $H^2(E) < \infty$ then

(1) $$H^2(E) \geqslant \int N(\mathfrak{x}', \Phi, E)\,dH^2 .$$

Proof. Obviously $N(\mathfrak{x}', \Phi, E)$ is an H^2 measurable function. There are only a denumerable number of planes π parallel to the coordinate planes such that $H^2(\pi \cap E) \neq 0$. Let K be a cube with sides parallel to the coordinate planes, containing E in its interior, and such that for each integer n it can be divided into n^3 congruent cubes $k_1^n, \ldots, k_{n^3}^n$ by planes π parallel to the coordinates planes for which $H^2(\pi \cap E) = 0$. For each integer n let $f_n(\mathfrak{x}')$ be the number of cubes $k_1^n, \ldots, k_{n^3}^n$ containing points of $\Phi^{-1}(\mathfrak{x}')$. Then

(2) $$H^2(E) = \sum_i H^2(E \cap k_i^n) \geqslant \sum_i H^2[\Phi(E \cap k_i^n)] = \int f_n(\mathfrak{x}')\,dH^2 .$$

Except for a set of H^2 measure zero, $f_n(x')$ converges monotonically upward to $N(\mathfrak{x}', \Phi, E)$. Thus (1) follows from (2).

5.21. – Lemma. Let T be a continuous mapping as given in 5.1 and let Φ be a Lipschitzian transformation with constant 1 from E_3 into $x'y'z'$-space E_3'. Then

(1) $$\int N(\mathfrak{x}, T, Q)\,dH^2 \geqslant \int N(\mathfrak{x}', \Phi T, Q)\,dH^2 .$$

Proof. Assume the integral on the left of (1) is finite (otherwise the inequality holds). For each integer n let A_n be the set of points $\mathfrak{x} \in E_3$ for which $N(\mathfrak{x}, T, Q) = n$ and let A_∞ be the set of points $\mathfrak{x} \in E_3$ for which $N(\mathfrak{x}, T, Q) = \infty$. Then $H^2(A_\infty) = 0$ and $H^2[\Phi(A_\infty)] = 0$. Thus

(2) $$\int N(\mathfrak{x}, T, Q)\,dH^2 = \sum_n n H^2(A_n)$$

and

(3) $$N(\mathfrak{x}', \Phi T, Q) = \sum_n n N(\mathfrak{x}', \Phi, A_n) , \qquad \mathfrak{x}' \notin \Phi(A_\infty) .$$

From (2), the Lemma in 5.20 and (3) we thus have

$$\int N(\mathfrak{x}, T, Q)\,\mathrm{d}H^2 = \sum_n nH^2(A_n) \geqslant \sum_n \int nN(\mathfrak{x}', \Phi, A_n)\,\mathrm{d}H^2 =$$

$$= \int \sum_n nN(\mathfrak{x}', \Phi, A_n)\,\mathrm{d}H^2 = \int N(\mathfrak{x}', \Phi T, Q)\,\mathrm{d}H^2 .$$

5.22. – Theorem (BESICOVITCH [1], FEDERER [4]; cf. the introduction). Let $f(x, y)$ be a continuous function defined for (x, y) in the unit square Q: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Then

$$A(Q) = H^2[\bar{f}(Q)] . \tag{1}$$

Proof. (a) Assume $A(Q) = \infty$ and $H^2[\bar{f}(Q)] = \infty$. Then (1) holds.

(b) Assume $A(Q) < \infty$. Then $f(x, y)$ is BVT and (1) follows from the Theorem in 5.19.

(c) Assume $H^2[\bar{f}(Q)] < \infty$. We note that $N(\mathfrak{x}, \bar{f}, Q)$ is 1 for $\mathfrak{x} \in \bar{f}(Q)$ and is 0 for $\mathfrak{x} \notin \bar{f}(Q)$. Let $a(T)$ denote the lower area of the surface represented by a continuous mapping as given in 5.1 (see LA V.1.7). Let P_1, P_2, P_3 denote respectively the projections of E_3 upon the yz-plane, zx-plane, xy-plane. Then by LA V.3.7, LA V.1.3, LA V.1.4 and the Lemma in 5.21 we have

$$A(Q) = a(\bar{f}) \leqslant a(P_1\bar{f}) + a(P_2\bar{f}) + a(P_3\bar{f}) \leqslant \iint N(y, z; P_1\bar{f}, Q)\,\mathrm{d}y\,\mathrm{d}z +$$

$$+ \iint N(z, x; P_2\bar{f}, Q)\,\mathrm{d}z\,\mathrm{d}x + \iint N(x, y; P_3\bar{f}, Q)\,\mathrm{d}x\,\mathrm{d}y \leqslant$$

$$\leqslant 3 \int N(\mathfrak{x}, \bar{f}, Q)\,\mathrm{d}H^2 = 3H^2[\bar{f}(Q)] < \infty .$$

Thus $A(Q) < \infty$ and (1) follows from (*b*).

5.23. – Let T be a continuous mapping as given in 5.1. T will be called a *quasi-Lipschitzian mapping* if the following conditions are satisfied. There exist a function $f(u, v)$ which is BVT on Q and a constant M such that for the continuous mapping

$$\bar{f}: \quad u = u , \quad v = v , \quad w = f(u, v) , \quad (u, v) \in Q ,$$

from Q into uvw-space, the inequality

$$d[T(u_1, v_1), T(u_2, v_2)] \leqslant M \cdot d[\bar{f}(u_1, v_1), \bar{f}(u_2, v_2)]$$

holds for every pair of points (u_1, v_1), (u_2, v_2) in Q.

If $f(u, v) \equiv 0$ the definition of a quasi-Lipschitzian mapping reduces to the ordinary definition of a Lipschitzian mapping.

5.24. – Lemma. Let T be a quasi-Lipschitzian mapping as given in 3.23. Then there exists a Lipschitzian transformation Φ with constant M from $\bar{f}(Q)$ onto $T(Q)$ such that $T = \Phi\bar{f}$.

Proof. For a point $\mathfrak{y} \in \bar{f}(Q)$, $\mathfrak{y} = \bar{f}(u, v)$ we set

$$\mathfrak{x} = \Phi(\mathfrak{y}) = T\bar{f}^{-1}(\mathfrak{y}) = T(u, v) .$$

For $\mathfrak{y}_1 = \bar{f}(u_1, v_1)$, $\mathfrak{y}_2 = \bar{f}(u_2, v_2)$, (u_1, v_1) and (u_2, v_2) in Q we have

$$M \cdot d(\mathfrak{y}_1, \mathfrak{y}_2) = M \cdot d[\bar{f}(u_1, v_1), \bar{f}(u_2, v_2)] \geqslant d[T(u_1, v_1), T(u_2, v_2)] = d[\Phi(\mathfrak{y}_1), \Phi(\mathfrak{y}_2)] .$$

Thus Φ is a Lipschitzian transformation from $\bar{f}(Q)$ onto $T(Q)$ and $T = T\bar{f}^{-1}\bar{f} = \Phi\bar{f}$.

5.25. – Let T be a quasi-Lipschitzian mapping as given in 5.23. By the Lemma in 5.24 there exists a Lipschitzian transformation Φ from $\bar{f}(Q)$ onto $T(Q)$ such that

$$(1) \qquad T = \Phi\bar{f} .$$

As anoted in 5.14 there exists a generalized conformal representation, of the surface represented by $\bar{f}$,

$$(2) \qquad \bar{f}^*: \quad u = u(\alpha, \beta) , \quad v = v(\alpha, \beta) , \quad w = w(\alpha, \beta) , \quad (\alpha, \beta) \in K ,$$

where K is the unit square $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$ in the $\alpha\beta$-plane. Then

$$(3) \qquad m: \quad u = u(\alpha, \beta) , \quad v = v(\alpha, \beta) , \quad (\alpha, \beta) \in K ,$$

is a continuous monotone mapping from K onto Q and

$$(4) \qquad \bar{f}^* = \bar{f}m .$$

We set

$$(5) \qquad T^* = Tm: \quad x = x^*(\alpha, \beta) , \quad y = y^*(\alpha, \beta) , \quad z = z^*(\alpha, \beta) , \quad (\alpha, \beta) \in K .$$

By LA II.1.66 T and T^* are representations of the same surface and hence

$$(6) \qquad A(T^*) = A(T) .$$

The mapping T^* can be written in the following forms:

$$(7) \qquad T^* = Tm = \Phi\bar{f}m = \Phi\bar{f}^* .$$

5.26. – Lemma. For the continuous mapping T^* defined in (5) of 5.25 each of the coordinate functions $x^*(\alpha, \beta)$, $y^*(\alpha, \beta)$, $z^*(\alpha, \beta)$ are ACT on K, and the first partial derivatives $x^*_\alpha, \ldots, z^*_\beta$ exist almost everywhere in K and are summable with their squares on K.

Proof. The fact that the coordinate functions in (5) of 5.25 are ACT on K follows immediately from the last form of T^* in (7) of 5.25 where Φ is a Lipschitzian transformation and from the fact that the coordinate functions of $\bar{f}^*$ in (2) of 5.25 are ACT on K.

Since the coordinate functions are ACT the partial derivatives $x_\alpha^*, \ldots, z_\beta^*$ exist almost everywhere on K. Let p and q be two points in the $\alpha\beta$-plane. Then

$$d[x^*(p), x^*(q)] \leqslant d[T^*(p), T^*(q)] \leqslant M d[\bar{f}^*(p), \bar{f}^*(q)] \leqslant$$
$$\leqslant M\{d[u(p), u(q)] + d[v(p), v(q)] + d[w(p), w(q)]\}.$$

Hence almost everywhere in K

$$(x_\alpha^*)^2 \leqslant 3M^2(u_\alpha^2 + v_\alpha^2 + w_\alpha^2).$$

Similar inequalities hold for $x_\beta^{*2}, \ldots, z_\beta^{*2}$. Since $\bar{f}^*$ is a generalized conformal mapping, $u_\alpha, \ldots, w_\beta$ are summable with their squares and hence $x_\alpha^*, \ldots, z_\beta^*$ are summable with their squares.

5.27. – Lemma. Let T be a quasi-Lipschitzian mapping as given in 5.23. Using the notations and mappings defined in 5.25 we have

$$N(\mathfrak{x}, T, Q) = N(\mathfrak{x}, T^*, K)$$

for every $\mathfrak{x} \in E_3$ except for a set of H^2 measure zero.

Proof. Let E_0 be the set-sum of all the non-degenerate maximal model continua determined by $\bar{f}^* = \bar{f}m$. Since $\bar{f}$ is topological this set is the same as the set-sum of the non-degenerate maximal model continua determined by m. Then by Lemma in 5.8

$$H^2[T^*(E_0)] = H^2[\Phi\bar{f}^*(E_0)] \leqslant M^2H^2[\bar{f}^*(E_0)] = 0.$$

Since m is one-to-one on $K - E_0$,

$$N(\mathfrak{x}, T, Q) = N(\mathfrak{x}, T^*, K) \quad \text{for} \quad \mathfrak{x} \notin T^*(E_0)$$

and the statement of the lemma follows.

5.28. – Theorem. Let T be a quasi-Lipschitzian mapping as given in 5.23. Then

$$A(T) = \int N(\mathfrak{x}, T, Q)\, dH^2.$$

Proof. Using the notation and mappings defined in 5.25 let W^* be the Jacobian of the mapping T^* and let E be the set of points where W^* does not exist. Then the LEBESGUE plane measure of E is zero and by the Lemma in 5.18 we have

(1) $$H^2[T^*(E)] = H^2[\Phi \mathfrak{f}^*(E)] \leqslant M^2 H^2[\mathfrak{f}^*(E)] = 0 .$$

From (1) we thus have that

(2) $$N(\mathfrak{x}, T^*, K - E) = N(\mathfrak{x}, T^*, K)$$

for $\mathfrak{x} \in E_3$ except for a set of H^2 measure zero. By the Lemma in 5.26 and by LA V.2.27 it follows that $A(T^*)$ is given by integrating W^* over K. Using the relationship (6) in 5.25, the theorem of FEDERER ([3], 6.3, p. 455), the relationship (2) of the present section, and the Lemmma in 5.27, we obtain the desired equality:

$$A(T) = A(T^*) = \iint_K W^* \, du \, dv = \iint_{K-E} W^* \, du \, dv = \int N(\mathfrak{x}, T^*, K - E) \, dH^2 =$$

$$= \int N(\mathfrak{x}, T^*, K) \, dH^2 = \int N(\mathfrak{x}, T, Q) \, dH^2 .$$

5.29. – Let T be a continuous mapping as given in 5.1. Assume that the coordinate functions satisfy the following conditions: (i) $x(u, v)$, $y(u, v)$ satisfy a LIPSCHITZ condition with constant $M \geqslant 1$. (ii) $z(u, v)$ is BVT on Q. In the terminology of 5.23 let $\mathfrak{f}$: $u = u$, $v = v$, $w = z(u, v)$. Then, for (u_1, v_1) and (u_2, v_2) in Q,

$$d[T(u_1, v_1), T(u_2, v_2)] \leqslant d[x(u_1, v_1), x(u_2, v_2)] + d[y(u_1, v_1), y(u_2, v_2)] +$$

$$+ d[z(u_1, v_1), z(u_2, v_2)] \leqslant 3M \cdot d[\mathfrak{f}(u_1, v_1), \mathfrak{f}(u_2, v_2)] .$$

Thus if T satisfies the conditions (i) and (ii) then T is a quasi-Lipschitzian mapping and by the Theorem in 5.28, $A(T) = \int N(\mathfrak{x}, T, Q) \, dH^2$. Let us note that in the special case $x \equiv u$, $y \equiv v$ this result reduces to the theorem stated in 5.19.

5.30. – Let T be a continuous mapping as given in 5.1. For $\mathfrak{x} \in E_3$ we set

$N_H(\mathfrak{x}, T, Q) =$ number of points in $T^{-1}(\mathfrak{x})$,

$N_Y(\mathfrak{x}, T, Q) =$ number of maximal model continua in $T^{-1}(\mathfrak{x})$,

$N_B(\mathfrak{x}, T, Q) =$ number of maximal model continua in $T^{-1}(\mathfrak{x})$ which do not intersect $Q - Q^0$.

We set

$$A_H(T) = \int N_H(\mathfrak{x}, T, Q) \, dH^2$$

and following the definitions of area of YOUNG [9] and BESICOVITCH [2] we set

$$A_Y(T) = \int N_Y(\mathfrak{x}, T, Q)\, dH^2,$$

$$A_B(T) = \int N_B(\mathfrak{x}, T, Q)\, dH^2.$$

5.31. – Lemma. If $A_H(T) < \infty$, then $A_H(T) = A_Y(T)$.

Proof. Assume $A_H(T) < \infty$. Let E_0 be the set-sum of all the non-degenerate maximal model continua determined by T. Then $H^2(T[E_0)] = 0$ and $N_H(\mathfrak{x}, T, Q) = N_Y(\mathfrak{x}, T, Q)$ except on a set of H^2 measure zero. Hence $A_H(T) = A_Y(T)$.

5.32. – Lemma. If $H^2[T(Q - Q^0)] = 0$ then $A_Y(T) = A_B(T)$.

Proof. If $H^2[T(Q - Q^0)] = 0$ then $N_Y(\mathfrak{x}, T, Q) = N_B(\mathfrak{x}, T, Q)$ except on a set of H^2 measure zero. Hence $A_Y(T) = A_B(T)$.

5.33. – Theorem. Let T be a quasi-Lipschitzian mapping as given in 5.23. Then (see 5.30)

$$A(T) = A_H(T) = A_Y(T) = A_B(T).$$

Proof. By (6) of 5.25, the Lemma in 5.26 and LA V.2.27 we have $A(T) < \infty$. The first two equalities then follow from the Theorem in 5.28 and the Lemma in 5.31. Using the notations and mappings in 5.25 we have

$$H^2[T(Q - Q^0)] = H^2[\Phi\bar{f}(Q - Q^0)] \leqslant M^2 H^2[\bar{f}(Q - Q^0)] = 0,$$

since clearly $H^2[\bar{f}(Q - Q^0)] = 0$. The third equality then follows from the Lemma in 5.32.

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